BILATERAL q -WATSON AND q -WHIPPLE SUMS

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ABSTRACT. Watson and Whipple (1925) discovered two summation formulae for a ${}_3F_2(1)$ -series. Their terminating q -analogues were found by Andrews (1976) and Jain (1981). As a common extension of these results, we prove a general bilateral series identity, which may also be considered as a full q -analogue of the ${}_3H_3$ -series identity due to M. Jackson (1949). This will be accomplished by means of the modified Abel lemma on summation by parts.

For two indeterminates x and q , the shifted-factorial of x with base q is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x) \quad \text{for } n \in \mathbb{N}.$$

When $|q| < 1$, we have two well-defined infinite products

$$(x; q)_\infty = \prod_{k=0}^{\infty} (1 - q^k x) \quad \text{and} \quad (x; q)_n = (x; q)_\infty / (q^n x; q)_\infty.$$

The product and fraction of shifted factorials are abbreviated respectively to

$$\begin{aligned} [\alpha, \beta, \dots, \gamma; q]_n &= (\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n, \\ \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \middle| q \right]_n &= \frac{(\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n}{(A; q)_n (B; q)_n \cdots (C; q)_n}. \end{aligned}$$

Following Gasper-Rahman [5], the unilateral and bilateral basic hypergeometric series are defined by

$$\begin{aligned} {}_{1+r}\phi_s \left[\begin{matrix} a_0, a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right] &= \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \left[\begin{matrix} a_0, a_1, \dots, a_r \\ q, b_1, \dots, b_s \end{matrix} \middle| q \right]_n z^n, \\ {}_r\psi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; z \right] &= \sum_{n=-\infty}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q \right]_n z^n, \end{aligned}$$

where the base q will be restricted to $|q| < 1$ for nonterminating q -series.

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In 1925, Watson [8] and Whipple [9] respectively found the following two ${}_3F_2(1)$ sums (cf. [2, SS3.3-3.4])

$$(1a) \quad {}_3F_2 \left[\begin{matrix} a, b, c \\ \frac{1+a+b}{2}, 2c \end{matrix} \middle| 1 \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1+a+b}{2})\Gamma(\frac{1-a-b}{2}+c)}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})\Gamma(\frac{1-a}{2}+c)\Gamma(\frac{1-b}{2}+c)},$$

$$(1b) \quad {}_3F_2 \left[\begin{matrix} a, 1-a, c \\ b, 1+2c-b \end{matrix} \middle| 1 \right] = \frac{2^{1-2c}\pi\Gamma(b)\Gamma(1+2c-b)}{\Gamma(\frac{a+b}{2})\Gamma(\frac{1+a-b+2c}{2})\Gamma(\frac{1-a+b}{2})\Gamma(\frac{2-a-b+2c}{2})},$$

where the convergence conditions for [8] and [9] are respectively $\Re(a+b-2c) < 1$ and $\Re(c) > 0$. After a half century, different q -analogues of the last two formulae were established respectively by Andrews [1] and Jain [7].

This short paper aims to prove a common extension of bilateral series for the q -Watson and q -Whipple sums, which provides also a full q -analogue of the ${}_3H_3$ -series identity due to M. Jackson [6, 1949]. A few summation and transformation formulae will be presented as consequences.

1. MAIN THEOREM AND CONSEQUENCES

Throughout this section, a, b, c, d are assumed to be four indeterminates such that the nonterminating series containing them as parameters are well defined.

Theorem 1. *Under the conditions $|a/b| < 1$ and $|c/d| < 1$, the following bilateral q -series transformation holds:*

$$\begin{aligned} {}_4\psi_4 \left[\begin{matrix} a, c, \sqrt{qbd}, -\sqrt{qbd} \\ qb, qd, \sqrt{qac}, -\sqrt{qac} \end{matrix} \middle| q; q \right] &= \left[\begin{matrix} a, 1/d \\ qb, q/c \end{matrix} \middle| q \right]_{\infty} \left[\begin{matrix} q^2, qbc, q/bc, q^2b/c, ac/bd \\ qac, a/b, a/d, c/d, q/bd \end{matrix} \middle| q^2 \right]_{\infty} \\ &\quad + \frac{1/d}{1-a/d} \left[\begin{matrix} a, c \\ qb, qd \end{matrix} \middle| q \right]_{\infty} \frac{(qbd; q^2)_{\infty}}{(qac; q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^2b/c; q^2)_k}{(q^2a/d; q^2)_k} \left(\frac{c}{d}\right)^k \\ &\quad + \frac{a}{1-a/d} \left[\begin{matrix} 1/b, 1/d \\ q/a, q/c \end{matrix} \middle| q \right]_{\infty} \frac{(q/ac; q^2)_{\infty}}{(q/bd; q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^2b/c; q^2)_k}{(q^2a/d; q^2)_k} \left(\frac{a}{b}\right)^k. \end{aligned}$$

As we shall show in Section 4, this theorem implies M. Jackson's identity of bilateral ${}_3H_3$ -series [6, Eq. 2.3]. Furthermore, this theorem contains the following four q -Watson and q -Whipple formulae as special cases, which were found by Andrews (1976) and Jain (1981) previously.

Firstly, letting $b = 1$, $c = q^{-n}$ in the last theorem and then renaming the parameters by $a \rightarrow q^{1+n}a$, $d \rightarrow c/q$, we recover Andrews' terminating q -Watson formula.

Corollary 2 (Andrews' terminating ${}_4\phi_3$ -series identity [1]).

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{1+n}a, \sqrt{c}, -\sqrt{c} \\ q\sqrt{a}, -q\sqrt{a}, c \end{matrix} \middle| q; q \right] = \begin{cases} 0, & n\text{-odd}; \\ c^m \left[\begin{matrix} q^2a/c, q \\ q^2a, qc \end{matrix} \middle| q^2 \right]_m, & n = 2m. \end{cases}$$

Instead, when $b \rightarrow 1$ and $d \rightarrow q^{-1-2n}$, we get another q -Watson formula.

Corollary 3 (Jain's terminating ${}_4\phi_3$ -series identity [7, Eq. 3.17]).

$${}_4\phi_3 \left[\begin{matrix} a, c, q^{-n}, -q^{-n} \\ q^{-2n}, \sqrt{qac}, -\sqrt{qac} \end{matrix} \middle| q; q \right] = \left[\begin{matrix} qa, qc \\ q, qac \end{matrix} \middle| q^2 \right]_n.$$

Then letting $a \rightarrow q^{1+n}$ and $c \rightarrow q^{-n}$ yields Andrews' q -Whipple formula.

Corollary 4 (Andrews' terminating ${}_4\phi_3$ -series identity [1]).

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, q^{1+n}, \sqrt{qbd}, -\sqrt{qbd} \\ -q, qb, qd \end{matrix} \middle| q; q \right] = \frac{[q^{1-n}b, q^{1-n}d; q^2]_n}{[qb, qd; q]_n} q^{\binom{n+1}{2}}.$$

Finally, when $c \rightarrow q/a$ and $d \rightarrow q^{-1-2n}/b$, we obtain another q -Whipple formula.

Corollary 5 (Jain's terminating ${}_4\phi_3$ -series identity [7, Eq. 3.19]).

$${}_4\phi_3 \left[\begin{matrix} a, q/a, q^{-n}, -q^{-n} \\ -q, qb, q^{-2n}/b \end{matrix} \middle| q; q \right] = \frac{[qab, q^2b/a; q^2]_n}{(qb; q)_{2n}}.$$

In addition, for $d = 1$ in Theorem 1, we have the following surprising formula which transforms the balanced ${}_4\phi_3$ -series into a simpler one.

Corollary 6 (Nonterminating series transformation: $|c| < 1$).

$${}_4\phi_3 \left[\begin{matrix} a, c, \sqrt{qb}, -\sqrt{qb} \\ qb, \sqrt{qac}, -\sqrt{qac} \end{matrix} \middle| q; q \right] = \left[\begin{matrix} qa, c \\ qb, q \end{matrix} \middle| q \right]_{\infty} \frac{(qb; q^2)_{\infty}}{(qac; q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{(q^2b/c; q^2)_k}{(q^2a; q^2)_k} c^k.$$

2. PROOF VIA ABEL'S LEMMA ON SUMMATION BY PARTS

In order to prove Theorem 1, we shall utilize the modified Abel lemma on summation by parts [3, 4]. Given an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and $\triangle$, respectively, by

$$\nabla \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \triangle \tau_k = \tau_k - \tau_{k+1},$$

where $\triangle$ differs from the usual operator Δ only in the minus sign.

Lemma 7 (Abel's lemma on summation by parts). *Let $\{A_k\}$ and $\{B_k\}$ be two sequences such that the following two limits exist:*

$$[AB]_+ := \lim_{n \rightarrow +\infty} A_n B_{n+1} \quad \text{and} \quad [AB]_- := \lim_{n \rightarrow -\infty} A_n B_{n+1}.$$

Then the following transformation formula holds:

$$\sum_{k=-\infty}^{+\infty} B_k \nabla A_k = [AB]_+ - [AB]_- + \sum_{k=-\infty}^{+\infty} A_k \triangle B_k$$

provided that one of the nonterminating series just displayed is convergent.

Proof. According to the definition of the backward difference, we have

$$\sum_{k=-m}^n B_k \nabla A_k = \sum_{k=-m}^n B_k \{A_k - A_{k-1}\} = \sum_{k=-m}^n A_k B_k - \sum_{k=-m}^n A_{k-1} B_k.$$

Replacing k by $k+1$ for the last sum, we can reformulate the equation as follows:

$$\begin{aligned} \sum_{k=-m}^n B_k \nabla A_k &= A_n B_{n+1} - A_{-m-1} B_{-m} + \sum_{k=-m}^n A_k \{B_k - B_{k+1}\} \\ &= A_n B_{n+1} - A_{-m-1} B_{-m} + \sum_{k=-m}^n A_k \triangle B_k. \end{aligned}$$

Letting $m, n \rightarrow \infty$ leads to the equality stated in the modified Abel lemma. $\square$

Now we are ready to prove Theorem 1. First, define the Ω -function by

$$(2) \quad \Omega(a, b, c, d) = \sum_{k=-\infty}^{\infty} \left[\begin{matrix} a, & c \\ qb, & qd \end{matrix} \middle| q \right]_k \frac{(qbd; q^2)_k}{(qac; q^2)_k} q^k.$$

For the two sequences given by

$$A_k = \frac{(qa; q)_k (q^3 bd; q^2)_k}{(qd; q)_k (q^3 ab; q^2)_k} \quad \text{and} \quad B_k = \frac{(c; q)_k (q^3 ab; q^2)_k}{(q^2 b; q)_k (qac; q^2)_k}$$

it is almost trivial to compute the relations

$$\begin{aligned} [AB]_+ &:= \lim_{m \rightarrow \infty} A_m B_{m+1} = \left[\begin{matrix} qa, & c \\ q^2 b, & qd \end{matrix} \middle| q \right]_{\infty} \frac{(q^3 bd; q^2)_{\infty}}{(qac; q^2)_{\infty}}, \\ [AB]_- &:= \lim_{m \rightarrow -\infty} A_m B_{m+1} = \left[\begin{matrix} 1/qb, & 1/d \\ 1/a, & q/c \end{matrix} \middle| q \right]_{\infty} \frac{(q/ac; q^2)_{\infty}}{(1/qbd; q^2)_{\infty}} \end{aligned}$$

and the following differences:

$$\begin{aligned} \nabla A_k &= \frac{(1-d/a)(1-q^{1+k}b)}{(1-qbd)(1-1/a)} \frac{(a; q)_k (qbd; q^2)_k}{(qd; q)_k (q^3 ab; q^2)_k} q^k, \\ \triangle B_k &= \frac{(1-c/q^2b)(1-q^{1+k}a)}{(1-qac)(1-1/q^2b)} \frac{(c; q)_k (q^3 ab; q^2)_k}{(q^3 b; q)_k (q^3 ac; q^2)_k} q^k. \end{aligned}$$

By means of the modified Abel lemma on summation by parts, we can manipulate the Ω -series as follows:

$$\Omega(a, b, c, d) \times \frac{(1-d/a)(1-qb)}{(1-qbd)(1-1/a)} = \sum_k B_k \nabla A_k = [AB]_+ - [AB]_- + \sum_k A_k \triangle B_k.$$

Writing explicitly the last sum

$$\sum_k A_k \triangle B_k = \frac{(1-qa)(1-c/q^2b)}{(1-qac)(1-1/q^2b)} \sum_k \left[\begin{matrix} q^2 a, & c \\ q^3 b, & qd \end{matrix} \middle| q \right]_k \frac{(q^3 bd; q^2)_k}{(q^3 ac; q^2)_k} q^k,$$

we find the following recurrence relation:

$$\begin{aligned} \Omega(a, b, c, d) &= \frac{c(1-a)(1-qa)(1-qbd)(1-q^2b/c)}{d(1-qb)(1-q^2b)(1-qac)(1-a/d)} \Omega(q^2a, q^2b, c, d) \\ &+ \frac{1/d}{1-a/d} \left[\begin{matrix} a, & c \\ qb, & qd \end{matrix} \middle| q \right]_{\infty} \frac{(qbd; q^2)_{\infty}}{(qac; q^2)_{\infty}} + \frac{a}{1-a/d} \left[\begin{matrix} 1/b, & 1/d \\ q/a, & q/c \end{matrix} \middle| q \right]_{\infty} \frac{(q/ac; q^2)_{\infty}}{(q/bd; q^2)_{\infty}}. \end{aligned}$$

Iterating this relation m times, we get

$$\begin{aligned} \Omega(a, b, c, d) &= \left(\frac{c}{d} \right)^m \left[\begin{matrix} a, & qa, & qbd, & q^2b/c \\ qb, & q^2b, & qac, & a/d \end{matrix} \middle| q^2 \right]_m \Omega(q^{2m}a, q^{2m}b, c, d) \\ &+ \sum_{k=0}^{m-1} \frac{1/d}{1-q^{2k}a/d} \left[\begin{matrix} q^{2k}a, & c \\ q^{1+2k}b, & qd \end{matrix} \middle| q \right]_{\infty} \frac{(q^{1+2k}bd; q^2)_{\infty}}{(q^{1+2k}ac; q^2)_{\infty}} \left(\frac{c}{d} \right)^k \left[\begin{matrix} a, & qa, & qbd, & q^2b/c \\ qb, & q^2b, & qac, & a/d \end{matrix} \middle| q^2 \right]_k \\ &+ \sum_{k=0}^{m-1} \frac{q^{2k}a}{1-q^{2k}a/d} \left[\begin{matrix} q^{-2k}/b, & 1/d \\ q^{1-2k}/a, & q/c \end{matrix} \middle| q \right]_{\infty} \frac{(q^{1-2k}/ac; q^2)_{\infty}}{(q^{1-2k}/bd; q^2)_{\infty}} \left(\frac{c}{d} \right)^k \left[\begin{matrix} a, & qa, & qbd, & q^2b/c \\ qb, & q^2b, & qac, & a/d \end{matrix} \middle| q^2 \right]_k. \end{aligned}$$

Simplifying the last equation, we establish the following transformation formula.

Proposition 8.

$$\begin{aligned}\Omega(a, b, c, d) &= \left(\frac{c}{d}\right)^m \left[\frac{a, qa, qbd, q^2b/c}{qb, q^2b, qac, a/d} \middle| q^2 \right]_m \Omega(q^{2m}a, q^{2m}b, c, d) \\ &+ \frac{1/d}{1-a/d} \left[\frac{a, c}{qb, qd} \middle| q \right]_\infty \frac{(qbd; q^2)_\infty}{(qac; q^2)_\infty} \sum_{k=0}^{m-1} \frac{(q^2b/c; q^2)_k}{(q^2a/d; q^2)_k} \left(\frac{c}{d}\right)^k \\ &+ \frac{a}{1-a/d} \left[\frac{1/b, 1/d}{q/a, q/c} \middle| q \right]_\infty \frac{(q/ac; q^2)_\infty}{(q/bd; q^2)_\infty} \sum_{k=0}^{m-1} \frac{(q^2b/c; q^2)_k}{(q^2a/d; q^2)_k} \left(\frac{a}{b}\right)^k.\end{aligned}$$

Suppose that $|a/b| < 1$ and $|c/d| < 1$. When $m \rightarrow \infty$, in order to examine the limit of $\Omega(q^{2m}a, q^{2m}b, c, d)$, we rewrite this series by splitting the bilateral sum into the following three parts:

$$\begin{aligned}(3a) \quad \left(\frac{c}{d}\right)^m \Omega(q^{2m}a, q^{2m}b, c, d) &= \left(\frac{c}{d}\right)^m \sum_{k>0} \left[\frac{q^{2m}a, c}{q^{1+2m}b, qd} \middle| q \right]_k \frac{(q^{1+2m}bd; q^2)_k}{(q^{1+2m}ac; q^2)_k} q^k \\ (3b) \quad &+ \left(\frac{c}{d}\right)^m \sum_{k<-2m} \left[\frac{q^{2m}a, c}{q^{1+2m}b, qd} \middle| q \right]_k \frac{(q^{1+2m}bd; q^2)_k}{(q^{1+2m}ac; q^2)_k} q^k \\ (3c) \quad &+ \left(\frac{c}{d}\right)^m \sum_{k=-2m}^0 \left[\frac{q^{2m}a, c}{q^{1+2m}b, qd} \middle| q \right]_k \frac{(q^{1+2m}bd; q^2)_k}{(q^{1+2m}ac; q^2)_k} q^k.\end{aligned}$$

It is obvious that the partial series with respect to k displayed in the first line is convergent. Therefore the right member in (3a) tends to zero as $m \rightarrow \infty$. For the partial series displayed in the middle line, performing the replacement $k \rightarrow -2m-k$ and then applying the relations

$$\begin{aligned}\left[\frac{q^{2m}a, c}{q^{1+2m}b, qd} \middle| q \right]_{-2m-k} &= \left[\frac{qb, 1/d}{a, q/c} \middle| q \right]_{2m} \left(\frac{qd}{c}\right)^{2m} \left[\frac{1/b, q^{2m}/d}{q/a, q^{1+2m}/c} \middle| q \right]_k \left(\frac{q^2bd}{ac}\right)^k, \\ \frac{(q^{1+2m}bd; q^2)_{-2m-k}}{(q^{1+2m}ac; q^2)_{-2m-k}} &= \left[\frac{qac, q/ac}{qbd, q/bd} \middle| q^2 \right]_m \left(\frac{ac}{bd}\right)^m \frac{(q^{1+2m}/ac; q^2)_k}{(q^{1+2m}/bd; q^2)_k} \left(\frac{ac}{bd}\right)^k,\end{aligned}$$

we can reformulate it as

$$\left[\frac{qb, 1/d}{a, q/c} \middle| q \right]_{2m} \left[\frac{qac, q/ac}{qbd, q/bd} \middle| q^2 \right]_m \left(\frac{a}{b}\right)^m \sum_{k>0} \left[\frac{1/b, q^{2m}/d}{q/a, q^{1+2m}/c} \middle| q \right]_k \frac{(q^{1+2m}/ac; q^2)_k}{(q^{1+2m}/bd; q^2)_k} q^k,$$

which tends to zero too when $m \rightarrow \infty$.

Finally for the truncated series displayed in (3c), changing the summation index by $k \rightarrow k-m$ and then utilizing the relations

$$\begin{aligned}\left[\frac{q^{2m}a, c}{q^{1+2m}b, qd} \middle| q \right]_{k-m} &= \left[\frac{q^{1+m}b, 1/d}{q^ma, q/c} \middle| q \right]_m \left(\frac{qd}{c}\right)^m \\ &\times \left[\frac{q^ma, q^{1+m-k}/c}{q^{1+m}b, q^{m-k}/d} \middle| q \right]_k \left(\frac{c}{qd}\right)^k, \\ \frac{(q^{1+2m}bd; q^2)_{k-m}}{(q^{1+2m}ac; q^2)_{k-m}} &= \frac{(qac; q^2)_m (qbd; q^2)_k}{(qbd; q^2)_m (qac; q^2)_k},\end{aligned}$$

we can express it as

$$\left[\frac{q^{1+m}b, 1/d}{q^ma, q/c} \middle| q \right]_m \frac{(qac; q^2)_m}{(qbd; q^2)_m} \sum_{k=-m}^m \left[\frac{q^ma, q^{1+m-k}/c}{q^{1+m}b, q^{m-k}/d} \middle| q \right]_k \frac{(qbd; q^2)_k}{(qac; q^2)_k} \left(\frac{c}{d}\right)^k.$$

When $m \rightarrow \infty$, the limit of the last bilateral sum can be evaluated by Ramanujan's ${}_1\psi_1$ -sum [5, II-29]

$${}_1\psi_1 \left[\begin{matrix} qbd \\ qac \end{matrix} \middle| q^2; \frac{c}{d} \right] = \left[\begin{matrix} q^2, ac/bd, q/bc, qbc \\ a/b, c/d, q/bd, qac \end{matrix} \middle| q^2 \right]_{\infty}, \quad \text{where } |a/b| < 1, |c/d| < 1.$$

Summing up, we have established the transformation stated in Theorem 1.

3. ANOTHER DIFFERENCE PAIR

Alternatively, define another pair of sequences by

$$A_k = \left[\begin{matrix} qa, qbd/a \\ qb, qd \end{matrix} \middle| q \right]_k \quad \text{and} \quad B_k = \left[\begin{matrix} c \\ bd/a \end{matrix} \middle| q \right]_k \left[\begin{matrix} qbd \\ qac \end{matrix} \middle| q^2 \right]_k.$$

It is not hard to verify the relations

$$\begin{aligned} [AB]_+ &:= \lim_{m \rightarrow \infty} A_m B_{m+1} = \frac{1}{1 - bd/a} \left[\begin{matrix} qa, c \\ qb, qd \end{matrix} \middle| q \right]_{\infty} \frac{(qbd; q^2)_{\infty}}{(qac; q^2)_{\infty}}, \\ [AB]_- &:= \lim_{m \rightarrow -\infty} A_m B_{m+1} = \frac{1}{1 - a/bd} \left[\begin{matrix} 1/b, 1/d \\ 1/a, q/c \end{matrix} \middle| q \right]_{\infty} \frac{(q/ac; q^2)_{\infty}}{(q/bd; q^2)_{\infty}} \end{aligned}$$

and the following differences:

$$\begin{aligned} \nabla A_k &= \frac{(1 - a/b)(1 - a/d)}{(1 - a)(1 - a/bd)} \left[\begin{matrix} a, bd/a \\ qb, qd \end{matrix} \middle| q \right]_k q^k, \\ \triangle B_k &= \frac{(1 - q^{1+k}a)(1 - ac/bd)}{(1 - qac)(1 - a/bd)} \left[\begin{matrix} c \\ qbd/a \end{matrix} \middle| q \right]_k \left[\begin{matrix} qbd \\ q^3ac \end{matrix} \middle| q^2 \right]_k q^k. \end{aligned}$$

According to the modified Abel lemma on summation by parts, we can reformulate the Ω -series as follows:

$$\Omega(a, b, c, d) \times \frac{(1 - a/b)(1 - a/d)}{(1 - a)(1 - a/bd)} = \sum_k B_k \nabla A_k = [AB]_+ - [AB]_- + \sum_k A_k \triangle B_k.$$

Observing that the last sum is equal to

$$\sum_k A_k \triangle B_k = \frac{(1 - qa)(1 - ac/bd)}{(1 - qac)(1 - a/bd)} \sum_k \left[\begin{matrix} q^2a, c \\ qb, qd \end{matrix} \middle| q \right]_k \frac{(qbd; q^2)_k}{(q^3ac; q^2)_k} q^k,$$

we derive the following recurrence relation:

$$\begin{aligned} \Omega(a, b, c, d) &= \frac{(1 - a)(1 - qa)(1 - ac/bd)}{(1 - a/b)(1 - a/d)(1 - qac)} \Omega(q^2a, b, c, d) \\ &\quad + \frac{a}{(1 - a/b)(1 - a/d)} \left[\begin{matrix} 1/b, 1/d \\ q/a, q/c \end{matrix} \middle| q \right]_{\infty} \frac{(q/ac; q^2)_{\infty}}{(q/bd; q^2)_{\infty}} \\ &\quad - \frac{a/bd}{(1 - a/b)(1 - a/d)} \left[\begin{matrix} a, c \\ qb, qd \end{matrix} \middle| q \right]_{\infty} \frac{(qbd; q^2)_{\infty}}{(qac; q^2)_{\infty}}. \end{aligned}$$

Iterating the last relation m -times leads us to the following transformation.

Proposition 9.

$$\begin{aligned}\Omega(a, b, c, d) &= \left[\begin{matrix} a, & qa, & ac/bd \\ a/b, & a/d, & qac \end{matrix} \middle| q^2 \right]_m \Omega(q^{2m}a, b, c, d) \\ &+ a \left[\begin{matrix} 1/b, & 1/d \\ q/a, & q/c \end{matrix} \middle| q \right]_\infty \frac{(q/ac; q^2)_\infty}{(q/bd; q^2)_\infty} \sum_{k=0}^{m-1} \frac{(ac/bd; q^2)_k q^{2\binom{k+1}{2}}}{[a/b, a/d; q^2]_{k+1}} \left(-\frac{a}{c}\right)^k \\ &- \frac{a}{bd} \left[\begin{matrix} a, & c \\ qb, & qd \end{matrix} \middle| q \right]_\infty \frac{(qbd; q^2)_\infty}{(qac; q^2)_\infty} \sum_{k=0}^{m-1} \frac{(ac/bd; q^2)_k}{[a/b, a/d; q^2]_{k+1}} q^{2k}.\end{aligned}$$

The asymptotic behavior of $\Omega(q^{2m}a, b, c, d)$ as $m \rightarrow \infty$ can be examined analogously. For this purpose, we rewrite the series by splitting it into four parts:

$$(4a) \quad \Omega(q^{2m}a, b, c, d) = \sum_{k \geq 0} \left[\begin{matrix} q^{2m}a, & c \\ qb, & qd \end{matrix} \middle| q \right]_k \frac{(qbd; q^2)_k}{(q^{1+2m}ac; q^2)_k} q^k$$

$$(4b) \quad + \sum_{k < -2m} \left[\begin{matrix} q^{2m}a, & c \\ qb, & qd \end{matrix} \middle| q \right]_k \frac{(qbd; q^2)_k}{(q^{1+2m}ac; q^2)_k} q^k$$

$$(4c) \quad + \sum_{k=-2m}^{-m-1} \left[\begin{matrix} q^{2m}a, & c \\ qb, & qd \end{matrix} \middle| q \right]_k \frac{(qbd; q^2)_k}{(q^{1+2m}ac; q^2)_k} q^k$$

$$(4d) \quad + \sum_{k=-m}^{-1} \left[\begin{matrix} q^{2m}a, & c \\ qb, & qd \end{matrix} \middle| q \right]_k \frac{(qbd; q^2)_k}{(q^{1+2m}ac; q^2)_k} q^k.$$

Obviously, when $m \rightarrow \infty$, the partial series displayed in (4a) tends to

$$\sum_{k \geq 0} \left[\begin{matrix} c \\ qb, qd \end{matrix} \middle| q \right]_k (qbd; q^2)_k q^k.$$

For the partial series displayed in (4b), performing the replacement $k \rightarrow -k$, we can reformulate it as

$$\frac{(qac; q^2)_m}{(a; q)_{2m}} \left(-\frac{q^{m-1}a}{c}\right)^m \sum_{k > 2m} \left[\begin{matrix} 1/b, & 1/d \\ q/c \end{matrix} \middle| q \right]_k \frac{(q/ac; q^2)_{k-m}}{(q/bd; q^2)_k (q/a; q)_{k-2m}} q^k,$$

which tends to zero when $m \rightarrow \infty$.

For the finite sum displayed in (4c), it can be restated, under the replacement $k \rightarrow -m - k$, as

$$\frac{(qac; q^2)_m}{(q^m a; q)_m} q^{\binom{m+1}{2}} c^{-m} \sum_{k=1}^m \left[\begin{matrix} 1/b, & 1/d \\ q/c \end{matrix} \middle| q \right]_{k+m} \frac{(q/ac; q^2)_k q^{mk - \binom{k}{2}} (-a)^k}{(q/bd; q^2)_{k+m} (q^{m-k} a; q)_k},$$

which also tends to zero when $m \rightarrow \infty$.

Finally making the replacement $k \rightarrow -k$, we can express (4d) as

$$\sum_{k=1}^m \left[\begin{matrix} 1/b, & 1/d \\ q/c, & q^{2m-k}a \end{matrix} \middle| q \right]_k \frac{(q^{1+2m-2k}ac; q^2)_k}{(q/bd; q^2)_k} q^{\binom{k+1}{2}} c^{-k},$$

which becomes, as $m \rightarrow \infty$, the series

$$\sum_{k=1}^{\infty} \left[\begin{matrix} 1/b, & 1/d \\ q/c \end{matrix} \middle| q \right]_k \frac{q^{\binom{k+1}{2}} c^{-k}}{(q/bd; q^2)_k} = \sum_{k=-\infty}^{-1} \left[\begin{matrix} c \\ qb, qd \end{matrix} \middle| q \right]_k (qbd; q^2)_k q^k.$$

Combining the first and fourth parts together yields a bilateral series. We have therefore derived the nonterminating transformation formula.

Theorem 10.

$$\begin{aligned} \Omega(a, b, c, d) &= \left[\begin{matrix} a, & qa, & ac/bd \\ a/b, & a/d, & qac \end{matrix} \middle| q^2 \right]_{\infty} \sum_{k=-\infty}^{\infty} \left[\begin{matrix} c \\ qb, & qd \end{matrix} \middle| q \right]_k (qbd; q^2)_k q^k \\ &+ a \left[\begin{matrix} 1/b, & 1/d \\ q/a, & q/c \end{matrix} \middle| q \right]_{\infty} \frac{(q/ac; q^2)_{\infty}}{(q/bd; q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{(ac/bd; q^2)_k q^{2\binom{k+1}{2}}}{[a/b, a/d; q^2]_{k+1}} \left(-\frac{a}{c} \right)^k \\ &- \frac{a}{bd} \left[\begin{matrix} a, & c \\ qb, & qd \end{matrix} \middle| q \right]_{\infty} \frac{(qbd; q^2)_{\infty}}{(qac; q^2)_{\infty}} \sum_{k=0}^{\infty} \frac{(ac/bd; q^2)_k}{[a/b, a/d; q^2]_{k+1}} q^{2k}. \end{aligned}$$

Compared with Theorem 1, the last one seems less useful. However, letting $d = ac/b$ in this theorem does result in an unusual identity on the bilateral balancing series

$$(5) \quad {}_2\psi_2 \left[\begin{matrix} a, & c \\ qb, & qac/b \end{matrix} \middle| q; q \right] = \frac{(1-b)(1-b/ac)}{(1-b/a)(1-b/c)} \left\{ \left[\begin{matrix} q/b, & qb/ac \\ q/a, & q/c \end{matrix} \middle| q \right]_{\infty} - \left[\begin{matrix} a, & c \\ b, & ac/b \end{matrix} \middle| q \right]_{\infty} \right\}.$$

Strangely enough, this identity also follows from the telescoping method. In fact, observing the relation

$$\left[\begin{matrix} a, & c \\ qb, & qac/b \end{matrix} \middle| q \right]_k q^k = \frac{(1-b)(1-b/ac)}{(1-b/a)(1-b/c)} \left\{ \left[\begin{matrix} a, & c \\ b, & ac/b \end{matrix} \middle| q \right]_k - \left[\begin{matrix} a, & c \\ b, & ac/b \end{matrix} \middle| q \right]_{k+1} \right\},$$

its sum with respect to k from $-\infty$ to $+\infty$ reads as

$$\sum_{k=-\infty}^{+\infty} \left[\begin{matrix} a, & c \\ qb, & qac/b \end{matrix} \middle| q \right]_k q^k = \frac{(1-b)(1-b/ac)}{(1-b/a)(1-b/c)} \left\{ \left[\begin{matrix} a, & c \\ b, & ac/b \end{matrix} \middle| q \right]_{-\infty} - \left[\begin{matrix} a, & c \\ b, & ac/b \end{matrix} \middle| q \right]_{+\infty} \right\}.$$

This confirms the identity (5) in view of the equation

$$\left[\begin{matrix} a, & c \\ b, & ac/b \end{matrix} \middle| q \right]_{-\infty} = \left[\begin{matrix} q/b, & qb/ac \\ q/a, & q/c \end{matrix} \middle| q \right]_{\infty}. \quad \square$$

4. REVIEW OF M. JACKSON'S IDENTITY OF ${}_3H_3$ -SERIES

For the two nonterminating series identities ([8]) and ([9]) discovered respectively by Watson and Whipple, there is a common generalization of bilateral series due to M. Jackson [6, Eq. 2.3]:

$$(6a) \quad \left[\begin{matrix} a, & c, & \frac{1+b+d}{2} \\ 1+b, & 1+d, & \frac{1+a+c}{2} \end{matrix} \middle| 1 \right] = 2^{a+c-b-d-1} \pi \left\{ \frac{\cos \frac{a-c}{2} \pi}{\cos \frac{a+c}{2} \pi} + \frac{\cos \frac{b-d}{2} \pi}{\cos \frac{b+d}{2} \pi} \right\}$$

$$(6b) \quad \times \Gamma \left[\begin{matrix} 1-a, 1-c, 1+b, 1+d, \frac{2-a-c+b+d}{2} \\ \frac{1-a-c}{2}, \frac{1+b+d}{2}, \frac{2-a+b}{2}, \frac{2-a+d}{2}, \frac{2-c+b}{2}, \frac{2-c+d}{2} \end{matrix} \right]$$

provided that $\Re(a+c-b-d) < 2$ for convergence. We shall show that this identity follows from Theorem 1 as $q \rightarrow 1^-$ after having replaced a, b, c, d respectively by their q -exponentials q^a, q^b, q^c, q^d .

By means of Heine's transformation (cf. [5, III-3])

$${}_2\phi_1 \left[\begin{matrix} a, & b \\ c \end{matrix} \middle| q; x \right] = \frac{(abx/c; q)_{\infty}}{(x; q)_{\infty}} {}_2\phi_1 \left[\begin{matrix} c/a, & c/b \\ c \end{matrix} \middle| q; abx/c \right]$$

the equation displayed in Theorem 1 can be reformulated as

$$\begin{aligned}
 (7a) \quad & {}_4\psi_4 \left[\begin{matrix} a, & c, & \sqrt{qbd}, & -\sqrt{qbd} \\ qb, & qd, & \sqrt{qac}, & -\sqrt{qac} \end{matrix} \middle| q; q \right] \\
 (7b) \quad & = \left[\begin{matrix} a, & 1/d \\ qb, & q/c \end{matrix} \middle| q \right]_{\infty} \left[\begin{matrix} q^2, & qbc, & q/bc, & q^2b/c, & ac/bd \\ qac, & a/b, & a/d, & c/d, & q/bd \end{matrix} \middle| q^2 \right]_{\infty} \\
 (7c) \quad & + \frac{1/d}{1-a/d} \left[\begin{matrix} a, & c \\ qb, & qd \end{matrix} \middle| q \right]_{\infty} \left[\begin{matrix} qbd, q^2b/a \\ qac, c/d \end{matrix} \middle| q^2 \right]_{\infty} {}_2\phi_1 \left[\begin{matrix} ac/bd, a/d \\ q^2a/d \end{matrix} \middle| q^2; q^2b/a \right] \\
 (7d) \quad & + \frac{a}{1-a/d} \left[\begin{matrix} 1/b, & 1/d \\ q/a, & q/c \end{matrix} \middle| q \right]_{\infty} \left[\begin{matrix} q/ac, q^2d/c \\ q/bd, a/b \end{matrix} \middle| q^2 \right]_{\infty} {}_2\phi_1 \left[\begin{matrix} ac/bd, a/d \\ q^2a/d \end{matrix} \middle| q^2; q^2d/c \right].
 \end{aligned}$$

This expression permits us to realize the limiting process as $q \rightarrow 1^-$. Recall the q -Gamma function (cf. [5, I-35])

$$\Gamma_q(x) = (1-q)^{1-x} \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} \quad \text{with} \quad \lim_{q \rightarrow 1^-} \Gamma_q(x) = \Gamma(x).$$

Replacing a, b, c, d by their q -exponentials q^a, q^b, q^c, q^d in the last transformation and then letting $q \rightarrow 1^-$, we may evaluate separately the expressions from (7a) to (7d) as follows. First for the ${}_4\psi_4$ -series in (7a), it is trivial to see that the series becomes

$${}_3H_3 \left[\begin{matrix} a, & c, & \frac{1+b+d}{2} \\ 1+b, & 1+d, & \frac{1+a+c}{2} \end{matrix} \middle| 1 \right],$$

which is convergent under the condition $\Re(a + c - b - d) < 2$. Under the same condition, the limits of both ${}_2\phi_1$ -series in (7c) and (7d) can be evaluated, through the Gauss summation theorem (cf. [2, §1.3]), as follows:

$${}_2F_1 \left[\begin{matrix} \frac{a+c-b-d}{2}, \frac{a-d}{2} \\ 1 + \frac{a-d}{2} \end{matrix} \middle| 1 \right] = \Gamma \left[\begin{matrix} \frac{2+a-d}{2}, \frac{2-a-c+b+d}{2} \\ \frac{2+b-c}{2} \end{matrix} \right].$$

For the Γ -function, keeping in mind the reciprocal relations

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{and} \quad \Gamma\left(\frac{1}{2} + x\right)\Gamma\left(\frac{1}{2} - x\right) = \frac{\pi}{\cos \pi x}$$

as well as Legendre's duplication formula

$$\Gamma(x)\Gamma\left(\frac{1}{2} + x\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x),$$

we can further derive the following expressions:

$$\begin{aligned}
 (7b) \quad & \Rightarrow \Gamma \left[\frac{1+b}{2}, \frac{2+b}{2}, \frac{1-c}{2}, \frac{2-c}{2}, \frac{1+a+c}{2}, \frac{1-b-d}{2}, \frac{a-b}{2}, \frac{a-d}{2}, \frac{c-d}{2} \right] \\
 & = \Gamma \left[\frac{1-a-c}{2}, \frac{1+b+d}{2}, \frac{2-a+b}{2}, \frac{2-a+d}{2}, \frac{2-c+b}{2}, \frac{2-c+d}{2} \right] \\
 & \quad \times \frac{\pi}{2^{2-a-c+b+d}} \frac{\sin a\pi \sin d\pi \sin \frac{b+d-a-c}{2}\pi \cos \frac{b+c}{2}\pi}{\sin \frac{a-b}{2}\pi \sin \frac{a-d}{2}\pi \sin \frac{c-d}{2}\pi \cos \frac{a+c}{2}\pi \cos \frac{b+d}{2}\pi}; \\
 (7c) \quad & \Rightarrow \Gamma \left[\frac{1+b}{2}, \frac{2+b}{2}, \frac{1+d}{2}, \frac{2+d}{2}, \frac{1+a+c}{2}, \frac{a-d}{2}, \frac{c-d}{2}, \frac{2-a-c+b+d}{2} \right] \\
 & = \Gamma \left[\frac{1-a-c}{2}, \frac{1+b+d}{2}, \frac{2-a+b}{2}, \frac{2-a+d}{2}, \frac{2-c+b}{2}, \frac{2-c+d}{2} \right] \\
 & \quad \times \frac{\pi}{2^{2-a-c+b+d}} \frac{\sin a\pi \sin c\pi}{\sin \frac{a-d}{2}\pi \sin \frac{c-d}{2}\pi \cos \frac{a+c}{2}\pi}; \\
 (7d) \quad & \Rightarrow \Gamma \left[\frac{1-a}{2}, \frac{2-a}{2}, \frac{1-c}{2}, \frac{2-c}{2}, \frac{1-b-d}{2}, \frac{a-b}{2}, \frac{a-d}{2}, \frac{2-a-c+b+d}{2} \right] \\
 & = \Gamma \left[\frac{1-a-c}{2}, \frac{1+b+d}{2}, \frac{2-a+b}{2}, \frac{2-a+d}{2}, \frac{2-c+b}{2}, \frac{2-c+d}{2} \right] \\
 & \quad \times \frac{\pi}{2^{2-a-c+b+d}} \frac{\sin b\pi \sin d\pi}{\sin \frac{a-b}{2}\pi \sin \frac{a-d}{2}\pi \cos \frac{b+d}{2}\pi}.
 \end{aligned}$$

Now, we just need to prove that the sum of the last three trigonometric fractions equals the trigonometric expression displayed in (6a), i.e.

$$\begin{aligned}
 (8a) \quad & 2 \frac{\cos \frac{a-c}{2}\pi}{\cos \frac{a+c}{2}\pi} + 2 \frac{\cos \frac{b-d}{2}\pi}{\cos \frac{b+d}{2}\pi} = \frac{\sin a\pi \sin d\pi \sin \frac{b+d-a-c}{2}\pi \cos \frac{b+c}{2}\pi}{\sin \frac{a-b}{2}\pi \sin \frac{a-d}{2}\pi \sin \frac{c-d}{2}\pi \cos \frac{a+c}{2}\pi \cos \frac{b+d}{2}\pi} \\
 (8b) \quad & + \frac{\sin a\pi \sin c\pi}{\sin \frac{a-d}{2}\pi \sin \frac{c-d}{2}\pi \cos \frac{a+c}{2}\pi} + \frac{\sin b\pi \sin d\pi}{\sin \frac{a-b}{2}\pi \sin \frac{a-d}{2}\pi \cos \frac{b+d}{2}\pi}.
 \end{aligned}$$

Multiplying a, b, c, d by $2/\pi$, we can equivalently rewrite the last equation as follows:

$$\begin{aligned}
 (9a) \quad & 2\sin(a-b)\sin(a-d)\sin(c-d) \times \{ \cos(a-c)\cos(b+d) + \cos(a+c)\cos(b-d) \} \\
 (9b) \quad & = \sin 2a \sin 2d \sin(b+d-a-c) \cos(b+c) \\
 (9c) \quad & + \sin 2a \sin 2c \cos(b+d) \sin(a-b) \\
 (9d) \quad & + \sin 2b \sin 2d \cos(a+c) \sin(c-d).
 \end{aligned}$$

This will be confirmed by utilizing the following elementary trigonometric formulae:

$$\begin{aligned}
 (10a) \quad & \sin 2a = 2 \sin a \cos a, \\
 (10b) \quad & \sin(a+b) = \sin a \cos b + \sin b \cos a, \\
 (10c) \quad & \cos(a+b) = \cos a \cos b - \sin a \sin b.
 \end{aligned}$$

Firstly, according to (10b), we have

$$\sin(b + d - a - c) = \sin(b - a) \cos(d - c) + \sin(d - c) \cos(b - a),$$

which splits the expression displayed in (9b) into two parts:

$$(11a) \quad (9b) = \sin 2a \sin 2d \sin(b - a) \cos(d - c) \cos(b + c)$$

$$(11b) \quad + \sin 2a \sin 2d \sin(d - c) \cos(b - a) \cos(b + c).$$

Then combining (11a) with (9c) and (11b) with (9d) respectively, we get the following two expressions:

$$(12a) \quad (9c) + (11a) = \sin 2a \sin(a - b) \{ \sin 2c \cos(b + d) - \sin 2d \cos(c - d) \cos(b + c) \},$$

$$(12b) \quad (9d) + (11b) = \sin 2d \sin(c - d) \{ \sin 2b \cos(a + c) - \sin 2a \cos(a - b) \cos(b + c) \}.$$

By means of (10a) and (10c), expanding the expression in the braces $\{\dots\}$ of (12a) and then canceling the common terms, we may simplify it into

$$\begin{aligned} & 2 \left\{ \sin c \cos b \cos c \cos^3 d - \sin d \cos b \cos^2 c \cos^2 d \right. \\ & \quad \left. + \sin b \sin^2 c \sin^2 d \cos d - \sin b \sin c \sin^3 d \cos c \right\} \\ & = 2 \sin(c - d) \{ \sin b \sin c \sin^2 d + \cos b \cos c \cos^2 d \}, \end{aligned}$$

which leads us further to the following equality:

$$(9c) + (11a) = 4 \sin(a - b) \sin(c - d) \sin a \cos a \{ \sin b \sin c \sin^2 d + \cos b \cos c \cos^2 d \}.$$

Observing that (12b) can be deduced from (12a) by interchanging the parameters $a \rightleftharpoons d$ and $b \rightleftharpoons c$, we have from the last expression

$$(9d) + (11b) = 4 \sin(b - a) \sin(c - d) \sin d \cos d \{ \sin^2 a \sin b \sin c + \cos^2 a \cos b \cos c \}.$$

From these two expressions just displayed, we find that

$$\begin{aligned} & (9b) + (9c) + (9d) = (11a) + (11b) + (9c) + (9d) \\ & = 4 \sin(a - b) \sin(c - d) \left\{ \sin a \cos a \cos b \cos c \cos^2 d - \sin d \cos^2 a \cos b \cos c \cos d \right. \\ & \quad \left. + \sin a \sin b \sin c \sin^2 d \cos a - \sin^2 a \sin b \sin c \sin d \cos d \right\} \\ & = 4 \sin(a - b) \sin(c - d) \sin(a - d) \{ \cos a \cos b \cos c \cos d - \sin a \sin b \sin c \sin d \}. \end{aligned}$$

Finally, we can easily verify, by (10c), the following equation:

$$\begin{aligned} & 2 \{ \cos a \cos b \cos c \cos d - \sin a \sin b \sin c \sin d \} \\ & = \cos(a - c) \cos(b + d) + \cos(a + c) \cos(b - d). \end{aligned}$$

This completes the proof of the trigonometric identity stated in (9a-9b-9c-9d) and consequently the identity (6a-6b) due to M. Jackson (1949). $\square$

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