ILL-POSEDNESS RESULTS FOR THE (GENERALIZED) BENJAMIN-ONO-ZAKHAROV-KUZNETSOV EQUATION

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ABSTRACT. Here we consider results concerning ill-posedness for the Cauchy problem associated with the Benjamin-Ono-Zakharov-Kuznetsov equation, namely,

$$(IVP) \quad \begin{cases} u_t - \mathcal{H}u_{xx} + u_{xyy} + u^k u_x = 0, & (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+, \\ u(x, y, 0) = \phi(x, y). \end{cases}$$

For $k = 1$, (IVP) is shown to be ill-posed in the class of anisotropic Sobolev spaces $H^{s_1, s_2}(\mathbb{R}^2)$, $s_1, s_2 \in \mathbb{R}$, while for $k \geq 2$ ill-posedness is shown to hold in $H^{s_1, s_2}(\mathbb{R}^2)$, $2s_1 + s_2 < 3/2 - 2/k$. Furthermore, for $k = 2, 3$, and some particular values of s_1, s_2 , a stronger result is also established.

1. INTRODUCTION

The propagation of two-dimensional dispersive weakly nonlinear waves is usually obtained by assuming nearly one-dimensional waves. As a result, there are several two-dimensional models which are generalizations of well-known one-dimensional nonlinear dispersive equations. The most known and studied ones are the Kadomtsev-Petviashvili (KP) and Zakharov-Kuznetsov equations, which are generalizations of the Korteweg-de Vries (KdV) equation.

The one-dimensional generalized Benjamin-Ono (BO) equation,

$$(1.1) \quad u_t - \mathcal{H}u_{xx} + u^k u_x = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+,$$

is a model for propagation of one-dimensional internal waves in an ideal deep stratified fluid (for $k = 1$). In this paper we are interested in a model which is a natural two-dimensional extension of (1.1), namely, the generalized Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equation,

$$(1.2) \quad u_t - \mathcal{H}u_{xx} + \varepsilon u_{xyy} + u^k u_x = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}^+.$$

Here, $k > 0$ is an integer number, the constant ε measures the transverse effects and it is normalized to ± 1 , and $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H}u(x, y, t) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(z, y, t)}{x - z} dz,$$

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where p.v. denotes the Cauchy principal value.

With $k = 1$, equation (1.2) was introduced recently in [15, 17], and it has applications to electromigration in thin nanoconductors on a dielectric substrate.

Throughout the paper we consider $\varepsilon = 1$, so that (1.2) reads as

$$(1.3) \quad u_t - \mathcal{H}u_{xx} + u_{xyy} + u^k u_x = 0.$$

Our main interest in the present paper lies in the study of the well-posedness (or ill-posedness) for the Cauchy problem associated with (1.3), so that we couple (1.3) with the Cauchy data

$$(1.4) \quad u(x, y, 0) = \phi(x, y), \quad (x, y) \in \mathbb{R}^2.$$

As usual, the well-posedness is taken to be in Kato's sense; that is, it includes existence, uniqueness, persistency property, and regularity (at least continuity) of the *flow-map data-solution*. Actually, our results will be negative ones in the sense that one cannot obtain *high regularity* of the map data-solution.

So far, not so much is known about equation (1.2), and only a few works are available in the literature. Indeed, concerning local well-posedness the best known result is the following.

Theorem 1.1. *Let $s > 2$. Then for any $\phi \in H^s(\mathbb{R}^2)$, there exist a positive $T = T(\|\phi\|_{H^s})$ and a unique solution $u \in C([0, T]; H^s(\mathbb{R}^2))$ of the Cauchy problem (1.2)–(1.4). In addition, the flow-map $\phi \mapsto u(t)$ is continuous in the H^s -norm.*

Theorem 1.1 will be not proved here, but it can be established just by using a *parabolic regularization argument* (see [19] for a similar result), so that it does not take into account the dispersive structure of the equation. It seems not to be easy to obtain a reasonable improvement of Theorem 1.1. We observe that recently Tao [28], Burq and Planchon [7], and Ionescu and Kenig [13] have obtained much stronger results for the Cauchy problem associated with the BO equation. However, it should be pointed out that their results are established by constructing appropriate gauge transformations. In the case of the BO-ZK equation, it is not clear how to get a suitable transformation, and we do not know if such an approach could be used to improve Theorem 1.1. On the other hand, we believe Theorem 1.1 can be improved by employing the technique introduced by Kenig [16] (see also [14] and [18]), which combines localized Strichartz estimates with some energy estimates. This will appear in a forthcoming paper.

In [8, 9], we studied the existence and stability of solitary-wave solutions of the form $u(x, y, t) = \varphi_c(x - ct, y)$, $c > 0$, in terms of the sign of ε and the values of k (even if k is a rational number). To be more precise, for $\varepsilon = 1$ and $0 < k < 4$, solitary waves do exist. By using the pioneer theory introduced by Cazenave and Lions, we have proved that such solitary waves are orbitally stable for $0 < k < 4/3$ (see [9]). It should be pointed out that Theorem 1.1 is not strong enough to consider global perturbations in the energy space $H^{1/2,1}(\mathbb{R}^2)$ (see the notation below). Thus, our stability result is in the sense that if a solution in $H^s(\mathbb{R}^2)$, $s > 2$, starts near the orbit generated by a solitary wave (in the $H^{1/2,1}(\mathbb{R}^2)$ -norm), then as long as the solution exists, it remains near the orbit (see also [1] and [2] and the references therein). Moreover, by using the adapted method put forward by the KdV equation, we proved that the solitary waves are orbitally unstable for $4/3 < k < 4$ (see [8]).

We now turn our attention to describing our results. To motivate our interest in showing a local well-posedness, we first note that (1.3) has formally two conserved

quantities, namely,

$$\mathcal{F}(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 \, dx dy$$

and

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(u_y^2 + u \mathcal{H} u_x - \frac{2}{(k+1)(k+2)} u^{k+2} \right) \, dx dy.$$

Thus, a local well-posedness result, in appropriate spaces, could lead to a global one.

Before trying to prove any result, we usually ask ourselves what would be the lowest isotropic (or anisotropic) Sobolev index where we can prove a local well-posedness result. To deal with this question, we perform a scaling argument by observing that if u solves the equation (1.3) with initial data ϕ , then

$$u_\lambda(x, y, t) = \lambda^{2/k} u(\lambda^2 x, \lambda y, \lambda^4 t)$$

also solves (1.3) with initial data $u_\lambda(x, y, 0) = \lambda^{2/k} \phi(\lambda^2 x, \lambda y)$, for any $\lambda > 0$. As a consequence, since

$$\|u_\lambda(\cdot, \cdot, 0)\|_{\dot{H}^{s_1, s_2}} = \lambda^{2s_1 + s_2 + 2/k - 3/2} \|\phi\|_{\dot{H}^{s_1, s_2}},$$

the scale-invariant Sobolev spaces for the BO-ZK equation are $\dot{H}^{s_1, s_2}(\mathbb{R}^2)$ with $2s_1 + s_2 = 3/2 - 2/k$ (see the notation below). Thus, the natural spaces for studying the local well-posedness of equation (1.2) are the spaces $H^{s_1, s_2}(\mathbb{R}^2)$ with $2s_1 + s_2 \geq 3/2 - 2/k$.

We begin our results by considering the case $k = 1$. Therefore, from the above scaling arguments, one expects to prove local well-posedness in the spaces $H^{s_1, s_2}(\mathbb{R}^2)$ with $2s_1 + s_2 \geq -1/2$. However, by requiring C^2 -regularity of the flow-map data-solution, we prove that one cannot obtain such results for any $s_1, s_2 \in \mathbb{R}$. More precisely, we prove the following.

Theorem 1.2. *Assume $k = 1$. Let $s_1, s_2 \in \mathbb{R}$ and $T > 0$ be fixed numbers. Then, there does not exist a space X_T continuously embedded in $C([0, T]; H^{s_1, s_2}(\mathbb{R}^2))$ such that there exists a constant $C > 0$ with*

$$(1.5) \quad \|U(t)\phi\|_{X_T} \leq C \|\phi\|_{H^{s_1, s_2}}, \quad \phi \in H^{s_1, s_2}(\mathbb{R}^2),$$

and

$$(1.6) \quad \left\| \int_0^t U(t-t')[u(t')u_x(t')]dt' \right\|_{X_T} \leq C \|u\|_{X_T}^2, \quad u \in X_T,$$

where $U(t)$ is the unitary group on $H^{s_1, s_2}(\mathbb{R}^2)$ defined, via its Fourier transform, by

$$\widehat{U(t)\phi}(\xi, \eta) = \exp(it(\xi|\xi| + \xi\eta^2))\widehat{\phi}(\xi, \eta).$$

Corollary 1.3. *Assume $k = 1$. Let $s_1, s_2 \in \mathbb{R}$ be fixed. Then there does not exist a $T > 0$ such that the Cauchy problem (1.3)–(1.4) has a unique local solution defined on the interval $[0, T]$ and such that the flow-map data-solution*

$$S_t : \phi \mapsto u(t), \quad t \in [0, T],$$

is C^2 -differentiable at the origin from $H^{s_1, s_2}(\mathbb{R}^2)$ to $H^{s_1, s_2}(\mathbb{R}^2)$.

The main point to prove Theorem 1.2 is to explore the known results for the BO equation (1.1). Indeed, by using the technique introduced in [6] (see also [27]), it was proved in [23] that the BO equation is ill-posed in $H^s(\mathbb{R})$, $s \in \mathbb{R}$ (in the C^2 -regularity sense). The main idea to prove such a result is to locate some special waves, in which, in some sense, the interaction between low and high frequencies behaves badly itself. Here, our insight comes from the physical viewpoint and, roughly speaking, it is the following. Since equation (1.3) is obtained under the assumption of nearly one-dimensional waves, we construct particular waves such that in the direction of propagation (the x -direction) they behave as the waves of the BO equation, whereas in the transverse direction (the y -direction), we localize them into small frequencies (see [19], [20] for similar ideas).

Remark 1.4. We note that the same proof of Theorem 1.2 (and Corollary 1.3) still holds when we replace the space $H^{s_1, s_2}(\mathbb{R}^2)$ with $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$.

Next, we consider $k \geq 2$. In this case, our results are not so strong as in Theorem 1.2. However, for indices below the critical ones (in the scaling argument sense), we are able to show an ill-posedness result.

Theorem 1.5. *Let $k \geq 2$ and $s_1, s_2 \in \mathbb{R}$ such that $2s_1 + s_2 < 3/2 - 2/k$. Then for any $T > 0$, the flow-map $\phi \mapsto u$ (if it exists) is not of class C^{k+1} from $H^{s_1, s_2}(\mathbb{R}^2)$ to $C([0, T]; H^{s_1, s_2}(\mathbb{R}^2))$ at the origin.*

As in Theorem 1.2, the insight to show Theorem 1.5 is to explore the results for the generalized BO equation [21] (see also [3] and [22]). Of course, some matters appear with the transverse direction, but, as in Theorem 1.2, we are able to handle them with suitable localizations.

Finally, we prove a result showing that for $k = 2, 3$ the flow-map data-solution cannot be uniformly continuous below the critical family of spaces for some values of $s_1, s_2 \in \mathbb{R}$.

Theorem 1.6. *Let $k = 2, 3$ and $s_1, s_2 \geq 0$ such that $2s_1 + s_2 = 3/2 - 2/k$. Then, the Cauchy problem (1.3)–(1.4) is ill-posed for data in $H^{s_1, s_2}(\mathbb{R}^2)$, in the sense that the flow-map data-solution, $\phi \mapsto u(t)$, is not uniformly continuous.*

The method to prove Theorem 1.6 goes back to the techniques introduced in [4] and [5], which consists in constructing a sequence of converging data but such that the corresponding sequence of solutions does not converge. Usually, and this is our case, such data are given in terms of solitary-wave solutions. We note that the restriction on the values of k lies in the fact that we do not know if solitary waves exist for $k \geq 4$ (see [8, 9]).

Remark 1.7. As is well known, our results imply that the Cauchy problem (1.3)–(1.4) cannot be solved by an iterative method (in the respective spaces).

Notation. Throughout this paper we shall refer to equations (1.2) or (1.3) as the BO-ZK equation. For a function $f = f(x, y)$, the function $\widehat{f} = \widehat{f}(\xi, \eta)$ denotes its Fourier transform, defined as

$$\widehat{f}(\xi, \eta) = \int_{\mathbb{R}^2} e^{-i(x\xi + y\eta)} f(x, y) \, dx dy.$$

For any $s \in \mathbb{R}$, the space $H^s := H^s(\mathbb{R}^2)$ denotes the usual isotropic Sobolev space. Let $s_1, s_2 \in \mathbb{R}$. We define the anisotropic Sobolev spaces $H^{s_1, s_2} := H^{s_1, s_2}(\mathbb{R}^2)$ to

be the set of all tempered distributions f such that

$$\|f\|_{H^{s_1, s_2}}^2 = \int_{\mathbb{R}^2} (1 + \xi^2)^{s_1} (1 + \eta^2)^{s_2} |\widehat{f}(\xi, \eta)|^2 d\xi d\eta < \infty.$$

The homogeneous anisotropic Sobolev space $\dot{H}^{s_1, s_2} := \dot{H}^{s_1, s_2}(\mathbb{R}^2)$ is defined to be the set of all tempered distributions f such that

$$\|f\|_{\dot{H}^{s_1, s_2}}^2 = \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} |\widehat{f}(\xi, \eta)|^2 d\xi d\eta < \infty.$$

2. PROOFS OF THEOREM 1.2 AND COROLLARY 1.3

In this section, we prove Theorem 1.2 and Corollary 1.3. As we already mentioned, we follow closely the arguments in [23].

Let us first consider the linear problem

$$(2.1) \quad \begin{cases} u_t - \mathcal{H}u_{xx} + u_{xyy} = 0, & (x, y) \in \mathbb{R}^2, \quad t > 0, \\ u(x, y, 0) = \phi(x, y). \end{cases}$$

It is easily seen that the solution of (2.1) is given by

$$(2.2) \quad u(t) = U(t)\phi(x, y) = \int_{\mathbb{R}^2} e^{i(t(\xi|\xi| + \xi\eta^2) + x\xi + y\eta)} \widehat{\phi}(\xi, \eta) d\xi d\eta.$$

Proof of Theorem 1.2. The proof is argued by contradiction. Indeed, assume that such a space does exist and that (1.5) and (1.6) are fulfilled. By taking $u(t) = U(t)\phi$ in (1.6), we obtain

$$\left\| \int_0^t U(t-t') [(U(t')\phi)(U(t')\phi_x)] dt' \right\|_{X_T} \leq C \|U(t)\phi\|_{X_T}^2.$$

Since X_T is continuously embedded in $C([0, T]; H^{s_1, s_2}(\mathbb{R}^2))$, (1.5) yields

$$(2.3) \quad \left\| \int_0^t U(t-t') [(U(t')\phi)(U(t')\phi_x)] dt' \right\|_{H^{s_1, s_2}} \leq C \|\phi\|_{H^{s_1, s_2}}^2.$$

The idea now is to show that (2.3) fails by constructing a particular ϕ .

For $0 < \alpha \ll 1$ and $N \gg 1$, we consider the (disjoint) rectangles

$$Q_1 = [\alpha/2, \alpha] \times [\sqrt{\alpha}/2, \sqrt{\alpha}], \quad Q_2 = [N, N + \alpha] \times [\sqrt{\alpha}/2, \sqrt{\alpha}],$$

and define ϕ , via its Fourier transform, by

$$(2.4) \quad \widehat{\phi}(\xi, \eta) := \alpha^{-\frac{3}{4}} \chi_{Q_1} + \alpha^{-\frac{3}{4}} N^{-s_1} \chi_{Q_2},$$

where χ_A denotes the characteristic function of the set A .

Lemma 2.1. *Let ϕ be as in (2.4) and $s_1, s_2 \in \mathbb{R}$. Then, there exists a constant $K > 0$, independent of N and α , such that*

$$\|\phi\|_{H^{s_1, s_2}} \leq K.$$

Proof. The proof is a straightforward calculation. □

The following lemma is useful.

Lemma 2.2. *Let*

$$p(\xi, \eta) = \xi|\xi| + \xi\eta^2.$$

The following identity holds:

$$\begin{aligned} & \int_0^t U(t-t') [(U(t')\phi)(U(t')\phi_x)] dt' \\ &= \int_{\mathbb{R}^4} e^{i(x\xi+y\eta+tp(\xi,\eta))} \xi \widehat{\phi}(\xi_1, \eta_1) \widehat{\phi}(\xi - \xi_1, \eta - \eta_1) \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} d\xi d\xi_1 d\eta d\eta_1, \end{aligned}$$

where

$$\Lambda(\xi, \xi_1, \eta, \eta_1) = p(\xi_1, \eta_1) + p(\xi - \xi_1, \eta - \eta_1) - p(\xi, \eta).$$

Proof. The proof is carried out in a similar way to that in [23, Lemma 1] and [24, Lemma 4]. $\square$

In view of Lemma 2.2, we write

$$(2.5) \quad \int_0^t U(t-t') [(U(t')\phi)(U(t')\phi_x)] dt' = f_1(x, y, t) + f_2(x, y, t) + f_3(x, y, t),$$

where

$$\begin{aligned} f_1(x, y, t) &= \frac{c}{\alpha^{3/2}} \int_{Q_{11}} e^{i(x\xi+y\eta+tp(\xi,\eta))} \xi \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} d\xi d\xi_1 d\eta d\eta_1, \\ f_2(x, y, t) &= \frac{c}{\alpha^{3/2} N^{2s_1}} \int_{Q_{22}} e^{i(x\xi+y\eta+tp(\xi,\eta))} \xi \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} d\xi d\xi_1 d\eta d\eta_1, \end{aligned}$$

and

$$f_3(x, y, t) = \frac{c}{\alpha^{3/2} N^{s_1}} \int_{Q_{12} \cup Q_{21}} e^{i(x\xi+y\eta+tp(\xi,\eta))} \xi \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} d\xi d\xi_1 d\eta d\eta_1.$$

Here, we have defined

$$\begin{aligned} Q_{11} &:= \{(\xi, \xi_1, \eta, \eta_1) \in \mathbb{R}^4; (\xi_1, \eta_1) \in Q_1, (\xi - \xi_1, \eta - \eta_1) \in Q_1\}, \\ Q_{22} &:= \{(\xi, \xi_1, \eta, \eta_1) \in \mathbb{R}^4; (\xi_1, \eta_1) \in Q_2, (\xi - \xi_1, \eta - \eta_1) \in Q_2\}, \\ Q_{12} &:= \{(\xi, \xi_1, \eta, \eta_1) \in \mathbb{R}^4; (\xi_1, \eta_1) \in Q_1, (\xi - \xi_1, \eta - \eta_1) \in Q_2\}, \end{aligned}$$

and

$$Q_{21} := \{(\xi, \xi_1, \eta, \eta_1) \in \mathbb{R}^4; (\xi_1, \eta_1) \in Q_2, (\xi - \xi_1, \eta - \eta_1) \in Q_1\}.$$

Note that taking the Fourier transform (in the variables x, y), we obtain

$$\begin{aligned} \widehat{f}_1(\xi, \eta, t) &= \frac{c}{\alpha^{3/2}} \xi e^{itp(\xi, \eta)} \int_{(\xi_1, \eta_1) \in Q_{11}} \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1, \\ \widehat{f}_2(\xi, \eta, t) &= \frac{c}{\alpha^{3/2} N^{2s_1}} \xi e^{itp(\xi, \eta)} \int_{(\xi_1, \eta_1) \in Q_{22}} \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1, \end{aligned}$$

and

$$\widehat{f}_3(\xi, \eta, t) = \frac{c}{\alpha^{3/2} N^{s_1}} \xi e^{itp(\xi, \eta)} \int_{(\xi_1, \eta_1) \in Q_{12} \cup Q_{21}} \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1.$$

Next, we analyze the supports of $\widehat{f}_j, j = 1, 2, 3$. Actually, a simple analysis shows us that

$$(1) \quad \text{supp}(\widehat{f}_1) \subset [\alpha, 2\alpha] \times [\sqrt{\alpha}, 2\sqrt{\alpha}],$$

- (2) $\text{supp}(\widehat{f_2}) \subset [2N, 2N + 2\alpha] \times [\sqrt{\alpha}, 2\sqrt{\alpha}]$,
 (3) $\text{supp}(\widehat{f_3}) \subset [N, N + 2\alpha] \times [\sqrt{\alpha}, 2\sqrt{\alpha}]$.

Since the supports of $\widehat{f_j}$, $j = 1, 2, 3$ are disjoint, from (2.5), we deduce

$$(2.6) \quad \left\| \int_0^t U(t-t') [(U(t')\phi)(U(t')\phi_x)] dt' \right\|_{H^{s_1, s_2}} \geq \|f_3(\cdot, \cdot, t)\|_{H^{s_1, s_2}}.$$

The aim now is to get some “good” lower bound for $\|f_3(\cdot, \cdot, t)\|_{H^s}$. We first note that if $(\xi, \xi_1, \eta, \eta_1) \in Q_{12} \cup Q_{21}$, then

$$\Lambda(\xi, \xi_1, \eta, \eta_1) = -[(\xi - \xi_1)(2\xi_1 + \eta\eta_1) + (\eta - \eta_1)(\xi\eta_1 + \xi_1\eta)].$$

Therefore, for $(\xi, \xi_1, \eta, \eta_1) \in Q_{12} \cup Q_{21}$, we have

$$c\alpha N \leq |\Lambda(\xi, \xi_1, \eta, \eta_1)| \leq C\alpha N,$$

where c and C are constants independent of α and N . Thus, we choose α and N such that

$$(2.7) \quad \alpha N = N^{-\epsilon}, \quad 0 < \epsilon \ll 1.$$

Now, for $(\xi, \xi_1, \eta, \eta_1) \in Q_{12} \cup Q_{21}$, from (2.7), we obtain

$$\left| \frac{e^{it\Lambda(\xi, \xi_1, \eta, \eta_1)} - 1}{\Lambda(\xi, \xi_1, \eta, \eta_1)} \right| = |t| + O(N^{-\epsilon}).$$

Hence, using the integral mean value theorem, we get the bound

$$(2.8) \quad \|f_3(\cdot, \cdot, t)\|_{H^{s, s_2}} \geq C \frac{NN^{s_1}\alpha^{3/2}\alpha^{3/4}}{N^{s_1}\alpha^{3/2}} = C\alpha^{3/4}N = CN^{(1-3\epsilon)/4}.$$

Finally, gathering together Lemma 2.1 and inequalities (2.3), (2.6), and (2.8), we get

$$K^2 \geq \|\phi\|_{H^{s_1, s_2}}^2 \geq \|f_3(\cdot, \cdot, t)\|_{H^{s_1, s_2}} \geq CN^{(1-3\epsilon)/4},$$

which is a contradiction for $N \gg 1$. The proof of Theorem 1.2 is now completed. $\square$

Proof of Corollary 1.3. This is well understood by now. If the flow-map data-solution was C^2 -differentiable at the origin from $H^{s_1, s_2}(\mathbb{R}^2)$ to $H^{s_1, s_2}(\mathbb{R}^2)$, we should obtain

$$\left\| \int_0^t U(t-t') [(U(t')\phi)(U(t')\phi_x)] dt' \right\|_{H^{s_1, s_2}} \leq C\|\phi\|_{H^{s_1, s_2}}^2.$$

But as we showed in Theorem 1.2, the above inequality fails. This proves Corollary 1.3. $\square$

2.1. Generalizations to a class of dispersive equations. In this subsection, we consider a general class of dispersive equations, for which the same conclusions of Theorem 1.2 and Corollary 1.3 hold. Indeed, we consider the Cauchy problem

$$(2.9) \quad \begin{cases} u_t - \mathcal{L}u + uu_x = 0, & (x, y) \in \mathbb{R}^2, \quad t > 0, \\ u(x, y, 0) = \phi(x, y), \end{cases}$$

where $\mathcal{L}$ is a Fourier multiplier given by

$$\widehat{\mathcal{L}f}(\xi, \eta) = ip(\xi, \eta)\widehat{f}(\xi, \eta).$$

The following assumptions will be needed in the remainder of this section.

- (H1) The symbol p is a continuous real-valued function on $\mathbb{R}^2$ and differentiable on $\mathbb{R}^+ \times \mathbb{R}^+$.

(H2) $p(0, 0) = 0$, and if $\nabla p = (p_1, p_2)$, then

$$|p_j(\xi, \eta)| \lesssim |\eta|^{a_j} (|\xi|^{\gamma_j} + |\eta|^{b_j}), \quad (\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

with $0 \leq \gamma_j < 4/3$, $b_j \geq 0$, $j = 1, 2$, $a_1 \geq 0$, and $a_2 \geq 1$.

Lemma 2.3. *Let assumptions (H1)–(H2) hold. Then for $(\xi, \xi_1, \eta, \eta_1) \in Q_{12} \cup Q_{21}$, we have*

$$|\Lambda(\xi, \xi_1, \eta, \eta_1)| \leq C\alpha N^\gamma, \quad \alpha \ll 1, \quad N \gg 1,$$

where $\gamma = \max\{\gamma_1, \gamma_2\}$, Q_{12}, Q_{21} are as in the previous section, and

$$\Lambda(\xi, \xi_1, \eta, \eta_1) = p(\xi_1, \eta_1) + p(\xi - \eta, \xi_1 - \eta_1) - p(\xi, \eta).$$

Proof. We first note that from the mean value theorem,

$$(2.10) \quad |p(\xi, \eta)| \lesssim |\eta|^{a_1} |\xi| (|\xi|^{\gamma_1} + |\eta|^{b_1}) + |\eta|^{a_2+1} (|\xi|^{\gamma_2} + |\eta|^{b_2})$$

for $(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+$. Now (as in [12]), we denote

$$|\xi_{\min}| = \min\{|\xi|, |\xi_1|, |\xi - \xi_1|\}, \quad |\xi_{\max}| = \max\{|\xi|, |\xi_1|, |\xi - \xi_1|\}.$$

Suppose first that $|\xi_{\min}| = \xi_1$. Then, for some $\theta \in [0, 1]$, we have

$$\begin{aligned} & |p(\xi, \eta) - p(\xi - \xi_1, \eta - \eta_1)| \\ & \lesssim |\eta - \theta\eta_1|^{a_1} |\xi_1| (|\xi - \theta\xi_1|^{\gamma_1} + |\eta - \theta\eta_1|^{b_1}) \\ & \quad + |\eta - \theta\eta_1|^{a_2} |\eta_1| (|\xi - \theta\xi_1|^{\gamma_2} + |\eta - \theta\eta_1|^{b_2}) \\ & \lesssim |\eta - \theta\eta_1|^{a_1} |\xi_{\min}| (|\xi_{\max}|^{\gamma_1} + |\eta - \theta\eta_1|^{b_1}) + |\eta - \theta\eta_1|^{a_2} |\eta_1| (|\xi_{\max}|^{\gamma_2} + |\eta - \theta\eta_1|^{b_2}), \end{aligned}$$

where we have used the fact that

$$\max_{\theta \in [0, 1]} |\xi - \theta\xi_1| = |\xi_{\max}|.$$

Moreover, from (2.10),

$$|p(\xi_1, \eta_1)| \lesssim |\eta_1|^{a_1} |\xi_{\min}| (|\xi_{\max}|^{\gamma_1} + |\eta_1|^{b_1}) + |\eta_1|^{a_2+1} (|\xi_{\max}|^{\gamma_2} + |\eta_1|^{b_2}).$$

Therefore, the structure of Q_{12} and Q_{21} imply that for $(\xi, \xi_1, \eta, \eta_1) \in Q_{12} \cup Q_{21}$,

$$|\Lambda(\xi, \xi_1, \eta, \eta_1)| \leq |p(\xi_1, \eta_1)| + |p(\xi - \eta, \xi_1 - \eta_1) - p(\xi, \eta)| \lesssim \alpha N^\gamma.$$

If $|\xi_{\min}| = |\xi - \xi_1|$, a similar analysis can be performed. Note that in $Q_{12} \cup Q_{21}$ we always have $|\xi| \geq \max\{|\xi_1|, |\xi - \xi_1|\}$. This completes the proof of the lemma. $\square$

Theorem 2.4. *Let assumptions (H1)–(H2) hold. Then the conclusions of Theorem 1.2 and Corollary 1.3 are valid for the Cauchy problem (2.9).*

Sketch of the proof. In this case one chooses α and N such that $\alpha N^\gamma = N^{-\epsilon}$, $0 < \epsilon \ll 1$. Now, similarly to the proof of Theorem 1.2, we obtain

$$K^2 \geq \|\phi\|_{H^{s_1, s_2}}^2 \geq C\alpha^{3/4} N = CN^{(4-3\gamma-3\epsilon)/4},$$

which is a contradiction under our assumptions. $\square$

2.1.1. *Examples.* It is easy to see that the BO-ZK equation (1.2) fits in the hypotheses of Theorem 2.4. Here, we give another example where assumptions **(H1)**–**(H2)** are verified. The Shrira equation (see [26])

$$u_t + \mathcal{H}\Delta u + uu_x = 0,$$

where $\Delta = \partial_x^2 + \partial_y^2$ denotes the Laplacian operator, is a model for the description of essentially two-dimensional weakly nonlinear long-wave perturbations on the background of a boundary-layer type plane-parallel shear flow without inflection points (see [25]). It also describes the amplitude of the perturbation of the horizontal velocity component of a sheared flow of electrons (see [11]). For existence and stability of solitary waves, see [10].

In this case, we have

$$p(\xi, \eta) = \text{sign}(\xi)(\xi^2 + \eta^2).$$

A trivial verification shows that assumptions **(H1)**–**(H2)** hold.

3. PROOF OF THEOREM 1.5

In this section, we show Theorem 1.5. As previously commented, we use the adapted method put forward in [21] for the generalized BO equation.

Let us consider the flow-map $\phi \mapsto u(t; \phi)$, and define u_{k+1} by

$$u_{k+1} = \frac{\partial^{k+1} u}{\partial^{k+1} \phi} \Big|_{\phi=0} (h_N, \dots, h_N),$$

where the sequence $\{h_N\}$ will be constructed below. Then, since $u(\cdot; 0) = 0$, by straightforward calculations, we see that

$$u_{k+1} = (k+1)! \int_0^t U(t-t') \partial_x ((U(t') h_N)^{k+1}) dt'.$$

Thus, if $\phi \mapsto u(\cdot; \phi)$ is of class C^{k+1} at the origin, we see that necessarily

$$(3.1) \quad \sup_{t \in [0, T]} \|u_{k+1}(t)\|_{H^{s_1, s_2}} \leq C \|h_N\|_{H^{s_1, s_2}}^{k+1}.$$

In the sequel, we show that (3.1) fails for a suitable sequence of functions $\{h_N\}_N$. Let A and B be positive real numbers (which will be chosen later) such that $A < B$ and $A > kB/(k+2)$. Consider the real-valued function h_N defined, via its Fourier transform, by

$$\widehat{h}_N(\xi, \eta) = N^{-(4s_1+2s_2+3)/4} (\psi_{1+} + \psi_{1-} + \psi_{2+} + \psi_{2-}) \left(\frac{\xi}{N}, \frac{\eta}{\sqrt{N}} \right),$$

where ψ_{1+} is a smooth nonnegative function supported in the set

$$\{(\xi, \eta) \in \mathbb{R}^2 \mid 0 < A \leq \xi, \eta \leq B\},$$

$\psi_{1+} \equiv 1$ on the square

$$\left[A + \frac{1}{4}(B-A), B - \frac{1}{4}(B-A) \right] \times \left[A + \frac{1}{4}(B-A), B - \frac{1}{4}(B-A) \right],$$

and

$$\psi_{1+}(\xi, \eta) = \psi_{1-}(-\xi, \eta) = \psi_{2+}(\xi, -\eta) = \psi_{2-}(-\xi, -\eta).$$

Note that by definition,

$$\|h_N\|_{H^{s_1, s_2}} \simeq 1.$$

On the other hand, we see that the Fourier transform of u_{k+1} can be computed as follows:

$$\begin{aligned}
 (3.2) \quad & \widehat{u}_{k+1}(t, \xi_0, \eta_0) \chi_{[kAN, kBN] \times [kA\sqrt{N}, kB\sqrt{N}]} \\
 &= N^{-(4s_1+2s_2+3)(k+1)/4} e^{itp(\xi_0, \eta_0)} \xi_0 \int_0^t e^{-it'p(\xi, \eta)} \mathcal{F}_{\xi, \eta}((U(t')\psi_{1+})^{k+1})(\xi_0, \eta_0) dt' \\
 &= N^{-(4s_1+2s_2+3)(k+1)/4} e^{itp(\xi_0, \eta_0)} i\xi_0 \int_{\mathbb{R}^{2k}} \frac{e^{-itF(\xi_0, \eta_0, \dots, \xi_k, \eta_k)} - 1}{F(\xi_0, \eta_0, \dots, \xi_k, \eta_k)} \\
 &\quad \times \psi_{1+}\left(\frac{\xi_0 - \xi_1}{N}, \frac{\eta_0 - \eta_1}{\sqrt{N}}\right) \cdots \psi_{1+}\left(\frac{\xi_{k-1} - \xi_k}{N}, \frac{\eta_{k-1} - \eta_k}{\sqrt{N}}\right) \\
 &\quad \times \psi_{1+}\left(\frac{\xi_k}{N}, \frac{\eta_k}{\sqrt{N}}\right) d\xi_1 d\eta_1 \cdots d\xi_k d\eta_k,
 \end{aligned}$$

where $p(\xi_0, \eta_0) = \xi_0|\xi_0| + \xi_0\eta_0^2$ and

$$\begin{aligned}
 F(\xi_0, \eta_0, \dots, \xi_k, \eta_k) &= \sum_{j=1}^k \xi_j [2(\xi_{j-1} - \xi_j) + (\eta_{j-1} - \eta_j)^2] \\
 &\quad + \sum_{j=1}^k (\xi_{j-1} - \xi_j) \eta_j^2 + 2\xi_{j-1} \eta_j (\eta_{j-1} - \eta_j).
 \end{aligned}$$

Note that in the subset of $\mathbb{R}^{2k}$, defined by

$$A \leq \frac{\xi_0 - \xi_1}{N} \leq B, \dots, A \leq \frac{\xi_{k-1} - \xi_k}{N} \leq B, \quad A \leq \frac{\xi_k}{N} \leq B$$

and

$$A \leq \frac{\eta_0 - \eta_1}{\sqrt{N}} \leq B, \dots, A \leq \frac{\eta_{k-1} - \eta_k}{\sqrt{N}} \leq B, \quad A \leq \frac{\eta_k}{\sqrt{N}} \leq B,$$

we have that

$$\frac{k(k+1)}{2} (A^2 + A^3) N^2 \leq F(\xi_0, \eta_0, \dots, \xi_k, \eta_k) \leq \frac{k(k+1)}{2} (B^2 + B^3) N^2.$$

Thus, it follows that for

$$t_N = \frac{2}{A^2 k(k+1)} N^{-2} \simeq N^{-2}$$

and A close enough to B ,

$$\begin{aligned}
 (3.3) \quad & \widehat{u}_{k+1}(t_N, \xi_0, \eta_0) \chi_{[kAN, kBN] \times [kA\sqrt{N}, kB\sqrt{N}]} \geq N^{-(4s_1+2s_2+3)(k+1)/4} N^{-2} \\
 & \times \int_{\mathbb{R}^{2k}} |\xi_0| \psi_{1+}\left(\frac{\xi_0 - \xi_1}{N}, \frac{\eta_0 - \eta_1}{\sqrt{N}}\right) \cdots \psi_{1+}\left(\frac{\xi_{k-1} - \xi_k}{N}, \frac{\eta_{k-1} - \eta_k}{\sqrt{N}}\right) \\
 & \times \psi_{1+}\left(\frac{\xi_k}{N}, \frac{\eta_k}{\sqrt{N}}\right) d\xi_1 d\eta_1 \cdots d\xi_k d\eta_k.
 \end{aligned}$$

By using that for any $a, b, c, d \in \mathbb{R}$,

$$\widehat{\chi}_{[a,b] \times [c,d]}(\xi, \eta) = 4 \frac{\sin((b-a)\xi/2) \sin((d-c)\eta/2)}{\xi \eta} e^{-i((a+b)\xi + (c+d)\eta)/2},$$

we obtain that, for any $\varrho > 0$,

$$\begin{aligned} & \chi_{[\omega_1 N, \omega_2 N] \times [\omega_1 \sqrt{N}, \omega_2 \sqrt{N}]} * \cdots * \chi_{[\omega_1 N, \omega_2 N] \times [\omega_1 \sqrt{N}, \omega_2 \sqrt{N}]} \left(N(\omega + \varrho), \sqrt{N}(\omega + \varrho) \right) \\ &= 2^{2k} N^{3(k-1)/2} \int_{\mathbb{R}^2} \cos(\varrho \xi) \cos(\varrho \eta) \left(\frac{\sin(\tau \xi) \sin(\tau \eta)}{\xi \eta} \right)^k d\xi d\eta, \end{aligned}$$

where

$$\omega_1 = A + \tau, \quad \omega_2 = B - \tau, \quad \tau = \frac{1}{4}(B - A), \quad \text{and} \quad \omega = k(\omega_1 + \omega_2)/2.$$

As in [22], this proves that there exists $\varrho > 0$, small enough, such that

$$\begin{aligned} (3.4) \quad & |\widehat{u}_{k+1}(t_N, \xi_0, \eta_0)| \chi_{[k \frac{A+B}{2} N - \varrho N, k \frac{A+B}{2} N + \varrho N] \times [k \frac{A+B}{2} \sqrt{N} - \varrho \sqrt{N}, k \frac{A+B}{2} \sqrt{N} + \varrho \sqrt{N}]} \\ & \geq N^{-(4s_1+2s_2+3)(k+1)/4} N^{-2} |\xi_0| N^{3k/2} \\ & \quad \times \chi_{[k \frac{A+B}{2} N - \varrho N, k \frac{A+B}{2} N + \varrho N] \times [k \frac{A+B}{2} \sqrt{N} - \varrho \sqrt{N}, k \frac{A+B}{2} \sqrt{N} + \varrho \sqrt{N}]}, \end{aligned}$$

which yields

$$\|u_{k+1}(t_N)\|_{H^{s_1, s_2}}^2 \geq C N^{-(4s_1+2s_2+3)(k+1)/2} N^{3k-4} N^{2s_1+3} N^{s_2+\frac{1}{2}}.$$

Hence, for $T > 0$ fixed,

$$\lim_{N \rightarrow +\infty} \sup_{t \in [0, T]} \|u_{k+1}(t)\|_{H^{s_1, s_2}} \rightarrow +\infty,$$

as soon as $2s_1 + s_2 < 3/2 - 2/k$. Thus, estimate (3.1) fails, and the proof of Theorem 1.5 is completed.

4. PROOF OF THEOREM 1.6

In this section, attention is given to the proof of Theorem 1.6. First, let us recall some facts about solitary-wave solutions for (1.3). In order to describe our results, the space $\mathcal{Z}$ shall denote the closure of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$(4.1) \quad \|\varphi\|_{\mathcal{Z}}^2 = \|\varphi\|_{L^2(\mathbb{R}^2)}^2 + \|\varphi_y\|_{L^2(\mathbb{R}^2)}^2 + \left\| \partial_x^{1/2} \varphi \right\|_{L^2(\mathbb{R}^2)}^2,$$

where $\partial_x^{1/2}$ denotes the fractional derivative of order $1/2$ with respect to x , defined, via its Fourier transform, by $\widehat{\partial_x^{1/2} \varphi}(\xi, \eta) = |\xi|^{1/2} \widehat{\varphi}(\xi, \eta)$.

Solitary waves are solutions of (1.3) of the form $u(x, y, t) = \varphi_c(x - ct, y)$, where $\varphi_c \in \mathcal{Z}$ and $c \neq 0$ is the wave speed. Thus, substituting this form of u in (1.3) and integrating once, we see that $\varphi := \varphi_c$ must satisfy

$$(4.2) \quad -c\varphi + \frac{1}{k+1} \varphi^{k+1} - \mathcal{H}\varphi_x + \varphi_{yy} = 0.$$

The next lemma is sufficient for our purpose. For the proof we refer the reader to [9], where additional properties of the solitary waves are given.

Lemma 4.1. *Assume $1 \leq k \leq 3$ and $c > 0$. Then equation (4.2) admits a nontrivial solution $\varphi_c \in \mathcal{Z}$. Moreover, $\varphi_c \in H^\infty(\mathbb{R}^2)$, and it decays algebraically in the x -direction and exponentially in the y -direction.*

Proof of Theorem 1.6. Assume $k = 2$ or $k = 3$. Let $s_1, s_2 \geq 0$ be such that $2s_1 + s_2 = 3/2 - 2/k$. It is easy to see that

$$\varphi_c(x, y) = c^{1/k} \varphi_1(cx, \sqrt{c}y), \quad c > 0.$$

Thus, a straightforward calculation reveals that

$$(4.3) \quad \begin{aligned} \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_c \right\|_{L^2} &= c^{2/k-3/2+2s_1+s_2} \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_1 \right\|_{L^2} \\ &= \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_1 \right\|_{L^2} = \vartheta. \end{aligned}$$

Note that the constant ϑ does not depend on c .

Next, for any $c > 0$ fixed, we consider

$$u_c(x, y, t) = \varphi_c(x - ct, y).$$

Hence, at $t = 0$, we have $u_c(0) = \varphi_c$. Moreover, for any $c_1, c_2 > 0$, we obtain

$$(4.4) \quad \begin{aligned} \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} (\varphi_{c_1} - \varphi_{c_2}) \right\|_{L^2}^2 &= \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_{c_1} \right\|_{L^2}^2 \\ &\quad + \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_{c_2} \right\|_{L^2}^2 - 2 \langle \varphi_{c_1}, \varphi_{c_2} \rangle_{s_1, s_2}. \end{aligned}$$

But, we find that

$$\begin{aligned} \langle \varphi_{c_1}, \varphi_{c_2} \rangle_{s_1, s_2} &= \int_{\mathbb{R}^2} (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_{c_1}(x, y) \overline{(-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_{c_2}(x, y)} \, dx dy \\ &= \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \widehat{\varphi_{c_1}}(\xi, \eta) \overline{\widehat{\varphi_{c_2}}(\xi, \eta)} \, d\xi d\eta \\ &= (c_1 c_2)^{\frac{1}{k} - \frac{3}{2}} \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \widehat{\varphi_1}\left(\frac{\xi}{c_1}, \frac{\eta}{\sqrt{c_1}}\right) \overline{\widehat{\varphi_1}\left(\frac{\xi}{c_2}, \frac{\eta}{\sqrt{c_2}}\right)} \, d\xi d\eta \\ &= \left(\frac{c_2}{c_1}\right)^{\frac{1}{k} - \frac{3}{2}} \int_{\mathbb{R}^2} |\xi|^{2s_1} |\eta|^{2s_2} \widehat{\varphi_1}(\xi, \eta) \overline{\widehat{\varphi_1}\left(\frac{c_1}{c_2}\xi, \sqrt{\frac{c_1}{c_2}}\eta\right)} \, d\xi d\eta. \end{aligned}$$

Therefore, as $\theta := c_1/c_2 \rightarrow 1$, we get

$$(4.5) \quad \lim_{\theta \rightarrow 1} \langle \varphi_{c_1}, \varphi_{c_2} \rangle_{s_1, s_2} = \vartheta^2.$$

As a consequence of (4.3)–(4.5), we then get

$$\lim_{\theta \rightarrow 1} \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} (\varphi_{c_1} - \varphi_{c_2}) \right\|_{L^2} = 0.$$

On the other hand, for any $t > 0$,

$$\begin{aligned} \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} (u_{c_1}(t) - u_{c_2}(t)) \right\|_{L^2}^2 &= \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} u_{c_1}(t) \right\|_{L^2}^2 \\ &\quad + \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} u_{c_2}(t) \right\|_{L^2}^2 - 2 \langle u_{c_1}(t), u_{c_2}(t) \rangle_{s_1, s_2}. \end{aligned}$$

But, since

$$\widehat{u_c(t)}(\xi, \eta) = c^{1/k-3/2} e^{-ic\xi t} \widehat{\varphi_1}\left(\frac{\xi}{c}, \frac{\eta}{\sqrt{c}}\right),$$

we deduce that

$$\begin{aligned}
& \langle u_{c_1}(t), u_{c_2}(t) \rangle_{s_1, s_2} \\
&= \int_{\mathbb{R}^2} (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_{c_1}(x - c_1 t, y) \overline{(-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} \varphi_{c_2}(x - c_2 t, y)} \, dx dy \\
&= (c_1 c_2)^{\frac{1}{k} - \frac{3}{2}} \int_{\mathbb{R}^2} e^{-it\xi(c_1 - c_2)} |\xi|^{2s_1} |\eta|^{2s_2} \widehat{\varphi_1} \left(\frac{\xi}{c_1}, \frac{\eta}{\sqrt{c_1}} \right) \overline{\widehat{\varphi_1} \left(\frac{\xi}{c_2}, \frac{\eta}{\sqrt{c_2}} \right)} \, d\xi d\eta \\
&= \left(\frac{c_2}{c_1} \right)^{\frac{1}{k} - \frac{3}{2}} \int_{\mathbb{R}^2} e^{-it\xi c_1(c_1 - c_2)} |\xi|^{2s_1} |\eta|^{2s_2} \widehat{\varphi_1}(\xi, \eta) \overline{\widehat{\varphi_1} \left(\frac{c_1}{c_2} \xi, \sqrt{\frac{c_1}{c_2}} \eta \right)} \, d\xi d\eta.
\end{aligned}$$

Choosing $c_1 = m + 1$ and $c_2 = m \in \mathbb{N}$ and letting $m \rightarrow \infty$, then by the Riemann-Lebesgue lemma, we obtain

$$\lim_{m \rightarrow \infty} \langle u_{c_1}(t), u_{c_2}(t) \rangle_{s_1, s_2} = 0.$$

Therefore, for any $t > 0$,

$$\lim_{\theta \rightarrow 1} \left\| (-\partial_x^2)^{s_1/2} (-\partial_y^2)^{s_2/2} (u_{c_1}(t) - u_{c_2}(t)) \right\|_{L^2} = \sqrt{2} \, \vartheta.$$

This completes the proof of the theorem. $\square$

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