LIMITS OF COALGEBRAS, BIALGEBRAS AND HOPF ALGEBRAS

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Dedicated to the memory of Professor S. Ianuş

ABSTRACT. We give the explicit construction of the product of an arbitrary family of coalgebras, bialgebras and Hopf algebras: it turns out that the product of an arbitrary family of coalgebras (resp. bialgebras, Hopf algebras) is the sum of a family of coalgebras (resp. bialgebras, Hopf algebras). The equalizers of two morphisms of coalgebras (resp. bialgebras, Hopf algebras) are also described explicitly. As a consequence the categories of coalgebras, bialgebras and Hopf algebras are shown to be complete, and a complete description for limits in the above categories is given.

INTRODUCTION

It is well known that the category $k\text{-Alg}$ of k -algebras is complete and cocomplete; that is, any functor $F : I \rightarrow k\text{-Alg}$ has a limit and a colimit for all small categories I . This is immediately implied by the existence of products, coproducts, equalizers and coequalizers in the category $k\text{-Alg}$. The categories of coalgebras, bialgebras or Hopf algebras have arbitrary coproducts and coequalizers (see [5, Proposition 1.4.19], [4, Proposition 2.10], [8, Corollary 2.6.6] and [2, Remark 2.1, Theorem 2.2]); hence these categories are cocomplete. Related to the question of whether these categories are complete (i.e. if they have arbitrary products and equalizers), we could not find similar results in the classical Hopf algebra textbooks ([1], [9]), not even in the more recent ones ([4], [5]). For example, [5, Proposition 1.4.21] proves only the existence of finite products (namely the tensor product of coalgebras) and only in the category of *cocommutative* coalgebras, as a dual result to the one concerning commutative algebras.

In this paper we shall fill this gap: using the fact that the forgetful functor from the category of coalgebras to the category of vector spaces has a right adjoint, namely the so-called cofree coalgebra, we shall construct explicitly the product of an arbitrary family of coalgebras. As a consequence, the product of an arbitrary family of bialgebras and Hopf algebras is constructed. The equalizers of two morphisms of

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coalgebras (bialgebras, Hopf algebras) are also described explicitly. Thus we shall obtain that the categories of coalgebras, bialgebras and Hopf algebras are complete.

Throughout this paper, k will be a field. Unless specified otherwise, all vector spaces, algebras, coalgebras, bialgebras, tensor products and homomorphisms are over k . Our notation for the standard categories is as follows: ${}_k\mathcal{M}$ (k -vector spaces), $k\text{-Alg}$ (associative unital k -algebras), $k\text{-CoAlg}$ (coalgebras over k), $k\text{-BiAlg}$ (bialgebras over k), $k\text{-HopfAlg}$ (Hopf algebras over k), $\mathcal{M}^C$ (right C -comodules). For a coalgebra C , we will use Sweedler's Σ -notation, that is, $\Delta(c) = c_{(1)} \otimes c_{(2)}$, $(I \otimes \Delta)\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$, etc. Given a vector space V , $(K(V), p)$ stands for the cofree coalgebra on V , where $K(V)$ is a coalgebra and $p : K(V) \rightarrow V$ is a k -linear map. We refer to [1], [5], [9] for further details concerning Hopf algebras. A category $\mathcal{C}$ is called *(co)complete* if any functor $F : I \rightarrow \mathcal{C}$ has (co)limits, where I is a small category. A category $\mathcal{C}$ is *(co)complete* if and only if $\mathcal{C}$ has (co)equalizers of all pairs of arrows and all (co)products [7, Theorem 6.10]. Given a morphism $f \in \mathcal{C}$ we denote by $\text{dom}(f)$ and $\text{cod}(f)$ the domain, respectively the codomain, of f . If $\mathcal{C}$ is a small category we denote by $\text{Hom}(\mathcal{C})$ the set of all morphisms of $\mathcal{C}$.

1. LIMITS FOR COALGEBRAS, BIALGEBRAS AND HOPF ALGEBRAS

First, we explicitly construct the product of an arbitrary family of coalgebras.

Theorem 1.1. *The category $k\text{-CoAlg}$ of coalgebras has arbitrary products and equalizers. In particular, the category $k\text{-CoAlg}$ of coalgebras is complete.*

Proof. Let $f, g : C \rightarrow D$ be two coalgebra maps and $S := \{c \in C \mid f(c) = g(c)\}$, which is a k -subspace of C . Let E be the sum of all subcoalgebras of C included in S . Note that the family of subcoalgebras of C included in S is not empty since it contains the null coalgebra. It is immediate that E is a subcoalgebra of C . We shall prove that (E, i) is the equalizer of the pair (f, g) in $k\text{-CoAlg}$, where $i : E \rightarrow C$ is the canonical inclusion. Let E' be a coalgebra and $h : E' \rightarrow C$ a coalgebra map such that $f \circ h = g \circ h$. Then $f(h(x)) = g(h(x))$ for all $x \in E'$; hence $h(E') \in S$. Since $h(E')$ is a subcoalgebra in C we obtain $h(E') \subseteq E$. Thus there exists a unique coalgebra map $h : E' \rightarrow E$ such that $i \circ h = h$. Hence (E, i) is the equalizer of the pair (f, g) in the category $k\text{-CoAlg}$ of coalgebras.

Now let $(C_i)_{i \in I}$ be a family of coalgebras and $(\prod_{i \in I} C_i, (\pi_i)_{i \in I})$ be the product of the k -modules $(C_i)_{i \in I}$. Let $(K(\prod_{i \in I} C_i), p)$ be the cofree coalgebra over the vector space $\prod_{i \in I} C_i$.

Let D be the sum of all subcoalgebras E of $K(\prod_{i \in I} C_i)$ such that $\pi_i \circ p \circ j_E$ is a coalgebra map for all $i \in I$, where $j_E : E \rightarrow K(\prod_{i \in I} C_i)$ is the canonical inclusion. The family of subcoalgebras of E satisfying this property is nonempty since it contains the null coalgebra. The k -linear map $\pi_i \circ p \circ j : D \rightarrow C_i$ is a coalgebra map for all $i \in I$, where $j : D \rightarrow K(\prod_{i \in I} C_i)$ is the canonical inclusion. We shall prove that $(D, (\pi_i \circ p \circ j)_{i \in I})$ is the product of the family of coalgebras $(C_i)_{i \in I}$ in $k\text{-CoAlg}$.

Indeed, let D' be a coalgebra and $g_i : D' \rightarrow C_i$, $i \in I$, be a family of coalgebra maps. Using the universal property of the product in ${}_k\mathcal{M}$ we obtain that there exists a unique k -linear map $\bar{g} : D' \rightarrow \prod_{i \in I} C_i$ such that $\pi_i \circ \bar{g} = g_i$, for all $i \in I$. Furthermore, since $(K(\prod_{i \in I} C_i), p)$ is the cofree coalgebra over the k -module $\prod_{i \in I} C_i$, there exists a unique coalgebra map $\bar{f} : D' \rightarrow K(\prod_{i \in I} C_i)$ such

that $p \circ \bar{f} = \bar{g}$. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
 D' & & & & & & \\
 & \searrow \bar{f} & & \searrow \bar{g} & & \searrow g_i & \\
 D & \xrightarrow{j} & K(\prod_{i \in I} C_i) & \xrightarrow{p} & \prod_{i \in I} C_i & \xrightarrow{\pi_i} & C_i
 \end{array}$$

It follows that $(\pi_i \circ p)(\bar{f}(x)) = g_i(x)$, for all $x \in D'$ and $i \in I$. So, since each g_i is a coalgebra map, we have $\bar{f}(D') \subseteq D$. Hence, we proved that for any coalgebra D' and any family $g_i : D' \rightarrow C_i$, $i \in I$, of coalgebra maps there exists a coalgebra map $\bar{f} : D' \rightarrow D$ such that $(\pi_i \circ p \circ j) \circ \bar{f} = g_i$, for all $i \in I$. Let $h : D' \rightarrow D$ be another coalgebra map such that $(\pi_i \circ p \circ j) \circ h = g_i$ for all $i \in I$. From the uniqueness of $\bar{g}$ we obtain $p \circ j \circ h = \bar{g}$. Moreover, from the uniqueness of $\bar{f}$ we obtain $j \circ h = \bar{f}$, hence $h = \bar{f}$. Thus $(D, (\pi_i \circ p \circ j)_{i \in I})$ is the product of the family $(C_i)_{i \in I}$ in the category $k\text{-CoAlg}$ of coalgebras. $\square$

Remark 1.2. In [3, Lemma 1.1.3] a description for the equalizers in the category $k\text{-HopfAlg}$ is given. We can use the same method in order to obtain another description for the equalizer of a pair of coalgebra (or bialgebra) maps. Let $f, g : C \rightarrow D$ be two coalgebra maps. It can be easily proved that (E, i) is the equalizer of the pair (f, g) in the category $k\text{-CoAlg}$ of coalgebras, where

$$E = \{c \in C \mid c_{(1)} \otimes f(c_{(2)}) \otimes c_{(3)} = c_{(1)} \otimes g(c_{(2)}) \otimes c_{(3)}\}$$

and $i : E \rightarrow C$ is the canonical inclusion. This equivalent description of equalizers in the category $k\text{-CoAlg}$ will turn out to be more efficient for computations.

Example 1.3. Let G be a multiplicative group and kG be the k -vector space with basis $\{g \mid g \in G\}$ endowed with the classical coalgebra structure : $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$. Thus any element $x \in kG$ has the form $x = \sum_{g \in G} k_g g$, where $(k_g)_{g \in G}$ is a family of elements in k with only a finite number of nonzero elements. We use the formal notation $x^{-1} := \sum_{g \in G} k_g g^{-1}$ and $0^{-1} = 0$. Consider the coalgebra maps $f = Id_{kG}$ and $h : kG \rightarrow kG$ given by $h(g) = g^{-1}$ for all $g \in G$. Then, in light of the above remark, it follows that the equalizer of the morphisms (f, g) is given by the pair (E, i) , where $E = \{x \in kG \mid x \otimes x \otimes x = x \otimes x^{-1} \otimes x\}$ and i is the canonical inclusion.

As an easy consequence of [6, Chapter 5 §2, Theorem 1] we obtain the following description for limits in the category $k\text{-CoAlg}$ of coalgebras.

Remark 1.4. Let J be a small category, $F : J \rightarrow k\text{-CoAlg}$ be a functor, $(\prod_{j \in J} F(j), (p_j)_{j \in J})$, $(\prod_{u \in \text{Hom}(J)} F(\text{cod}(u)), (p_u)_{u \in \text{Hom}(J)})$ be the product in $k\text{-CoAlg}$ of the families $(F(j))_{j \in J}$, respectively $(F(\text{cod}(u)))_{u \in \text{Hom}(J)}$, and $f, g : \prod_{j \in J} F(j) \rightarrow \prod_{u \in \text{Hom}(J)} F(\text{cod}(u))$ be the unique coalgebra maps such that $p_u \circ f = p_{\text{cod}(u)}$ and $p_u \circ g = F(u) \circ p_{\text{dom}(u)}$ for all $u \in \text{Hom}(J)$. We define

$$D = \{x \in \prod_{j \in J} F(j) \mid x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes g(x_{(2)}) \otimes x_{(3)}\}.$$

Then the pair $(D, (\varphi_j = p_j \circ e)_{j \in J})$ is the limit of the functor F , where $e : D \rightarrow \prod_{j \in J} F(j)$ is the canonical inclusion.

In what follows we will make use of Theorem 1.1 in order to construct the product in the category of $k\text{-BiAlg}$ of bialgebras.

Theorem 1.5. *The category $k\text{-BiAlg}$ of bialgebras has arbitrary products and equalizers. In particular, the category $k\text{-BiAlg}$ of bialgebras is complete.*

Proof. Let $(B_i, m_i, \eta_i, \Delta_i, \varepsilon_i)_{i \in I}$ be a family of bialgebras and let $((\prod_{i \in I} B_i, \Delta, \varepsilon), (\pi_i)_{i \in I})$ be the product of this family in the category $k\text{-CoAlg}$ of coalgebras. Since $(B_i, m_i, \eta_i, \Delta_i, \varepsilon_i)$ is a bialgebra it follows that $m_i : B_i \otimes B_i \rightarrow B_i$ and $\eta_i : k \rightarrow B_i$ are coalgebra maps for all $i \in I$. Then there exists a unique coalgebra map $\eta : k \rightarrow \prod_{i \in I} B_i$ such that the diagram

$$(1) \quad \begin{array}{ccc} k & & \\ \eta \downarrow & \searrow \eta_i & \\ \prod_{i \in I} B_i & \xrightarrow{\pi_i} & B_i \end{array}$$

is commutative for all $i \in I$. Also there exists a unique coalgebra map $m : \prod_{i \in I} B_i \otimes \prod_{i \in I} B_i \rightarrow \prod_{i \in I} B_i$ for which the diagram

$$(2) \quad \begin{array}{ccc} \prod_{i \in I} B_i \otimes \prod_{i \in I} B_i & & \\ m \downarrow & \searrow m_i \circ (\pi_i \otimes \pi_i) & \\ \prod_{i \in I} B_i & \xrightarrow{\pi_i} & B_i \end{array}$$

is commutative for all $i \in I$.

First, we will prove that $(\prod_{i \in I} B_i, m, \eta)$ is a k -algebra. Since $\pi_i \circ m \circ (m \otimes Id)$ is a coalgebra map, by the universal property of the product we obtain that there exists a unique coalgebra map $\psi : (\prod_{i \in I} B_i)^{\otimes 3} \rightarrow \prod_{i \in I} B_i$ such that the diagram

$$\begin{array}{ccc} (\prod_{i \in I} B_i)^{\otimes 3} & & \\ \psi \downarrow & \searrow \pi_i \circ m \circ (m \otimes Id) & \\ \prod_{i \in I} B_i & \xrightarrow{\pi_i} & B_i \end{array}$$

is commutative for all $i \in I$. It is easy to see that the coalgebra map $m \circ (m \otimes Id)$ makes the above diagram commute. Thus, using the uniqueness of ψ , in order to prove that $m \circ (m \otimes Id) = m \circ (Id \otimes m)$ it is enough to show that $\pi_i \circ m \circ (m \otimes Id) = \pi_i \circ m \circ (Id \otimes m)$ for all $i \in I$. We have

$$\begin{aligned} \pi_i \circ m \circ (m \otimes Id) &\stackrel{(2)}{=} m_i \circ (\pi_i \otimes \pi_i) \circ (m \otimes Id) \\ &= m_i \circ ((\pi_i \circ m) \otimes \pi_i) \\ &\stackrel{(2)}{=} m_i \circ [(m_i \circ (\pi_i \otimes \pi_i)) \otimes \pi_i] \\ &= m_i \circ (m_i \otimes Id) \circ (\pi_i \otimes \pi_i \otimes \pi_i) \\ &= m_i \circ (Id \otimes m_i) \circ (\pi_i \otimes \pi_i \otimes \pi_i) \\ &= m_i \circ [\pi_i \otimes (m_i \circ (\pi_i \otimes \pi_i))] \\ &\stackrel{(2)}{=} m_i \circ (\pi_i \otimes (\pi_i \circ m)) \\ &= m_i \circ (\pi_i \otimes \pi_i) \circ (Id \otimes m) \\ &\stackrel{(2)}{=} \pi_i \circ m \circ (Id \otimes m). \end{aligned}$$

Hence $m \circ (m \otimes Id) = m \circ (Id \otimes m)$; i.e. m is associative.

Now consider the coalgebra map $\pi_i \circ m \circ (\eta \otimes Id)$. From the universal property of the product, we obtain that there exists a unique coalgebra map $\varphi : k \otimes \prod_{i \in I} B_i \rightarrow \prod_{i \in I} B_i$ such that the diagram

$$\begin{array}{ccc} k \otimes \prod_{i \in I} B_i & & \\ \downarrow \varphi & \searrow \pi_i \circ m \circ (\eta \otimes Id) & \\ \prod_{i \in I} B_i & \xrightarrow{\pi_i} & B_i \end{array}$$

is commutative for all $i \in I$. By the argument above, in order to prove that $m \circ (\eta \otimes Id) = s$ it will be enough to show that $\pi_i \circ m \circ (\eta \otimes Id) = \pi_i \circ s$, where we denote by s the scalar multiplication. We have

$$\begin{aligned} \pi_i \circ m \circ (\eta \otimes Id) &\stackrel{(2)}{=} m_i \circ (\pi_i \otimes \pi_i) \circ (\eta \otimes Id) \\ &= m_i \circ ((\pi_i \circ \eta) \otimes \pi_i) \\ &\stackrel{(1)}{=} m_i \circ (\eta_i \otimes \pi_i) \\ &= m_i \circ (\eta_i \otimes Id) \circ (Id \otimes \pi_i) \\ &= s \circ (Id \otimes \pi_i). \end{aligned}$$

Furthermore, let $k_1 \otimes b \in k \otimes \prod_{i \in I} B_i$. Having in mind that π_i is a k -linear map we obtain

$$\begin{aligned} s \circ (Id \otimes \pi_i)(k_1 \otimes b) &= k_1 \pi_i(b) \\ &= \pi_i(k_1 b) \\ &= \pi_i \circ s(k_1 \otimes b). \end{aligned}$$

Thus we have proved that $\pi_i \circ m \circ (\eta \otimes Id) = \pi_i \circ s$. In the same way it follows that $m \circ (Id \otimes \eta) = s$. Hence $(\prod_{i \in I} B_i, m, \eta)$ is an algebra, and since m and η are coalgebra maps, we obtain that $(\prod_{i \in I} B_i, m, \eta, \Delta, \varepsilon)$ is actually a bialgebra.

To end the proof we still need to show that $(\prod_{i \in I} B_i, m, \eta, \Delta, \varepsilon)$ is the product of the family $(B_i, m_i, \eta_i, \Delta_i, \varepsilon_i)_{i \in I}$ in the category $k\text{-BiAlg}$. Let $(B, m_B, \eta_B, \Delta_B, \varepsilon_B)$ be a bialgebra and $(g_i)_{i \in I}$ be a family of bialgebra maps, $g_i : B \rightarrow B_i$ for all $i \in I$. From the universal property of the product, we obtain that there exists a unique coalgebra map $\theta : B \rightarrow \prod_{i \in I} B_i$ such that the following diagram commutes:

$$(3) \quad \begin{array}{ccc} B & & \\ \downarrow \theta & \searrow g_i & \\ \prod_{i \in I} B_i & \xrightarrow{\pi_i} & B_i \end{array}$$

We only need to prove that θ is also an algebra map. By the argument used above, it is enough to show that

$$(4) \quad \pi_i \circ \theta \circ m_B = \pi_i \circ m \circ (\theta \otimes \theta) \quad \text{and} \quad \pi_i \circ \theta \circ \eta_B = \pi_i \circ \eta.$$

Having in mind that g_i is an algebra map, we have

$$\begin{aligned}
 \pi_i \circ m \circ (\theta \otimes \theta) &\stackrel{(2)}{=} m_i \circ (\pi_i \otimes \pi_i) \circ (\theta \otimes \theta) \\
 &= m_i \circ ((\pi_i \circ \theta) \otimes (\pi_i \circ \theta)) \\
 &\stackrel{(3)}{=} m_i \circ (g_i \otimes g_i) \\
 &= g_i \circ m_B \\
 &\stackrel{(3)}{=} \pi_i \circ \theta \circ m_B.
 \end{aligned}$$

Moreover, $\pi_i \circ \theta \circ \eta_B \stackrel{(3)}{=} g_i \circ \eta_B = \eta_i \stackrel{(1)}{=} \pi_i \circ \eta$; hence (4) holds.

In what follows we construct equalizers. Let $(A, m_A, \eta_A, \Delta_A, \varepsilon_A)$ and $(B, m_B, \eta_B, \Delta_B, \varepsilon_B)$ be two bialgebras and $f, g : B \rightarrow A$ be two bialgebra maps. We denote this by $S := \{b \in B \mid f(b) = g(b)\}$. Let D be the sum of all subcoalgebras of B contained in S . We already noticed before that the family of subcoalgebras of B with this property is nonempty and that D is a subcoalgebra of B . The pair (D, i) is the equalizer of the morphisms (f, g) in $k\text{-BiAlg}$, where $i : D \rightarrow B$ is the canonical inclusion. We only need to prove that D is actually a subbialgebra of B . Consider $q = \sum_{k=1}^n d_{i_k} \otimes d_{j_k} \in D \otimes D$. We then have

$$\begin{aligned}
 m_A \circ (f \otimes f)(q) &= m_A \left(\sum_{k=1}^n f(d_{i_k}) \otimes f(d_{j_k}) \right) \\
 &= m_A \left(\sum_{k=1}^n g(d_{i_k}) \otimes g(d_{j_k}) \right) \\
 &= m_A \circ (g \otimes g)(q).
 \end{aligned}$$

Now having in mind that f and g are algebra maps we obtain $f(m_B(D \otimes D)) = g(m_B(D \otimes D))$. Hence $m_B(D \otimes D) \subseteq S$, and since $m_B(D \otimes D)$ is a subcoalgebra, it follows that $m_B(D \otimes D) \subseteq D$. Thus D is a subbialgebra of B , and it can be shown as in Theorem 1.1 that the pair (D, i) is the equalizer of the morphisms (f, g) in $k\text{-BiAlg}$. $\square$

As remarked before, we can obtain a description for the equalizers in $k\text{-BiAlg}$ similar to the one in Remark 1.2. Thus, we have the following description for limits in $k\text{-BiAlg}$:

Remark 1.6. Let J be a small category, $F : J \rightarrow k\text{-BiAlg}$ be a functor, $(\prod_{j \in J} F(j), (p_j)_{j \in J})$, $(\prod_{u \in \text{Hom}(J)} F(\text{cod}(u)), (p_u)_{u \in \text{Hom}(J)})$ be the product in $k\text{-BiAlg}$ of the families $(F(j))_{j \in J}$, respectively $(F(\text{cod}(u)))_{u \in \text{Hom}(J)}$, and $f, g : \prod_{j \in J} F(j) \rightarrow \prod_{u \in \text{Hom}(J)} F(\text{cod}(u))$ be the unique bialgebra maps such that $p_u \circ f = p_{\text{cod}(u)}$ and $p_u \circ g = F(u) \circ p_{\text{dom}(u)}$ for all $u \in \text{Hom}(J)$. We define

$$D = \{x \in \prod_{j \in J} F(j) \mid x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes g(x_{(2)}) \otimes x_{(3)}\}.$$

Then the pair $(D, (\varphi_j = p_j \circ e)_{j \in J})$ is the limit of the functor F , where $e : D \rightarrow \prod_{j \in J} F(j)$ is the canonical inclusion.

Theorem 1.7. *The category $k\text{-HopfAlg}$ of Hopf algebras has arbitrary products and equalizers. In particular, the category $k\text{-HopfAlg}$ of Hopf algebras is complete.*

Proof. Let $(H_i, m_i, \eta_i, \Delta_i, \varepsilon_i, S_i)_{i \in I}$ be a family of Hopf algebras and let $((B := \prod_{i \in I} H_i, \Delta, \varepsilon, m, \eta), (\pi_i)_{i \in I})$ be the product of this family in the category $k\text{-BiAlg}$

of bialgebras. The universal property of the product yields a unique bialgebra map $S : B^{op, cop} \rightarrow B$ such that the following diagram commutes for all $i \in I$:

$$(5) \quad \begin{array}{ccc} B & \xrightarrow{\pi_i} & B_i \\ S \uparrow & & \uparrow S_i \\ B^{op, cop} & \xrightarrow{\pi_i} & B_i^{op, cop} \end{array}$$

Let H be the sum of all subcoalgebras C of the bialgebra B such that $S(c_{(1)})c_{(2)} = c_{(1)}S(c_{(2)}) = \eta \circ \varepsilon(c)$ for all $c \in C$. The family of subcoalgebras C satisfying the above property is nonempty by the same argument used in the proof of Theorem 1.1. Moreover, it is easy to see that

$$(6) \quad S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \eta \circ \varepsilon(h)$$

for all $h \in H$. We will prove that H is a bialgebra, and it will follow by (6) that H is actually a Hopf algebra with the antipode $S|_H$. First note that $\eta(k) = k1_B \subseteq H$. Let $h, g \in H$. We then have

$$\begin{aligned} S((hg)_{(1)})(hg)_{(2)} &= S(h_{(1)}g_{(1)})h_{(2)}g_{(2)} \\ &= S(g_{(1)})S(h_{(1)})h_{(2)}g_{(2)} \\ &= S(g_{(1)})(\eta \circ \varepsilon)(h)g_{(2)} \\ &= (\eta \circ \varepsilon(h))(\eta \circ \varepsilon(g)) = \eta \circ \varepsilon(hg). \end{aligned}$$

In the same way it can be proved that $(hg)_{(1)}S((hg)_{(2)}) = \eta \circ \varepsilon(hg)$. Thus $hg \in H$, and H is indeed a bialgebra. In order to conclude that $S|_H$ is an antipode for H we need to prove that $S(H) \subseteq H$. Let $h \in H$; we obtain

$$\begin{aligned} S(S(h)_{(1)})S(h)_{(2)} &= S(S(h_{(2)}))S(h_{(1)}) \\ &= S(h_{(1)}S(h_{(2)})) \\ &= S(\eta \circ \varepsilon(h)) \\ &= \eta \circ \varepsilon(h) = \eta \circ \varepsilon(S(h)). \end{aligned}$$

A similar computation shows that we also have $S(h)_{(1)}S(S(h)_{(2)}) = \eta \circ \varepsilon(S(h))$ for all $h \in H$. Hence H is a Hopf algebra with $S|_H$ as antipode.

To end the proof we still need to show that $((H, m, \eta, \Delta, \varepsilon, S|_H), (q_i)_{i \in I})$ is the product of the family $(H_i, m_i, \eta_i, \Delta_i, \varepsilon_i, S_i)_{i \in I}$ in the category $k\text{-HopfAlg}$, where $q_i := \pi_i \circ j$ for all $i \in I$ and $j : H \rightarrow B$ is the canonical inclusion.

Let K be a Hopf algebra with antipode S_K and $f_i : K \rightarrow H_i$ be a family of Hopf algebra maps for all $i \in I$. Since B is the product in $k\text{-BiAlg}$ of the above family of Hopf algebras, there exist a unique morphism of bialgebras $\bar{f} : K \rightarrow B$ such that the following diagram commutes:

$$(7) \quad \begin{array}{ccc} K & & \\ \bar{f} \downarrow & \searrow f_i & \\ B & \xrightarrow{\pi_i} & H_i \end{array}$$

Using the fact that f_i is a Hopf algebra map we have

$$\begin{aligned}\pi_i \circ S \circ \bar{f} &\stackrel{(5)}{=} S_i \circ \pi_i \circ \bar{f} \\ &\stackrel{(7)}{=} S_i \circ f_i \\ &= f_i \circ S_K \\ &\stackrel{(7)}{=} \pi_i \circ \bar{f} \circ S_K\end{aligned}$$

for all $i \in I$. By the same argument used in the proof of Theorem 1.5 it follows that

$$(8) \quad S \circ \bar{f} = \bar{f} \circ S_K.$$

Thus, for all $k \in K$ we have

$$\begin{aligned}S(\bar{f}(k)_{(1)})\bar{f}(k)_{(2)} &= S(\bar{f}(k_{(1)}))\bar{f}(k_{(2)}) \\ &\stackrel{(8)}{=} \bar{f}(S_K(k_{(1)}))\bar{f}(k_{(2)}) \\ &= \bar{f}(S_K(k_{(1)})k_{(2)}) \\ &= \bar{f}(k).\end{aligned}$$

Hence $\bar{f}(K) \subseteq H$. Thus, we obtained a unique Hopf algebra map $\bar{f} : K \rightarrow H$ such that $q_i \circ \bar{f} = f_i$ for all $i \in I$.

Now, since the forgetful functor $U : k\text{-HopfAlg} \rightarrow k\text{-BiAlg}$ has a left adjoint (see [10]), it follows that, in particular, U preserves products. That is, $H = B$ and the map S obtained in (5) is actually an antipod for B . Thus, $((B, m, \eta, \Delta, \varepsilon, S), (\pi_i)_{i \in I})$ is the product of the family $(H_i, m_i, \eta_i, \Delta_i, \varepsilon_i, S_i)_{i \in I}$ in the category $k\text{-HopfAlg}$.

Now let $f, g : H \rightarrow K$ be two Hopf algebra morphisms and let $S := \{h \in H \mid f(h) = g(h)\}$, which is just a k -subspace of H . Let D be the sum of all subcoalgebras of H contained in S . Again, the family of subcoalgebras of H included in S is not empty by the same argument used in Theorem 1.1. A simple computation shows that D is in fact a Hopf subalgebra of H . Moreover, (D, i) is the equalizer in the category $k\text{-HopfAlg}$ of the pair (f, g) , where $i : D \rightarrow H$ is the canonical inclusion. $\square$

Remark 1.8. Let J be a small category, $F : J \rightarrow k\text{-HopfAlg}$ be a functor, $(\Pi_{j \in J} F(j), (p_j)_{j \in J})$, $(\Pi_{u \in \text{Hom}(J)} F(\text{cod}(u)), (p_u)_{u \in \text{Hom}(J)})$ be the product in $k\text{-HopfAlg}$ of the families $(F(j))_{j \in J}$, respectively $(F(\text{cod}(u)))_{u \in \text{Hom}(J)}$, and $f, g : \Pi_{j \in J} F(j) \rightarrow \Pi_{u \in \text{Hom}(J)} F(\text{cod}(u))$ be the unique Hopf algebra maps such that $p_u \circ f = p_{\text{cod}(u)}$ and $p_u \circ g = F(u) \circ p_{\text{dom}(u)}$ for all $u \in \text{Hom}(J)$. We define

$$D = \{x \in \Pi_{j \in J} F(j) \mid x_{(1)} \otimes f(x_{(2)}) \otimes x_{(3)} = x_{(1)} \otimes g(x_{(2)}) \otimes x_{(3)}\}.$$

Then the pair $(D, (\varphi_j = p_j \circ e)_{j \in J})$ is the limit of the functor F , where $e : D \rightarrow \Pi_{j \in J} F(j)$ is the canonical inclusion.

The key role in the construction of the product in the category of coalgebras was played by the fact that the forgetful functor from the category of coalgebras to the category of vector spaces has a right adjoint. It is therefore natural to ask if the conclusion remains true for the category of R -corings ([4]). Let R be a ring, $R\text{-Corings}$ be the category of R -corings and ${}_R\mathcal{M}_R$ be the category of R -bimodules.

Problem. Does there exist a right adjoint for the forgetful functor $F : R - \text{Corings} \rightarrow {}_R\mathcal{M}_R$?

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