STRICTLY POSITIVE DEFINITE FUNCTIONS ON COMPACT ABELIAN GROUPS

JAN EMONDS AND HARTMUT FÜHR

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ABSTRACT. We study the Fourier characterisation of strictly positive definite functions on compact abelian groups. Our main result settles the case $G = F \times \mathbb{T}^r$, with $r \in \mathbb{N}$ and where F is a finite abelian group. The characterisation obtained for these groups does not extend to arbitrary compact abelian groups; it fails in particular for all torsion-free groups.

1. INTRODUCTION

Let G be a compact abelian group. A complex-valued function f on G is called positive definite if for all $x_1, \dots, x_n$ in G and all $c_1, \dots, c_n$ in $\mathbb{C} \setminus \{0\}$ we have

$$(1.1) \quad \sum_{i,j=1}^n c_i \overline{c_j} f(x_j^{-1} x_i) \geq 0.$$

If the inequality above becomes strict whenever the x_i are distinct, we call f strictly positive definite. By $\mathfrak{P}(G)$ we denote the set of continuous positive definite functions on G and by $\mathfrak{P}^+(G)$ the subset of strictly positive definite functions.

Bochner's theorem [R, Theorem 1.4.3] provides a neat characterisation of the elements of $\mathfrak{P}(G)$ via the Fourier transform: $f \in \mathfrak{P}(G)$ iff $\hat{f} \geq 0$. For $\mathfrak{P}^+(G)$, however, no simple general characterisation is known. For the compact group $\mathbb{T}$ of complex numbers with modulus one, the question has been studied in [ER, P, Su] and solved in [ER, P]. Partial results for general, not necessarily abelian, compact groups may be found in [AP].

It is an easy observation to make that in order to decide whether $f \in \mathfrak{P}(G)$ is in fact in $\mathfrak{P}^+(G)$, only the support of $\hat{f}$ needs to be known; see Theorem 2.2 for a precise statement. Thus, strict positive definiteness translates to a property of subsets of the dual group $\hat{G}$, and we accordingly call $K \subset \hat{G}$ **strictly positive definite** if it is the support of $\hat{f}$, for some $f \in \mathfrak{P}^+(G)$.

This paper compares this notion to another property, called ubiquity: A subset $K \subset \hat{G}$ is called **ubiquitous** if for all $H < \hat{G}$ of finite index and all $\gamma \in \hat{G}$, the intersection $\gamma H \cap K$ is nonempty. It is fairly easy to see that K is ubiquitous if

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it is strictly positive definite; see Lemma 2.5 below. The chief result of our paper states that the converse is true for $G = F \times \mathbb{T}^r$:

Theorem 1.1. *Let $G = F \times \mathbb{T}^r$, and let $K \subset \widehat{G}$. Then K is strictly positive definite iff K is ubiquitous.*

The case $r = 1$, F trivial was settled in [ER, P]. The paper [Su] established partial results, its author apparently unaware of the previous source. While strictly speaking the results of [Su] are contained in the earlier paper, we have found [Su] to be a useful source of ideas; in particular, the notion of ubiquity is taken from that paper.

This paper proceeds as follows: Section 2 collects general remarks and definitions relating to strict positive definiteness. We observe that if $\mathfrak{P}^+(G)$ is nonempty, then G is metrisable (Corollary 2.4). We then prove the implication “strict positive definite $\Rightarrow$ ubiquitous” (Lemma 2.5). A closer look at the torsion subgroup of G allows us to determine interesting classes of examples: The converse of Lemma 2.5 is true for all torsion groups and fails for all torsion-free groups. In the final section we focus on the proof of Theorem 1.1.

2. PRELIMINARIES AND GENERALITIES

Throughout this paper, G denotes a compact abelian group, and $\widehat{G}$ its character group. Throughout this section, we will write the group operations in G and $\widehat{G}$ multiplicatively. In the context of compact groups, Bochner’s theorem [R, 1.4.3] yields that every function f in $\mathfrak{P}(G)$ has a uniformly converging Fourier series with positive coefficients. That is, there is a subset K of $\widehat{G}$ and a sequence $(a_\gamma)_{\gamma \in K}$ of strictly positive numbers such that

$$(2.1) \quad f(x) = \sum_{\gamma \in K} a_\gamma \gamma(x).$$

Let $F = \{x_1, \dots, x_n\}$ be a subset of G and $c = (c_1, \dots, c_n)^T$ a vector in $\mathbb{C}^n$. A trigonometric polynomial $p_{c,F}$ on $\widehat{G}$ is a finite linear combination of characters of $\widehat{G}$, i.e. an assignment

$$(2.2) \quad \widehat{G} \rightarrow \mathbb{C}, \gamma \mapsto \sum_{i=1}^n c_i \gamma(x_i).$$

Note that the space of trigonometric polynomials is closed under addition, multiplication and complex conjugation. Furthermore, since characters over abelian groups are linearly independent, any trigonometric polynomial arising from pairwise different x_i with nonzero coefficients c_i will be nonzero. If f is given by (2.1), then one has

$$(2.3) \quad \sum_{i,j=1}^n c_i \overline{c_j} f(x_j^{-1} x_i) = \sum_{\gamma \in K} a_\gamma \left| \sum_{i=1}^n c_i \gamma(x_i) \right|^2 = \sum_{\gamma \in K} a_\gamma |p_{c,F}(\gamma)|^2.$$

In particular, (2.3) vanishes iff the trigonometric polynomial $p_{c,F}$ vanishes on K . This observation motivates the following definition:

Definition 2.1. A subset K of $\widehat{G}$ is called strictly positive definite if there is no trigonometric polynomial vanishing on K except for the zero polynomial.

The above calculations have established the following result:

Theorem 2.2. *A function $f \in \mathfrak{P}(G)$ is in $\mathfrak{P}^+(G)$ iff the support of $\hat{f}$ is strictly positive definite.*

Let us now collect some basic properties of strictly positive definite sets. The following arguments will rely mainly on duality theory. In particular, we recall the notion of annihilator subgroups: For $M \subset \hat{G}$, let $M^\perp = \{x \in G : \gamma(x) = 1, \forall \gamma \in M\}$. Likewise, $N^\perp = \{\gamma \in \hat{G} : \gamma(x) = 1, \forall x \in N\}$ for $N \subset G$.

Lemma 2.3. *Let $K \subset \hat{G}$ be strictly positive definite. Then K generates $\hat{G}$.*

Proof. Assume that $H = \langle K \rangle$ is a proper subgroup. Pick a nontrivial character $\tilde{\chi}$ of the quotient group $\hat{G}/H$; then $\chi(\gamma) = \tilde{\chi}(\gamma H)$ defines a character of $\hat{G}$. By Pontryagin duality there exists $x \in G$ such that $\chi(\gamma) = \gamma(x)$. The nonzero trigonometric polynomial $p(\gamma) = \gamma(x) - 1$ vanishes on $H \supset K$, proving that K is not strictly positive definite. $\square$

Corollary 2.4. *$\mathfrak{P}^+(G)$ is nonempty iff G is metrisable.*

Proof. Note that by [R, Theorem 2.2.6], G is metrisable iff $\hat{G}$ is countable. Now if $\mathfrak{P}^+(G)$ is nonempty, there exists a strictly positive definite $K \subset \hat{G}$. Since K is the support of a converging Fourier series, K is countable. But then $\hat{G} = \langle K \rangle$ is countable.

For the converse, we pick a summable nowhere vanishing family $(a_\gamma)_{\gamma \in \hat{G}}$ of positive numbers, which exists by countability of $\hat{G}$. Define f according to (2.1), with $K = \hat{G}$. Now (2.3), with $K = \hat{G}$, implies that f is strictly positive definite. $\square$

Let us next establish the general implication between strict positive definiteness and ubiquity. The central question of this paper is when the converse of this result holds.

Lemma 2.5. *If $K \subset \hat{G}$ is strictly positive definite, it is ubiquitous.*

Proof. First we prove that for proper subgroups $H < \hat{G}$ of finite index and $\gamma \in \hat{G}$ there exists a trigonometric polynomial vanishing precisely on γH : By duality, $H^\perp \cong (\hat{G}/H)^\wedge$ is finite, and thus $H = H^{\perp\perp} = \{x_1, \dots, x_n\}^\perp$, with $x_1, \dots, x_n \in G$. Hence, if we define a trigonometric polynomial p by

$$(2.4) \quad p(\mu) = \sum_{i=1}^n |\overline{\gamma(x_i)}\mu(x_i) - 1|^2, \quad (\mu \in \hat{G}),$$

we find that $p(\mu) = 0$ iff $\gamma^{-1}\mu \in \{x_1, \dots, x_n\}^\perp$ iff $\mu \in \gamma H$.

Now, if K is not ubiquitous, then $K \cap \gamma H = \emptyset$ for suitable H of finite index, and $\gamma \in \hat{G}$. Write $\hat{G} \setminus \gamma H = \bigcup_{i=1}^m \mu_i H$, and pick trigonometric polynomials p_i vanishing precisely on $\mu_i H$. Then $\prod_{i=1}^m p_i$ is a trigonometric polynomial vanishing precisely outside of γH . It is therefore nonzero and vanishes on K , which then cannot be strictly positive definite. $\square$

Next we characterise finite strictly positive definite subsets. As a byproduct, we clarify the case of finite groups.

Lemma 2.6. *Let $K \subset \hat{G}$ be finite. Then K is strictly positive definite iff G is finite and $K = \hat{G}$.*

Proof. Note that by definition, K is strictly positive definite iff the restriction map $p \mapsto p|_K$, defined on the space of trigonometric polynomials, has trivial kernel. If G is infinite, the space of trigonometric polynomials on $\widehat{G}$ is infinite-dimensional, precluding the existence of finite strictly positive definite sets. Thus, if K is finite, G has to be finite as well, and ubiquity of $K \subset \widehat{G}$ implies $K = \widehat{G}$. The converse is obvious. $\square$

Some clarification concerning the converse of Lemma 2.5 is obtained by considering the torsion subgroup G_t of G , defined as

$$G_t = \{x \in G : x^n = e_G, \text{ for suitable } n \in \mathbb{N}\}.$$

The torsion subgroup is usually not closed; for instance, the torsion subgroup of the torus group is dense. The following observation indicates how the torsion subgroup is related to ubiquity.

Lemma 2.7. *Let*

$$H_0 = \bigcap_{H < \widehat{G}, [\widehat{G}:H] < \infty} H.$$

Then

$$H_0 = G_t^\perp, \quad H_0^\perp = \overline{G_t}.$$

Proof. We first prove $G_t \subset H_0^\perp$: By the isomorphism theorem for groups, $\gamma \in H_0$ iff $\phi(\gamma) = e$, for all group homomorphisms ϕ with finite image. In particular, if $x \in G_t$, the mapping $\widehat{G} \ni \gamma \mapsto \gamma(x) \in \mathbb{T}$ has finite image, since $\gamma(x)^n = \gamma(x^n) = 1$. It follows for $\gamma \in H_0$ that $\gamma(x) = 1$, which means $x \in H_0^\perp$.

For the proof of $G_t^\perp \subset H_0$ let $H < \widehat{G}$ be of finite index. By duality theory, $H^\perp \cong (\widehat{G}/H)^\wedge$ is finite, hence a subgroup of G_t . But then $G_t^\perp \subset H^{\perp\perp} = H$. Since $H < \widehat{G}$ was chosen arbitrarily of finite index, it follows that $G_t^\perp \subset H_0$.

Both inclusions shown so far imply $H_0 = H_0^{\perp\perp} \subset G_t^\perp \subset H_0$, and thus $H_0 = G_t^\perp$. The second equality follows from this. $\square$

We now settle the extreme cases $G_t = G$ and $G_t = \{e\}$. First the good news.

Theorem 2.8. *Let G be a torsion group, and let $K \subset \widehat{G}$. Then K is strictly positive definite iff K is ubiquitous.*

Proof. Only the “if” part needs to be shown. Suppose that $K \subset \widehat{G}$ is not strictly positive definite. Hence there is a nontrivial trigonometric polynomial

$$p : \widehat{G} \ni \gamma \mapsto \sum_{i=1}^n c_i \gamma(x_i)$$

of $\widehat{G}$ vanishing on K . Since G is a torsion group, $\langle x_1, \dots, x_n \rangle$ is finite, and by duality theory, $H = \{x_1, \dots, x_n\}^\perp \subset \widehat{G}$ has finite index. Furthermore, for any $\gamma \in \widehat{G}$ and $\eta \in H$, one has

$$p(\gamma\eta) = \sum_{i=1}^n c_i \gamma(x_i) \eta(x_i) = p(\gamma),$$

implying that p vanishes on an H -invariant set. In particular, since p is nonzero and vanishes on K , $\widehat{G} \setminus K$ contains an H -coset. Thus K is not ubiquitous. $\square$

The theorem applies to groups of the form $G = \prod_{i=1}^{\infty} F_i$, with finite groups F_i of bounded order. The other extreme provides a whole class of examples for which the converse of Lemma 2.5 fails.

Theorem 2.9. *Suppose that G is nontrivial and torsion-free. Then every nonempty subset of $\widehat{G}$ is ubiquitous, but finite subsets are not strictly positive definite.*

Proof. Suppose G_t is trivial. With H_0 as defined in Lemma 2.7, one obtains from Lemma 2.7 that $\widehat{G}/H_0$ is trivial also. Hence $\widehat{G}$ has no proper finite index subgroups, and then every nonempty subset is ubiquitous. Since G is torsion-free and nontrivial, it is infinite, and then Lemma 2.6 implies that no finite $K \subset \widehat{G}$ is strictly positive definite. $\square$

This result applies, for instance, to the group $\mathbb{Z}_p$ of p -adic integers.

3. THE CASE $G = F \times \mathbb{T}^r$

The remainder of the paper is reserved for $G = F \times \mathbb{T}^r$. We identify the dual group of G in the canonical way with $F \times \mathbb{Z}^r$ and write the latter additively. At first we will deal with $\mathbb{T}^r$ separately. Here we will need a fairly deep theorem from number theory.

3.1. Products of $\mathbb{T}$. We start with two lemmata that will be needed in the proof of the main result. The first one is a fact from elementary group theory.

Lemma 3.1. *Every finite intersection of subgroups of finite index is a subgroup of finite index as well.*

Proof. This follows by induction and:

- (1) If A and B are subgroups of a group G , then $[B : A \cap B] \leq [G : A]$.
- (2) If $B < G$ and $A < B$, then $[G : A] = [G : B][B : A]$.

$\square$

Lemma 3.2. *Let H be subgroup of infinite index in $\mathbb{Z}^r$ and let $y \in \mathbb{Z}^r \setminus H$. Then there is a subgroup G of finite index such that $H \subset G$ and $y \notin G$.*

Proof. Let H be a subgroup of infinite index in $\mathbb{Z}^r$ and let $y \notin H$. Then there is a basis $x_1, \dots, x_r$ of $\mathbb{Z}^r$ and some $\alpha_1, \dots, \alpha_r$ in $\mathbb{Z} \setminus \{0\}$ such that $\alpha_1 x_1, \dots, \alpha_s x_s$ form a basis of H and $(\bigoplus_{i=1}^s \mathbb{Z} x_i)/H \cong \bigoplus_{i=1}^s (\mathbb{Z}/\alpha_i \mathbb{Z})$; see [B, 2.9.2]. In particular,

we have $s < r$. Now y can be expressed as $y = \sum_{i=1}^r \beta_i x_i$ with unique integers $\beta_1, \dots, \beta_r$. If $\beta_{s+1} = \dots = \beta_r = 0$, we define G to be the subgroup generated by $\alpha_1 x_1, \dots, \alpha_s x_s, x_{s+1}, \dots, x_r$ and note that $G \supset H$ is of finite index and $y \notin G$ due to the uniqueness of the coefficients. If $\beta_i \neq 0$ for some $i \in \{s+1, \dots, r\}$, then G is defined as the subgroup generated by $\alpha_1 x_1, \dots, \alpha_r x_r$ with $\alpha_{s+1}, \dots, \alpha_r$ in $\mathbb{Z} \setminus \{0\}$ and α_i not a divisor of β_i . Then once again H is a subset of G , G is of finite index and y is not an element of G by construction. $\square$

The main device for showing sufficiency of ubiquity is the following theorem due to Laurent; see [L]. For a partition $\mathcal{P}$ of $\{1, \dots, n\}$ we write $\gamma \in \text{Null}(p^{\mathcal{P}})$ if

$$(3.1) \quad 0 = \sum_{k \in P} c_k \gamma(x_k)$$

for all $P \in \mathcal{P}$. Clearly, $\text{Null}(p^{\mathcal{P}}) \subset \text{Null}(p) = p^{-1}(\{0\})$. A partition $\mathcal{P}'$ is called finer than $\mathcal{P}$ if $\mathcal{P}'$ is a partition of $\{1, \dots, n\}$ and for all $P' \in \mathcal{P}'$ there is $P \in \mathcal{P}$ such that $P' \subset P$. In short, we write $\mathcal{P}' < \mathcal{P}$ if $\mathcal{P}' \neq \mathcal{P}$ and $\mathcal{P}'$ is finer than $\mathcal{P}$. Furthermore, $\gamma \in \text{Null}(p)$ is called maximal with respect to $\mathcal{P}$ if $\gamma \in \text{Null}(p^{\mathcal{P}})$ and $\gamma \notin \text{Null}(p^{\mathcal{P}'})$ for every $\mathcal{P}' < \mathcal{P}$.

By $H_{\mathcal{P}}$ we denote the subgroup of $\mathbb{Z}^r$ defined by

$$\begin{aligned} H_{\mathcal{P}} &= \bigcap_{P \in \mathcal{P}} \{\gamma \in \mathbb{Z}^r : \gamma(x_k) = \gamma(x_l) \text{ for } k, l \in P\} \\ &= \bigcap_{P \in \mathcal{P}} \{\gamma \in \mathbb{Z}^r : \gamma(x_k x_l^{-1}) = 1 \text{ for } k, l \in P\} \\ &= \bigcap_{P \in \mathcal{P}} \{x_k x_l^{-1} : k, l \in P\}^{\perp}. \end{aligned}$$

Finally, let $S_{\mathcal{P}} \subset \text{Null}(p)$ be the set which is maximal with respect to $\mathcal{P}$. Then one has the following theorem due to Laurent, [L].

Theorem 3.3. *$S_{\mathcal{P}}$ is a finite union of $H_{\mathcal{P}}$ -cosets or it is empty.*

This theorem can be regarded as a generalisation of a number theoretical result of Skolem, Mahler and Lech on linear recurrences; see [P]. We are interested in $\text{Null}(p)$, so we need only a corollary. By definition we have

$$(3.2) \quad \text{Null}(p) \supset \bigcup_{\mathcal{P} \text{ partition of } \{1, \dots, n\}} S_{\mathcal{P}}.$$

Conversely, $\gamma \in \text{Null}(p)$ implies $\gamma \in \text{Null}(p^{\mathcal{P}})$, where $\mathcal{P} = \{\{1, \dots, n\}\}$. Suppose now $\gamma \notin S_{\mathcal{P}}$ for every partition $\mathcal{P}$. That is, for every partition $\mathcal{P}$ the fact that $\gamma \in \text{Null}(p^{\mathcal{P}})$ implies that $\gamma \in \text{Null}(p^{\mathcal{P}'})$ for some $\mathcal{P}' < \mathcal{P}$. But there are only finitely many partitions of $\{1, \dots, n\}$, where $\mathcal{P}_0 = \{\{1\}, \dots, \{n\}\}$ is the finest partition with respect to $<$. In particular, $\gamma \in \text{Null}(p^{\mathcal{P}_0})$, but $\gamma \notin S_{\mathcal{P}_0}$ leads to a contradiction. Hence, it follows that $\gamma \in S_{\mathcal{P}}$ for some partition $\mathcal{P}$. That is,

$$(3.3) \quad \text{Null}(p) = \bigcup_{\mathcal{P} \text{ partition of } \{1, \dots, n\}} S_{\mathcal{P}}$$

and we get:

Corollary 3.4. *Let p be a nontrivial trigonometric polynomial on $\mathbb{Z}^r$ such that $\text{Null}(p) \neq \emptyset$. There are finitely many subgroups $G_1, \dots, G_n$ of $\mathbb{Z}^r$ and $x_1, \dots, x_n \in \mathbb{Z}^r$ such that*

$$(3.4) \quad \text{Null}(p) = \bigcup_{i=1}^n x_i + G_i.$$

Theorem 3.5. *Let K be a subset of $\mathbb{Z}^r$. If K is ubiquitous, then it is also strictly positive definite.*

Proof. Assume that K is not strictly positive definite. Then there exists a nonzero trigonometric polynomial p such that $K \subset S$, where S denotes the set of its zeros. By Corollary 3.4 we know that S can be written as $\bigcup_{i=1}^n \gamma_i + H_i$ for some $\gamma_1, \dots, \gamma_n$ in $\mathbb{Z}^r$ and subgroups $H_1, \dots, H_n$. Without loss of generality we can assume that

$H_1, \dots, H_m$ are of finite index and $H_{m+1}, \dots, H_n$ are of infinite index. Since p is nonzero there is some γ in $\mathbb{Z}^r \setminus S$. But then

$$(3.5) \quad H' = \bigcap_{i=1}^m H_i$$

satisfies

$$(3.6) \quad \gamma + H' \cap \bigcup_{i=1}^m \gamma_i + H_i = \emptyset$$

and is of finite index; see Lemma 3.1. Furthermore, for every $i = m+1, \dots, n$ we pick by virtue of Lemma 3.2 a subgroup I_i of finite index such that $H_i \subset I_i$ and $\gamma - \gamma_i \notin I_i$. Now if we put

$$(3.7) \quad H = H' \cap \bigcap_{i=m+1}^n I_i,$$

then H is still of finite index. Furthermore, for each $i \in \{1, \dots, m\}$,

$$\gamma + H \cap \gamma_i + H_i \subset \gamma + H_i \cap \gamma_i + H_i = \emptyset$$

by choice of γ , whereas for $i \in \{m+1, \dots, n\}$,

$$\gamma + H \cap \gamma_i + H_i \subset \gamma + I_i \cap \gamma_i + I_i = \emptyset$$

by choice of I_i . Hence finally,

$$(3.8) \quad \gamma + H \cap K \subset \gamma + H \cap S = \emptyset,$$

which shows that K is not ubiquitous. $\square$

3.2. Strict positive definiteness over direct products. It remains to combine the results for the factors F and $\mathbb{T}^r$, obtained in Lemma 2.6 and Theorem 3.5 respectively. The following somewhat technical result illustrates that the transfer of results for the factors to the product group is not entirely trivial.

Theorem 3.6. *Let $G = G_1 \times G_2$, with compact groups G_1 and G_2 . Let $K \subset \widehat{G_1 \times G_2}$. For $\gamma \in \widehat{G_1}$ let*

$$(3.9) \quad K_2(\gamma) = \{\omega : (\gamma, \omega) \in K\}$$

and

$$(3.10) \quad K_1 = \{\gamma : K_2(\gamma) \text{ is strictly positive definite}\}.$$

Finally, let

$$(3.11) \quad \tilde{K} = \bigcup_{\gamma \in K_1} \{\gamma\} \times K_2(\gamma).$$

If K_1 is strictly positive definite, then so is $\tilde{K}$ and in particular K .

Proof. For $\gamma \in K_1$ and $\omega \in K_2(\gamma)$ let positive real numbers a_γ , resp. b_ω , be given such that $\sum_{\gamma \in K_1} a_\gamma \left(\sum_{\omega \in K_2(\gamma)} b_\omega \right)$ is convergent. If we put $f = \sum_{\gamma \in K_1} a_\gamma \gamma \left(\sum_{\omega \in K_2(\gamma)} b_\omega \omega \right)$ = $\sum_{(\gamma, \omega) \in \tilde{K}} a_\gamma b_\omega \gamma \omega$, then f converges absolutely and unconditionally on $G_1 \times G_2$

by Fubini's theorem; see [HS, (21.13)]. Now suppose that for distinct $z_1 = (x_1, y_1), \dots, z_n = (x_n, y_n)$ in $G_1 \times G_2$ and some complex $c_1, \dots, c_n$ we have

$$(3.12) \quad 0 = \sum_{i,j=1}^n c_i \overline{c_j} f(z_j^{-1} z_i) = \sum_{(\gamma, \omega) \in \tilde{K}} a_\gamma b_\omega \left| \sum_{i=1}^n c_i \gamma(x_i) \omega(y_i) \right|^2.$$

Without loss of generality we can assume that $y_1, \dots, y_m$ are distinct and we put $I_l = \{k : y_k = y_l\}$. Thus $I_1, \dots, I_m$ form a partition of $\{1, \dots, n\}$ and for $(\gamma, \omega) \in \tilde{K}$ (3.12) reads:

$$(3.13) \quad 0 = \sum_{i=1}^n c_i \gamma(x_i) \omega(y_i) = \sum_{l=1}^m \left(\sum_{k \in I_l} c_k \gamma(x_k) \right) \omega(y_l).$$

Since $K_2(\gamma)$ is strictly positive definite this implies $0 = \sum_{k \in I_l} c_k \gamma(x_k)$ for $(\gamma, \omega) \in \tilde{K}$ and $l = 1, \dots, m$. But this again leads to $c_k = 0$ for $k \in I_l$ and $l = 1, \dots, m$, since K_1 is strictly positive definite. $\square$

We do not know of an exhaustive characterisation of strictly positive definite subsets of the product group $\widehat{G}_1 \times \widehat{G}_2$ in terms of strictly positive definite subsets of $\widehat{G}_1$ and $\widehat{G}_2$. It is fairly easy to see that the sufficient condition of the theorem is not necessary: For a counterexample, consider the case $G_1 = G_2 = \mathbb{T}$. Let

$$K = \bigcup_{n=0}^{\infty} \{n\} \times \{-n, \dots, n\},$$

and let K_1 be defined as in the theorem. Then Lemma 2.6 implies that $K_1 = \emptyset$. But K is strictly positive definite, which can be easily seen by applying the theorem with the roles of $\widehat{G}_1$ and $\widehat{G}_2$ interchanged and using the observation that

$$K = \bigcup_{m=-\infty}^{\infty} \{k : k \geq |m|\} \times \{m\}.$$

One could formulate a version of the theorem that covers this example as well, e.g. by introducing a condition that is symmetric with respect to the roles of $\widehat{G}_1$ and $\widehat{G}_2$. More generally, since strict positive definiteness is clearly preserved by the action of a group automorphism, the condition would have to be invariant under automorphisms of $\widehat{G}_1 \times \widehat{G}_2$ as well. The counterexample illustrates that the sufficient condition is not invariant under the automorphism $(\gamma, \omega) \mapsto (\omega, \gamma)$. It seems open to us whether a clear-cut characterisation working for all product groups is available.

3.3. Proof of Theorem 1.1. By Lemma 2.5 it remains to prove the “if” direction. Assume that $K \subset F \times \mathbb{Z}^r$ is ubiquitous, and define $K_2(\gamma)$, for arbitrary $\gamma \in F$, according to Theorem 3.6. It suffices to show, for all $\gamma \in F$, that $K_2(\gamma) \subset \mathbb{Z}^r$ is strictly positive definite. Assume $\gamma \in F$, H is a subgroup of finite index in $\mathbb{Z}^r$ and $\omega \in \mathbb{Z}^r$. Then $\{e\} \times H$ is a subgroup of finite index in $F \times \mathbb{Z}^r$, and by assumption the intersection

$$(3.14) \quad K \cap (\gamma, \omega)(\{e\} \times H) = K \cap \{\gamma\} \times \omega H = \{(\gamma, \chi) : \chi \in K_2(\gamma) \cap \omega H\}$$

is not empty. Hence, the projection thereof to the second variable is also not empty and this projection is nothing but the intersection of $K_2(\gamma)$ with ωH . That is,

$K_2(\gamma)$ is ubiquitous for arbitrary $\gamma \in F$. Now Theorem 3.5 implies that $K_2(\gamma)$ is strictly positive definite, and we are done. $\square$

REFERENCES

- [AP] M. Allali and T. Przebinda, *Strictly positive definite functions on a compact group*, Proc. Amer. Math. Soc. 129 (2001), 1459-1462. MR1814173 (2002f:43004)
- [B] S. Bosch, *Algebra*, 5. Auflage, Springer-Verlag, Berlin (2004).
- [ER] G. A. Edgar and J. M. Rosenblatt, *Difference equations over locally compact abelian groups*, Trans. Amer. Math. Soc. 253 (1979), 273-289. MR536947 (80i:39001)
- [HS] E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, New York (1965). MR0367121 (51:3363)
- [L] M. Laurent, *Equations exponentielles-polynômes et suites récurrentes linéaires. II*, J. Number Th. 31 (1989), 24-53. MR978098 (90b:11023)
- [P] A. Pinkus, *Strictly Hermitian positive definite functions*, J. d'Analyse Math. 94 (2004), 293-318. MR2124464 (2005k:43004)
- [R] W. Rudin, *Fourier Analysis on Groups*, Wiley-Interscience, New York (1962). MR0152834 (27:2808)
- [Su] X. Sun, *Strictly positive definite functions on the unit circle*, Math. Comp. 74 (2005), 709-721. MR2114644 (2005i:41003)

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT PADERBORN, D-33098 PADERBORN, GERMANY
E-mail address: `emonds@math.upb.de`

LEHRSTUHL A FÜR MATHEMATIK, RWTH AACHEN, D-52056 AACHEN, GERMANY
E-mail address: `fuehr@matha.rwth-aachen.de`