NON-HAUSDORFF ÉTALE GROUPOIDS

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ABSTRACT. We present examples of non-Hausdorff, étale, essentially principal groupoids for which three results, known to hold in the Hausdorff case, fail. These results are: (A) the subalgebra of continuous functions on the unit space is maximal abelian within the reduced groupoid C^* -algebra, (B) every nonzero ideal of the reduced groupoid C^* -algebra has a nonzero intersection with the subalgebra of continuous functions on the unit space, and (C) the open support of a normalizer is a bisection.

1. INTRODUCTION

This paper is concerned with *étale groupoids* [10, 1, 9, 11, 4]. A topological groupoid G is said to be étale if its unit space $G^{(0)}$ is locally compact and Hausdorff, and if the range map “ r ” (and consequently also the source map “ s ”) is a local homeomorphism.

One may or may not assume the global topology of G to be Hausdorff, but, while non-Hausdorff *topological spaces* may be safely ignored in numerous applications of Topology, non-Hausdorff groupoids do occur in many essential situations, such as the holonomy groupoid of a foliation [2] or the groupoid of germs of a pseudogroup of local homeomorphisms on a topological space [11, Section 3].

Therefore, rather than dismissing non-Hausdorff groupoids as a nuisance, it is highly desirable to embrace them in the general theory.

An étale groupoid G is said to be *principal* if its *isotropy group bundle*, namely

$$G' := \{\gamma \in G : s(\gamma) = r(\gamma)\},$$

coincides with the unit space $G^{(0)}$, and it is said to be *essentially principal* if the *interior* of G' coincides with $G^{(0)}$. Principal groupoids correspond to *free* group actions, while the essentially principal ones correspond to *topologically free* actions, hence the relevance of these concepts.

Among the important consequences of the property of being essentially principal in the Hausdorff case, the following stand out:

- (A) $C_0(G^{(0)})$ is maximal abelian within the reduced groupoid C^* -algebra $C_r^*(G)$ [11, 4.2].
- (B) Every nonzero ideal of $C_r^*(G)$ has a nonzero intersection with $C_0(G^{(0)})$ (see the appendix for a precise statement).

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- (C) If u is a normalizer of $C_0(G^{(0)})$ within $C_r^*(G)$, then the *open support* of u , namely

$$\text{supp}'(u) = \{\gamma \in G : u(\gamma) \neq 0\},$$

is a bisection [11, Proposition 4.7].

These results underlie mainstream developments in the theory of C^* -algebras: (B) is related to uniqueness theorems for Cuntz-Krieger algebras [3, 2.15], [5, 13.2] and graph algebras [8], as well as to results on reduced crossed products by partial group actions [6, 2.6], while (A) and (C) are related to Cartan subalgebras [7, 11].

It should be stressed that (A), (B) and (C) are only known to hold under the assumption that G is Hausdorff!

In trying to embrace non-Hausdorff groupoids within the general theory, the author (and quite likely many other people) has spent a lot of energy in an effort to generalize the above facts beyond the Hausdorff case. After having failed to do so he has found examples of non-Hausdorff étale groupoids which provide counterexamples for all of the above statements. In what follows we shall discuss these examples in detail.

Our first example is related to an example by G. Skandalis [12] built with a different purpose, namely that of exhibiting a minimal foliation whose C^* -algebra is not simple.

The author would like to thank Jean Renault for many fruitful discussions and for bringing Skandalis' example to his attention. He would also like to thank Alcides Buss for many interesting discussions while the author was searching for the second example below.

2. THE FIRST EXAMPLE

Consider the following subsets of $\mathbb{R}^2$:

$$\begin{aligned} X &= [-1, 1] \times \{0\}, \\ Y &= \{0\} \times [-1, 1], \\ Z &= X \cup Y. \end{aligned}$$

Clearly Z is invariant under the action of the subgroup $H \subseteq GL_2(\mathbb{R})$ generated by

$$\sigma_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let G be the groupoid of germs for the action of H on Z (see Section (3) of [11] for the definition of the groupoid of germs for a given pseudogroup). As is the case for every groupoid of germs, G is essentially principal [11, 3.4].

We shall adopt a slightly simplified notation in relation to [11], namely the germ of the transformation φ at the point x will be denoted by $[\varphi, x]$, as opposed to Renault's notation $[y, \varphi, x]$, where $y = \varphi(x)$.

In the present case it is interesting to observe that

$$\begin{aligned} [\sigma_y, x] &= [I, x], & [\sigma_x \sigma_y, x] &= [\sigma_x, x], \\ [\sigma_x, y] &= [I, y], & [\sigma_x \sigma_y, y] &= [\sigma_y, y], \end{aligned}$$

for all $x \in X^* := X \setminus \{0\}$, and all $y \in Y^* := Y \setminus \{0\}$, where “ I ” stands for the identity map and where we denote the zero vector of $\mathbb{R}^2$ simply by “0”. We therefore see

that G consists of the following distinct elements:

- $[I, x]$, for $x \in X^*$,
- $[\sigma_x, x]$, for $x \in X^*$,
- $[I, 0]$,
- $[\sigma_y, 0]$,
- $[I, y]$, for $y \in Y^*$,
- $[\sigma_y, y]$, for $y \in Y^*$,
- $[\sigma_x, 0]$,
- $[\sigma_x \sigma_y, 0]$.

Observe that the isotropy group bundle G' is formed by the last three elements listed above, in addition to the units.

Recall that a *bisection* is a subset of G restricted to which both the range and source maps are injective. Consider the following open bisections of G :

- $U_1 = \{[I, z] : z \in Z\} = G^{(0)}$,
- $U_x = \{[\sigma_x, z] : z \in Z\}$,
- $U_y = \{[\sigma_y, z] : z \in Z\}$,
- $U_{xy} = \{[\sigma_x \sigma_y, z] : z \in Z\}$.

Let $f_1, f_x, f_y, f_{xy} \in C_c(G)$ (for the definition of $C_c(G)$ see [2], [9] or [4, 3.9]) be the characteristic function of U_1, U_x, U_y , and U_{xy} , respectively. Finally put

$$f = f_1 - f_x - f_y + f_{xy}.$$

By direct computation one checks that

- $f([I, 0]) = 1$,
- $f([\sigma_x, 0]) = -1$,
- $f([\sigma_y, 0]) = -1$,
- $f([\sigma_x \sigma_y, 0]) = 1$

and that f vanishes on all other points of G . In particular notice that the support of f (the set of points where f does not vanish, no closure) is the set

$$(2.1) \quad \{[I, 0], [\sigma_x, 0], [\sigma_y, 0], [\sigma_x \sigma_y, 0]\} = r^{-1}(\{0\}) = s^{-1}(\{0\}),$$

which is contained in G' .

2.1. Proposition. *For every $g \in C_c(G)$ one has that*

$$g * f = f * g = \lambda(g)f,$$

where $\lambda(g)$ is the scalar given by

$$\lambda(g) = g([I, 0]) - g([\sigma_x, 0]) - g([\sigma_y, 0]) + g([\sigma_x \sigma_y, 0]).$$

Proof. Recall that for every $\gamma \in G$ one has

$$(f * g)(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta), \quad \forall \gamma \in G.$$

If the above sum is nonzero, then there exists at least one pair (α, β) such that $\alpha\beta = \gamma$ and $f(\alpha) \neq 0$. As seen in (2.1), this implies that $r(\alpha) = 0$, and hence necessarily $r(\gamma) = 0$, as well. Therefore $f * g$ is supported in $r^{-1}(\{0\})$. A similar reasoning and the same conclusion apply to $g * f$.

We leave it to the reader to compute $(f * g)(\gamma)$ and $(g * f)(\gamma)$ for the four elements γ in $r^{-1}(\{0\})$, after which the result will become apparent. $\square$

The first conclusion to be drawn from the above result is:

2.2. Proposition. *Even though G is essentially principal (every groupoid of germs is essentially principal by [11, 3.4]), there is a nonzero ideal $J \subseteq C_r^*(G)$ for which $J \cap C_0(G^{(0)}) = \{0\}$.*

Proof. By Proposition 2.2 one has that $J := \mathbb{C}f$ is an ideal in $C_r^*(G)$. Since f is not in $C_0(G^{(0)})$, the intersection of J with $C_0(G^{(0)})$ is trivial. $\square$

The second conclusion is:

2.3. Proposition. *Even though G is essentially principal, one has that $C_0(G^{(0)})$ is not maximal abelian within $C_r^*(G)$.*

Proof. It is enough to notice that by Proposition 2.2 one has that f is a central element of $C_r^*(G)$, and hence commutes with every element of $C_0(G^{(0)})$, but f is not in $C_0(G^{(0)})$. $\square$

Since the support of f is contained in G' , and in view of [11, 4.2], it is not surprising that f commutes with every element of $C_0(G^{(0)})$.

3. STRANGE NORMALIZERS

Let n be a positive integer and let $I = I_n = \{1, 2, \dots, n\}$ be seen as a discrete topological space. On the product space $[0, 1] \times I$, we consider the equivalence relation “ $\sim$ ” according to which

$$(0, i) \sim (0, j), \quad \forall i, j \in I,$$

and such that no other pairs of points are related except for each point with itself. The quotient topological space

$$X = ([0, 1] \times I) / \sim$$

therefore looks like a star with n edges. Incidentally, in the special and very relevant case $n = 4$, notice that X is homeomorphic to the space Z of the previous section.

The equivalence class of $(0, i)$ will be denoted simply by 0, and if $t > 0$, the equivalence class of (t, i) , namely the singleton $\{(t, i)\}$, will be denoted by (t, i) by abuse of language.

Let S_n be the group of permutations of I and consider the action of S_n on $[0, 1] \times I$, where each $\sigma \in S_n$ acts by

$$(t, i) \mapsto (t, \sigma(i)).$$

The equivalence relation “ $\sim$ ” above is clearly left invariant by this action, so we get an action of S_n on X . Considering the subgroup $A_n \subseteq S_n$ formed by all even permutations, we may restrict the above action to A_n , and we shall let

$$G = G(X, A_n)$$

be the corresponding groupoid of germs. The unit space of G is therefore homeomorphic to X , and we shall tacitly identify these from now on.

Notice that if $x = (t, i)$, with $t > 0$, and if $\sigma, \sigma' \in A_n$, then

$$[\sigma, (t, i)] = [\sigma', (t, i)] \iff \sigma(i) = \sigma'(i).$$

In other words, the germ of σ at (t, i) depends only on $j := \sigma(i)$. We may therefore denote this germ simply by (t, i, j) . On the other hand, it is easy to see that

$$[\sigma, 0] = [\sigma', 0] \iff \sigma = \sigma'.$$

We may then describe G as being the set

$$\{(t, i, j) : t \in (0, 1], i, j \in I\} \cup \{[\sigma, 0] : \sigma \in A_n\}.$$

The main technical result of this section is in order:

3.1. Theorem. *Assuming that $n \geq 4$, and given any $\tau \in S_n$ (not necessarily in A_n), there exists a unitary element u in $C_r^*(G)$ such that for all $t > 0$ and all $i, j \in I$, one has*

$$u(t, i, j) = \begin{cases} 1, & \text{if } \tau(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $u^*fu = f \circ \tau$ for every $f \in C_0(X)$.

Proof. For each $\sigma \in A_n$, let $U_\sigma = \{[\sigma, x] : x \in X\}$ be the canonical bisection associated to σ . Since U_σ is compact, one has that its characteristic function, here denoted 1_σ , is an element of $C_r^*(G)$, which is easily seen to be unitary. Moreover, the correspondence

$$\sigma \in A_n \mapsto 1_\sigma \in C_r^*(G)$$

is a unitary representation of A_n in $C_r^*(G)$, which therefore integrates to a *-homomorphism

$$\phi : C^*(A_n) \rightarrow C_r^*(G).$$

Given a generic element

$$(3.1.1) \quad a = \sum_{\sigma \in A_n} a_\sigma \delta_\sigma \in C^*(A_n)$$

and any $(t, i, j) \in G$, with $t > 0$, observe that

$$(3.1.2) \quad \phi(a)(t, i, j) = \sum_{\sigma \in A_n} a_\sigma 1_\sigma(t, i, j) = \sum_{\sigma(i)=j} a_\sigma.$$

Changing subjects slightly, consider the representation π of S_n on the Hilbert space $\mathbb{C}^n$, where each $\sigma \in S_n$ is mapped to the unitary operator $\pi(\sigma)$ defined on the canonical basis $\{e_i\}_{i \in I}$ of $\mathbb{C}^n$ by

$$\pi(\sigma)e_i = e_{\sigma(i)}.$$

Denote by $\tilde{\pi}$ the corresponding integrated representation of $C^*(S_n)$ on $\mathbb{C}^n$, and observe that for each a as in (3.1.1), one has that

$$\langle \tilde{\pi}(a)e_i, e_j \rangle = \sum_{\sigma \in A_n} a_\sigma \langle \pi(\sigma)e_i, e_j \rangle = \sum_{\sigma(i)=j} a_\sigma,$$

so we have by (3.1.2) that

$$(*) \quad \phi(a)(t, i, j) = \langle \tilde{\pi}(a)e_i, e_j \rangle,$$

for all $a \in C^*(A_n)$, all $t \in (0, 1]$, and all $i, j \in I$.

Assuming that $n \geq 4$, one may prove that the commutant in $\mathcal{B}(\mathbb{C}^n)$ of both $\tilde{\pi}(C^*(S_n))$ and $\tilde{\pi}(C^*(A_n))$ coincide with the set of all matrices of the form

$$\begin{pmatrix} z & y & y & \dots & y \\ y & z & y & \dots & y \\ y & y & z & \dots & y \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & y & \dots & z \end{pmatrix},$$

where $z, y \in \mathbb{C}$. The crucial point in doing this is that A_n acts *bi-transitively* on I , meaning that given $i_1 \neq i_2$ and $j_1 \neq j_2$, there exists $\sigma \in A_n$ such that $\sigma(i_1) = j_1$ and $\sigma(i_2) = j_2$. Incidentally, this is not true for $n < 4$.

By the double commutant theorem we conclude that $\tilde{\pi}(C^*(S_n)) = \tilde{\pi}(C^*(A_n))$. Given $\tau \in S_n$, as in the theorem, we therefore have that $\pi(\tau) \in \tilde{\pi}(C^*(A_n))$, so there exists some $v \in C^*(A_n)$ such that $\tilde{\pi}(v) = \pi(\tau)$. Absent any K-theoretic obstructions, we may assume that v is unitary.

The element u of which the theorem speaks is the unitary element $u := \phi(v) \in C_r^*(G)$. To see that it satisfies the required conditions, notice that for all $t > 0$ and $i, j \in I$ we have that

$$u(t, i, j) = \phi(v)(t, i, j) = \langle \tilde{\pi}(v)e_i, e_j \rangle = \langle \pi(\tau)e_i, e_j \rangle = \langle e_{\tau(i)}, e_j \rangle,$$

proving the first assertion. It follows that the *open support* of u , namely

$$\text{supp}'(u) = \{\gamma \in G : u(\gamma) \neq 0\}$$

(cf. [11]), consists precisely of all germs $(t, i, \tau(i))$, where $t > 0$, besides a few other germs at 0. Therefore the range of any $\gamma \in \text{supp}'(u)$ coincides with the image of its source under the action of τ . From this it immediately follows that

$$u^*fu = f \circ \tau, \quad \forall f \in C_0(X). \quad \square$$

The relevance of this result is in relation to [11, Proposition 4.7], where it is proved that the open support of a normalizer of $C_0(X)$ in $C_r^*(G)$ is a bisection. In the present non-Hausdorff situation this fails:

3.1. Proposition. *Let $n \geq 4$, let $\tau \in S_n \setminus A_n$, and let u be given as in Theorem 3.1. Then u is a normalizer of $C_0(G^{(0)})$ within $C_r^*(G)$, but even though G is essentially principal, the open support of u is not a bisection.*

Proof. Recall from the proof of Theorem 3.1 that $u = \phi(v)$, where v is a unitary element in $C^*(A_n)$ such that $\tilde{\pi}(v) = \pi(\tau)$. Assuming by contradiction that $\text{supp}'(u)$ is a bisection and noticing that the germs $[\sigma, 0]$ all have range and source equal to 0, we deduce that there is at most one $\sigma \in A_n$ for which $u([\sigma, 0]) \neq 0$.

Write $v = \sum_{\sigma \in A_n} a_\sigma \delta_\sigma$, as in (3.1.1), so that $u = \sum_{\sigma \in A_n} a_\sigma 1_\sigma$, and hence

$$u([\sigma, 0]) = a_\sigma, \quad \forall \sigma \in A_n.$$

We then see that there is only one $\sigma \in A_n$ for which $a_\sigma \neq 0$, which implies that $v = a_\sigma \delta_\sigma$. Consequently

$$\pi(\tau) = \tilde{\pi}(v) = a_\sigma \pi(\sigma),$$

which contradicts the fact that $\tau \notin A_n$. $\square$

Another dilemma presented by this example is related to the program initiated by Kumjian in [7], and recently continued by Renault in [11], attempting to classify Cartan sub-algebras of C^* -algebras. Given a commutative subalgebra B of a C^* -algebra A and a normalizer $u \in N(B)$, Kumjian [7] showed the existence of a partial homeomorphism θ_u of the spectrum X of B , such that for all $x \in X$, with $(u^*u)(x) \neq 0$, one has

$$(u^*bu)(x) = (u^*u)(x)b(\theta_u(x)), \quad \forall b \in B.$$

When A is the reduced C^* -algebra of an essentially principal, Hausdorff, étale groupoid G , and $B = C_0(G^{(0)})$, Renault showed [11] that one can reconstruct G from the inclusion “ $B \subseteq A$ ”, as the germs of the partial homeomorphisms θ_u , where u ranges in the set of all normalizers.

In the present example, if one attempted to reconstruct G from the inclusion “ $C_0(X) \subseteq C_r^*(G)$ ” using the above method, the presence of the strange normalizers

u above would lead us to consider the germ at zero of every $\tau \in S_n$ by the last assertion of Theorem 3.1, but the isotropy group $G(0)$ is only as large as $A_n!$

4. APPENDIX (THE INTERSECTION PROPERTY FOR IDEALS IN ESSENTIALLY PRINCIPAL, HAUSDORFF GROUPOIDS)

In this section we prove result (B) stated in the introduction. Although this result has been used in several contexts under various guises (see the introduction for some references), it seems not to have appeared in the literature in quite the general form we have in mind. We begin with some elementary considerations about representations of commutative C^* -algebras.

Let X be a locally compact Hausdorff space and let π be a representation of $C_0(X)$ on a Hilbert space H . As any ideal of $C_0(X)$, the kernel of π must be of the form $C_0(U)$, for some open set $U \subseteq X$.

4.1. Definition. Given a representation π of $C_0(X)$, with $\text{Ker}(\pi) = C_0(U)$, we will refer to $X \setminus U$ as the *support* of π .

4.2. Lemma. *If π is a representation of $C_0(X)$ on a Hilbert space H , and if x lies in the support of π , then $|f(x)| \leq \|\pi(f)\|$, for every f in $C_0(X)$.*

Proof. This is left to the reader. $\square$

Given a groupoid G , for every $x \in G^{(0)}$ we denote by $G(x)$ (cf. [10, I.1.1]) the *isotropy group at x* , namely

$$G(x) = \{\gamma \in G : s(\gamma) = r(\gamma) = x\}.$$

Obviously $x \in G(x)$, but in case $G(x) = \{x\}$ we say that x *has no isotropy*.

The following result gives the key inequality from which the next theorem will be deduced. It is roughly based on [10, II.4.4].

4.3. Lemma. *Let G be an étale, Hausdorff groupoid and let π be a representation of $C^*(G)$ on a Hilbert space H . Suppose in addition that we are given $x \in G^{(0)}$ such that*

- (i) x has no isotropy,
- (ii) x lies in the support of $\pi|_{C_0(G^{(0)})}$.

Then for every $f \in C_c(G)$, one has that $|f(x)| \leq \|\pi(f)\|$.

Proof. Let $\mathcal{V}$ be the collection of all open neighborhoods of x within $G^{(0)}$. We will view $\mathcal{V}$ as a directed set under the order relation

$$v \leq w \iff v \supseteq w, \quad \forall v, w \in \mathcal{V}.$$

For each $v \in \mathcal{V}$, choose $v_1, v_2 \in \mathcal{V}$, relatively compact, and such that $\bar{v}_2 \subseteq v_1 \subseteq \bar{v}_1 \subseteq v$. By Uryshon's Lemma let

$$g_v : G^{(0)} \rightarrow [0, 1]$$

be a continuous function whose restriction to $\bar{v}_2$ is identically equal to 1 and which vanishes off v_1 . The support of g_v is contained in $\bar{v}_1$, which is compact, so $g_v \in C_c(G^{(0)})$.

We claim that there exists $\xi_v \in H$, with $\|\xi_v\| = 1$, and such that

$$\pi(g_v)\xi_v = \xi_v.$$

In order to prove it, use Uryshon's Lemma again to produce a continuous function $h_v : G^{(0)} \rightarrow [0, 1]$, vanishing off v_2 and such that $h_v(x) \neq 0$. Observe that, since g_v is identically equal to 1 on v_2 , we have that

$$(4.3.1) \quad g_v h_v = h_v.$$

From (ii) and Lemma 4.2 it follows that

$$0 < |h_v(x)| \leq \|\pi(h_v)\|,$$

so $\pi(h_v) \neq 0$, and one may pick $\eta_v \in H$ such that $\|\pi(h_v)\eta_v\| = 1$. Setting $\xi_v = \pi(h_v)\eta_v$, we have that

$$\pi(g_v)\xi_v = \pi(g_v)\pi(h_v)\eta_v = \pi(g_v h_v)\eta_v \stackrel{(4.3.1)}{=} \pi(h_v)\eta_v = \xi_v,$$

proving our claim.

We next claim that

$$(4.3.2) \quad \lim_{v \in \mathcal{V}} \langle \pi(f)\xi_v, \xi_v \rangle = f(x), \quad \forall f \in C_c(G).$$

Without loss of generality we will suppose that there are $K, U \subseteq G$ such that K is compact, U is an open bisection, $K \subseteq U$, and f vanishes outside of K . We will further denote by α_U the homeomorphism from $s(U)$ to $r(U)$ given by

$$\alpha_U(s(\gamma)) = r(\gamma), \quad \forall \gamma \in U.$$

The proof of (4.3.2) will be broken up in the following three cases:

- (a) $x \notin s(U)$,
- (b) $x \in s(U)$, and $\alpha_U(x) \neq x$,
- (c) $x \in s(U)$, and $\alpha_U(x) = x$.

Proof of (4.3.2) under (a). Noticing that $x \notin s(K)$, there exists some $v_0 \in \mathcal{V}$, with $v_0 \cap s(K) = \emptyset$. For every $v \subseteq v_0$, one then has that

$$(fg_v)(\gamma) = f(\gamma)g_v(s(\gamma)) = 0, \quad \forall \gamma \in G,$$

because either $\gamma \notin K$ or $s(\gamma) \in s(K)$. Therefore $\pi(f)\xi_v = \pi(fg_v)\xi_v = 0$, proving that the left-hand side of (4.3.2) vanishes. Observing that $x \notin K$ (or else $x = s(x) \in s(K) \subseteq s(U)$), we see that the right-hand side of (4.3.2) also vanishes.

Proof of (4.3.2) under (b). Let A and B be pairwise disjoint open subsets of $G^{(0)}$ such that $x \in A$ and $\alpha_U(x) \in B$. Setting $v_0 = A \cap \alpha_U^{-1}(B)$, notice that $x \in v_0$ and that $v_0 \cap \alpha_U(v_0) = \emptyset$. For every $v \subseteq v_0$ we have that

$$(g_v^* f g_v)(\gamma) = \overline{g_v(r(\gamma))} f(\gamma) g_v(s(\gamma)), \quad \forall \gamma \in G.$$

If the above is nonzero for some γ , then $\gamma \in U$ and both $s(\gamma)$ and $r(\gamma)$ lie in v . Therefore

$$r(\gamma) = \alpha_U(s(\gamma)) \in v \cap \alpha_U(v) \subseteq v_0 \cap \alpha_U(v_0) = \emptyset,$$

which is impossible. So $g_v^* f g_v = 0$, and hence

$$\langle \pi(f)\xi_v, \xi_v \rangle = \langle \pi(f)\pi(g_v)\xi_v, \pi(g_v)\xi_v \rangle = \langle \pi(g_v^* f g_v)\xi_v, \xi_v \rangle = 0,$$

again proving that the left-hand side of (4.3.2) vanishes. As for the right-hand side, notice that $x \notin U$, because otherwise

$$\alpha_U(x) = \alpha_U(s(x)) = r(x) = x,$$

which is not in accordance with (b). Thus $f(x) = 0$, and (4.3.2) is verified in the present case.

Proof of (4.3.2) under (c). Given that $\alpha_U(x) = x$, there exists $\gamma \in U$ such that $s(\gamma) = r(\gamma) = x$. Thus

$$\gamma \in G(x) = \{x\},$$

so $\gamma = x$, by (i), and hence $x \in U$. As G is assumed to be Hausdorff, we have that f is continuous,¹ so, given $\varepsilon > 0$, we may choose a neighborhood v_0 of x , contained in $U \cap G^{(0)}$ and such that

$$y \in v_0 \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

For every $v \subseteq v_0$ we have that

$$(fg_v)(\gamma) = f(\gamma)g_v(s(\gamma)), \quad \forall \gamma \in G.$$

If the above is nonzero for some γ , then $\gamma \in U$ and $s(\gamma) \in v$. Thus, both γ and $s(\gamma)$ lie in U , and since these have the same source, we deduce that $\gamma = s(\gamma)$, and hence that $\gamma \in v$. It follows that fg_v vanishes outside v and, in particular, $fg_v \in C_0(G^{(0)})$. On the other hand, for every $y \in v$, one has that

$$|(fg_v)(y) - f(x)g_v(y)| = |f(y) - f(x)||g_v(y)| \leq \varepsilon \|g_v\| = \varepsilon,$$

which gives $\|fg_v - f(x)g_v\| \leq \varepsilon$. Therefore, for v as above,

$$\begin{aligned} |\langle \pi(f)\xi_v, \xi_v \rangle - f(x)| &= |\langle \pi(f)\xi_v, \xi_v \rangle - \langle f(x)\xi_v, \xi_v \rangle| \\ &= |\langle \pi(fg_v)\xi_v, \xi_v \rangle - \langle f(x)\pi(g_v)\xi_v, \xi_v \rangle| \\ &= |\langle \pi(fg_v - f(x)g_v)\xi_v, \xi_v \rangle| \leq \|fg_v - f(x)g_v\| \|\xi_v\| \|\xi_v\| \leq \varepsilon, \end{aligned}$$

proving (4.3.2) under the last case. We then finally get

$$|f(x)| \stackrel{(4.3.2)}{=} \lim_{v \in \mathcal{V}} |\langle \pi(f)\xi_v, \xi_v \rangle| \leq \|\pi(f)\|. \quad \square$$

We are now ready to prove the precise form of result (B) stated in the introduction.

4.4. Theorem. *Let G be an étale, Hausdorff, essentially principal, second countable groupoid.*

- (a) *If π is a representation of $C_r^*(G)$ such that π is faithful on $C_0(G^{(0)})$, then π is faithful.*
- (b) *If J is a nonzero ideal in $C_r^*(G)$, then $J \cap C_0(G^{(0)})$ is nonzero.*

Proof. We address (a) first. Since π is assumed to be faithful on $C_0(G^{(0)})$, the support of $\pi|_{C_0(G^{(0)})}$ is the whole of $G^{(0)}$. Given $f \in C_c(G)$ one then has by Lemma 4.3 that

$$(4.4.1) \quad |f(x)| \leq \|\pi(f)\|,$$

for every x in $G^{(0)}$ without entropy. Employing [11, 3.1] we see that the set of such x 's is dense in $G^{(0)}$, and since the restriction of f to $G^{(0)}$ is continuous,² we conclude that in fact (4.4.1) holds for every $x \in G^{(0)}$, so

$$(4.4.2) \quad \sup_{x \in G^{(0)}} |f(x)| \leq \|\pi(f)\|.$$

¹On a non-Hausdorff groupoid the accepted definition of $C_c(G)$ (see [2], [9] or [4, 3.9]) includes functions which are discontinuous, so we truly need to assume G to be Hausdorff here.

²Here we again need to assume that G is Hausdorff.

Let E be the standard conditional expectation from $C_r^*(G)$ to $C_0(G^{(0)})$ [10, II.4.8], [11, 4.3]. For f in $C_c(G)$ recall that $E(f)$ coincides with the restriction of f to $G^{(0)}$, so we may write (4.4.2) as

$$(4.4.3) \quad \|E(f)\| \leq \|\pi(f)\|, \quad \forall f \in C_c(G).$$

Letting B be the range of π , we claim that there exists a bounded linear map F from B to $C_0(G^{(0)})$ such that the diagram

$$\begin{array}{ccc} C_r^*(G) & \xrightarrow{\pi} & B \\ E \downarrow & \nearrow F & \\ C_0(G^{(0)}) & & \end{array}$$

commutes. We first define F on the dense *-subalgebra $\pi(C_c(G)) \subseteq B$ by

$$F(\pi(f)) = E(f), \quad \forall f \in C_c(G).$$

By (4.4.3) this is well defined and bounded, and hence may be continuously extended to the whole of B . The extension will then clearly satisfy the required conditions.

Let $a \in C_r^*(G)$ be such that $\pi(a) = 0$. Then

$$0 = F(\pi(a^*)\pi(a)) = F(\pi(a^*a)) = E(a^*a).$$

Since E is faithful [11, 4.3ii], we deduce that $a = 0$, hence concluding the proof of (a).

We now turn to proving (b). Consider a representation π of $C_r^*(G)$ whose kernel coincides with J . Such a representation may be obtained by faithfully embedding $C_r^*(G)/J$ as an algebra of operators on a Hilbert space.

Arguing by contradiction, if the intersection of J with $C_0(G^{(0)})$ is zero, then the restriction $\pi|_{C_0(G^{(0)})}$ is faithful and hence π itself is faithful by (a), from which one would deduce that J is zero. $\square$

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