ZAGIER DUALITY FOR HARMONIC WEAK MAASS FORMS OF INTEGRAL WEIGHT

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(Communicated by Ken Ono)

ABSTRACT. We show the existence of “Zagier duality” between vector valued harmonic weak Maass forms and vector valued weakly holomorphic modular forms of integral weight. This duality phenomenon arises naturally in the context of harmonic weak Maass forms as developed in recent works by Bruinier, Funke, Ono, and Rhoades. Concerning the isomorphism between the spaces of scalar and vector valued harmonic weak Maass forms of integral weight, Zagier duality between scalar valued ones is derived.

1. INTRODUCTION AND STATEMENT OF RESULTS

For an integer k , let $M_{k+\frac{1}{2}}^{!+}$ be the space of weakly holomorphic modular forms $f(\tau)$ of weight $k + \frac{1}{2}$ for $\Gamma_0(4)$ satisfying Kohnen’s plus space condition; that is, $f(\tau)$ has a Fourier expansion of the form

$$f(\tau) = \sum_{(-1)^k n \equiv 0, 1 \pmod{4}} c_f(n) q^n.$$

For $d \geq 0$ with $d \equiv 0, 3 \pmod{4}$ there is a unique modular form $f_d \in M_{\frac{1}{2}}^{!+}$ having a Fourier expansion of the form

$$f_d(\tau) = q^{-d} + \sum_{D>0} c_{f_d}(D) q^D.$$

On the other hand, for $D > 0$ with $D \equiv 0, 1 \pmod{4}$ there is also a unique modular form $g_D \in M_{\frac{3}{2}}^{!+}$ having a Fourier expansion of the form

$$g_D(\tau) = q^{-D} + \sum_{d \geq 0} c_{g_D}(d) q^d.$$

As proven by Zagier [21, Theorem 1] the $g_1(\tau)$ is essentially the generating function for the traces of singular moduli. He also proved the so-called “Zagier duality” [21, Theorem 4] relating the Fourier coefficients of $f_d(\tau)$ and $g_D(\tau)$:

Received by the editors March 14, 2010.

2010 *Mathematics Subject Classification*. Primary 11F11, 11F30; Secondary 11F37, 11F50.

The first author was partially supported by BK21 at POSTECH, the Tae-Joon Park POSTECH Postdoctoral Fellowship, and NRF 2010-0008426.

The second author was partially supported by NRF20090083919 and NRF2009-0094069.

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Theorem 1.1 (Zagier).

$$c_{f_d}(D) = -c_{g_D}(d) \quad \text{for all } D \text{ and } d.$$

Zagier's results have inspired vast research subjects (for instance see [3, 12, 13, 15, 16, 17, 18, 20]) and have been extended to study duality properties on the space of harmonic weak Maass forms $H_k(\Gamma_0(N))$. For instance, in terms of the weight (higher weight) and space (weak Maass forms) aspects, K. Bringmann and K. Ono [3, Theorem 1.1] showed Zagier duality between certain Maass-Poincaré series of weight $-k+3/2$ and Poincaré series of weight $k+1/2$ for $\Gamma_0(4)$ with $k \geq 1$. In fact, their Fourier coefficients are traces of singular moduli of certain weight 0 Maass forms. A. Folsom and K. Ono [15] found Zagier duality between certain weight $1/2$ harmonic weak Maass forms and weight $3/2$ weakly holomorphic modular forms on $\Gamma_0(144)$ with Nebentypus $(\frac{12}{\cdot})$. The holomorphic part of their initial Maass form is essentially Ramanujan's mock theta function $f(q)$. Also, C. H. Kim [18] proved Zagier duality between certain weakly holomorphic modular forms of weight $1/2$ and $3/2$ on $\Gamma_0(4p)$. Here p is a prime such that the genus of the Fricke group $\Gamma_0(p)^*$ equals 0. In his result, those Fourier coefficients are essentially traces of singular moduli of the Hauptmodul $j_p^*(\tau)$ for $\Gamma_0(p)^*$.

Now we introduce the results of integral weight cases. J. Rouse [20, Theorem 1] proved Zagier duality between certain weakly holomorphic modular forms of weight 0 and 2 on $\Gamma_0(p)$ with Nebentypus $(\frac{\cdot}{p})$, where $p = 5, 13, 17$. D. Choi [11, Theorem 1.2] gave a simple proof of Rouse's result by using the residue theorem, and extended it to any odd prime level p . Note that their results concern weakly holomorphic modular forms. Recently P. Guerzhoy [16, Theorem 1] showed that there is Zagier duality between certain harmonic weak Maass forms of weight k and weakly holomorphic modular forms of weight $2-k$ for $\text{SL}_2(\mathbb{Z})$, where $k \leq 0$ is even.

The purpose of this paper is to derive Zagier duality for harmonic weak Maass forms, and, as a result, this extends the previously known Zagier duality between integral weight forms. To this end, we first show that duality holds between the space H_{k,ρ_L} of vector valued harmonic weak Maass forms and the space $M_{2-k,\bar{\rho}_L}^1$ of vector valued weakly holomorphic modular forms with $k \leq 0$ an integer (Theorem 1.2). This duality phenomenon arises naturally in the context of harmonic weak Maass forms as developed in recent works by Bruinier, Funke, Ono, and Rhoades. [8, 9]. By taking L in our result as an even unimodular lattice, we immediately recover the recent result by Guerzhoy [16] (Corollary 1.3). Now we take L so that a certain space $H_k^\epsilon(\Gamma_0(p), (\frac{\cdot}{p}))$ of scalar valued harmonic weak Maass forms is isomorphic to H_{k,ρ_L} (Proposition 1.4). Then, as another corollary, for any odd prime p we have Zagier duality between $H_k^\epsilon(\Gamma_0(p), (\frac{\cdot}{p}))$ and a certain space $M_{2-k}^{\text{!}\delta}(\Gamma_0(p), (\frac{\cdot}{p}))$ of scalar valued weakly holomorphic modular forms (Corollary 1.5).

To state our main theorem, let L be a non-degenerate even lattice of signature (b^+, b^-) , and let L' be its dual lattice. We denote the standard basis elements of the group algebra $\mathbb{C}[L'/L]$ by $\mathbf{e}_\gamma$ for $\gamma \in L'/L$. Let ρ_L be the Weil representation associated to the discriminant form $(L'/L, Q)$, and let $\bar{\rho}_L$ be its dual representation. We write M_{k,ρ_L}^1 for the space of $\mathbb{C}[L'/L]$ -valued weakly holomorphic modular forms of weight k and type ρ_L , and H_{k,ρ_L} for the space of $\mathbb{C}[L'/L]$ -valued harmonic weak Maass forms of weight k and type ρ_L (see Section 2). In the following theorem we obtain Zagier duality between the space H_{k,ρ_L} of vector valued harmonic weak

Maass forms and the space $M_{2-k, \bar{\rho}_L}^!$ of vector valued weakly holomorphic modular forms with $k \leq 0$ an integer.

Theorem 1.2. *Let L be a non-degenerate even lattice of signature (b^+, b^-) , and let $k \leq 0$ be an integer such that $2k - b^+ + b^- \equiv 0 \pmod{2}$. Let $\alpha, \beta \in L'/L$ and $m \in \mathbb{Z} - Q(\alpha)$, $n \in \mathbb{Z} + Q(\beta)$ with $m, n > 0$. Then there exist $f_{\alpha, m} \in H_{k, \rho_L}$ and $g_{\beta, n} \in M_{2-k, \bar{\rho}_L}^!$ with Fourier expansions of the form*

$$f_{\alpha, m}(\tau) = f_{\alpha, m}^-(\tau) + q^{-m} \mathfrak{e}_{\alpha} + (-1)^{k + \frac{b^- - b^+}{2}} q^{-m} \mathfrak{e}_{-\alpha} + \sum_{\gamma \in L'/L} \sum_{\substack{l \in \mathbb{Z} + Q(\gamma) \\ l \geq 0}} c_{f_{\alpha, m}}^+(\gamma, l) q^l \mathfrak{e}_{\gamma},$$

$$g_{\beta, n}(\tau) = q^{-n} \mathfrak{e}_{\beta} + (-1)^{k + \frac{b^+ - b^-}{2}} q^{-n} \mathfrak{e}_{-\beta} + \sum_{\gamma \in L'/L} \sum_{\substack{l \in \mathbb{Z} - Q(\gamma) \\ l > 0}} c_{g_{\beta, n}}(\gamma, l) q^l \mathfrak{e}_{\gamma}$$

such that

$$c_{f_{\alpha, m}}^+(\beta, n) = -c_{g_{\beta, n}}(\alpha, m).$$

If $k < 0$, then $f_{\alpha, m}$, $g_{\beta, n}$ are uniquely determined.

Remark 1. (1) We can include the case $n = 0$ by taking $g_{\beta, 0}$ as an Eisenstein series E_{β} ([6, Theorem 1.6]).

(2) For $k < 0$ one can use [6, Proposition 1.10] to construct $f_{\alpha, m}$ and $g_{\beta, n}$ explicitly.

If we take L in Theorem 1.2 as a unimodular one, we can recover Guerzhoy's result [16, Theorem 1].

Corollary 1.3 (Guerzhoy). *For $m, n \in \mathbb{Z}_{>0}$ there exist $f_m \in H_k(\mathrm{SL}_2(\mathbb{Z}))$, $g_n \in M_{2-k}^!(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansions of the form*

$$f_m(\tau) = f_m^-(\tau) + q^{-m} + \sum_{l \in \mathbb{Z}_{\geq 0}} c_{f_m}^+(l) q^l,$$

$$g_n(\tau) = q^{-n} + \sum_{l \in \mathbb{Z}_{>0}} c_{g_n}(l) q^l$$

such that

$$c_{f_m}^+(n) = -c_{g_n}(m).$$

If $k < 0$, then f_m , g_n are uniquely determined.

Of course we may take L as another suitable lattice. Let p be an odd prime. We write $M_k^!(\Gamma_0(p), (\frac{\cdot}{p}))$ for the space of weakly holomorphic modular forms of integral weight k for $\Gamma_0(p)$ with Nebentypus $(\frac{\cdot}{p})$. For $\epsilon \in \{\pm 1\}$ we define the subspace

$$M_k^{\epsilon}(\Gamma_0(p), (\frac{\cdot}{p})) := \{f = \sum_{n \in \mathbb{Z}} c_f(n) q^n \in M_k^!(\Gamma_0(p), (\frac{\cdot}{p})) \mid c_f(n) = 0 \text{ if } (\frac{n}{p}) = -\epsilon\}.$$

Suppose that the discriminant group L'/L is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Then $b^+ - b^-$ is even, and the quadratic form on L'/L is equivalent to $Q(\gamma) = \frac{\lambda \gamma^2}{p}$ for $\lambda, \gamma \in \mathbb{Z}/p\mathbb{Z}$ with $\lambda \neq 0$.

Now we put $\epsilon = (\frac{\lambda}{p})$, $\delta = (\frac{-1}{p})\epsilon$, and assume that $k \equiv (b^+ - b^-)/2 \pmod{2}$. Then Bruinier and Bundschuh [7, Theorem 5] showed that the space $M_k^{\epsilon}(\Gamma_0(p), (\frac{\cdot}{p}))$ (resp. $M_k^{\delta}(\Gamma_0(p), (\frac{\cdot}{p}))$) of scalar valued weakly holomorphic modular forms is isomorphic to the space $M_{k, \rho_L}^!$ (resp. $M_{k, \bar{\rho}_L}^!$) of vector valued ones.

We observe that Bruinier and Bundschuh's argument can also be applied to the spaces of scalar and vector valued harmonic weak Maass forms. Let $H_k(\Gamma_0(p), (\frac{\cdot}{p}))$ denote the space of harmonic weak Maass forms of weight k for $\Gamma_0(p)$ with Nebentypus $(\frac{\cdot}{p})$ (see Section 2). In particular $f \in H_k(\Gamma_0(p), (\frac{\cdot}{p}))$ has a unique decomposition $f = f^+ + f^-$, where

$$\begin{aligned} f^+(\tau) &= \sum_{n \gg -\infty} c_f^+(n) q^n, \\ f^-(\tau) &= \sum_{n < 0} c_f^-(n) \Gamma(1-k, 4\pi|n|y) q^n. \end{aligned}$$

Here $\Gamma(a, y) = \int_y^\infty e^{-t} t^{a-1} dt$ denotes the incomplete Gamma function. For $\epsilon \in \{\pm 1\}$ we define the subspace

$$H_k^\epsilon(\Gamma_0(p), (\frac{\cdot}{p})) := \{f \in H_k(\Gamma_0(p), (\frac{\cdot}{p})) \mid c_f^\pm(n) = 0 \text{ if } (\frac{n}{p}) = -\epsilon\}.$$

For a given $f \in H_k(\Gamma_0(p), (\frac{\cdot}{p}))$ we define a $\mathbb{C}[L'/L]$ -valued function $F = \sum_{\gamma \in \mathbb{Z}/p\mathbb{Z}} F_\gamma \mathfrak{e}_\gamma$ by

$$F_\gamma(\tau) := s(\gamma) \sum_{\substack{n \in \mathbb{Z} \\ n \equiv pQ(\gamma) \pmod{p}}} c_f(n, \frac{y}{p}) q^{n/p}.$$

Here $c_f(n, y) := c_f^+(n) + c_f^-(n) \Gamma(1-k, 4\pi|n|y)$, and for $l \in \mathbb{Z}$,

$$s(l) := \begin{cases} 2 & \text{if } l \equiv 0 \pmod{p}, \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 1.4. *Let L be a non-degenerate even lattice of signature (b^+, b^-) such that the discriminant group L'/L is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with p an odd prime. Put $\epsilon = (\frac{\lambda}{p})$ and $\delta = (\frac{-1}{p})\epsilon$. We assume that $k \equiv \frac{b^+ - b^-}{2} \pmod{2}$. Then the map $f \mapsto F$ defines an isomorphism of $H_k^\epsilon(\Gamma_0(p), (\frac{\cdot}{p}))$ onto H_{k, ρ_L} and of $H_k^\delta(\Gamma_0(p), (\frac{\cdot}{p}))$ onto $H_{k, \bar{\rho}_L}$.*

Remark 2. A similar result for the half integral weight case can be found in [10].

As another corollary of Theorem 1.2 combining with Proposition 1.4 we obtain Zagier duality between the space $H_k^\epsilon(\Gamma_0(p), (\frac{\cdot}{p}))$ of scalar valued harmonic weak Maass forms and the space $M_{2-k}^{l\delta}(\Gamma_0(p), (\frac{\cdot}{p}))$ of scalar valued weakly holomorphic modular forms.

Corollary 1.5. *With the notation and assumption as in Proposition 1.4, we further assume that $k \leq 0$. For $m, n \in \mathbb{Z}_{>0}$ with $(\frac{-m}{p}) \neq -\epsilon$ and $(\frac{-n}{p}) \neq -\delta$, there exist $f_m \in H_k^\epsilon(\Gamma_0(p), (\frac{\cdot}{p}))$ and $g_n \in M_{2-k}^{l\delta}(\Gamma_0(p), (\frac{\cdot}{p}))$ with Fourier expansions of the form*

$$\begin{aligned} f_m(\tau) &= f_m^-(\tau) + q^{-m} + \sum_{l \geq 0} c_{f_m}^+(l) q^l, \\ g_n(\tau) &= q^{-n} + \sum_{l > 0} c_{g_n}(l) q^l \end{aligned}$$

such that

$$s(n) c_{f_m}^+(n) = -s(m) c_{g_n}(m).$$

If $k < 0$, then f_m, g_n are uniquely determined.

Remark 3. (1) In [16] Guerzhoy called such a pair (f_m, g_n) a “grid”.
 (2) This extends the result given by Rouse [20, Theorem 1].

2. PRELIMINARIES

2.1. Scalar valued modular forms. Let $\tau = x + iy \in \mathbb{H}$, the complex upper half plane, with $x, y \in \mathbb{R}$. Let $k \in \mathbb{Z}$, N a positive integer, and let χ be a Dirichlet character modulo N .

Recall that *weakly holomorphic modular forms of weight k for $\Gamma_0(N)$ with Nebentypus χ* are holomorphic functions $f : \mathbb{H} \rightarrow \mathbb{C}$ which satisfy the following conditions:

- (i) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau);$$

- (ii) f has a Fourier expansion of the form

$$f(\tau) = \sum_{\substack{n \in \mathbb{Z} \\ n \gg -\infty}} c_f(n) q^n,$$

and analogous conditions are required at all cusps. Here $q = e^{2\pi i \tau}$ as usual.

We write $M_k^!(\Gamma_0(N), \chi)$ for the space of these weakly holomorphic modular forms.

A smooth function $f : \mathbb{H} \rightarrow \mathbb{C}$ is called a *harmonic weak Maass form of weight k for $\Gamma_0(N)$ with Nebentypus χ* if it satisfies the following conditions:

- (i) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^k f(\tau).$$

- (ii) $\Delta_k f = 0$, where Δ_k is the weight k hyperbolic Laplace operator defined by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

- (iii) There is a Fourier polynomial $P_f(\tau) = \sum_{-\infty \ll n \leq 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(\tau) = P_f(\tau) + O(e^{-\varepsilon y})$ as $y \rightarrow \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

We denote the space of these harmonic weak Maass forms by $H_k(\Gamma_0(N), \chi)$. This space can be denoted by $H_k^+(\Gamma_0(N), \chi)$ in the context of [8]. Here we follow notation given in [9]. Then the space $H_k(\Gamma_0(N), \chi)$ contains the space $M_k^!(\Gamma_0(N), \chi)$ of weakly holomorphic modular forms of weight k for $\Gamma_0(N)$ with Nebentypus χ . The polynomial $P_f \in \mathbb{C}[q^{-1}]$ is called the principal part of f at the corresponding cusps. In particular $f \in H_k(\Gamma_0(N), \chi)$ has a unique decomposition $f = f^+ + f^-$, where

$$\begin{aligned} f^+(\tau) &= \sum_{n \gg -\infty} c_f^+(n) q^n, \\ f^-(\tau) &= \sum_{n < 0} c_f^-(n) \Gamma(1 - k, 4\pi|n|y) q^n. \end{aligned}$$

2.2. Vector valued modular forms. We write $\mathrm{Mp}_2(\mathbb{R})$ for the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$. The elements are pairs (M, ϕ) , where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\phi : \mathbb{H} \rightarrow \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. The multiplication is defined by

$$(M, \phi(\tau))(M', \phi'(\tau)) = (MM', \phi(M'\tau)\phi'(\tau)).$$

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ we use the notation $\tilde{M} := (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}) \in \mathrm{Mp}_2(\mathbb{R})$. We denote by $\mathrm{Mp}_2(\mathbb{Z})$ the integral metaplectic group, that is, the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $\mathrm{Mp}_2(\mathbb{R}) \rightarrow \mathrm{SL}_2(\mathbb{R})$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $T := (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1)$ and $S := (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau})$.

Let (V, Q) be a non-degenerate rational quadratic space of signature (b^+, b^-) . Let $L \subset V$ be an even lattice with dual L' . We denote the standard basis elements of the group algebra $\mathbb{C}[L'/L]$ by $\mathbf{e}_\gamma$ for $\gamma \in L'/L$ and write $\langle \cdot, \cdot \rangle$ for the standard scalar product, antilinear in the second entry, such that $\langle \mathbf{e}_\gamma, \mathbf{e}_{\gamma'} \rangle = \delta_{\gamma, \gamma'}$. There is a unitary representation ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$, called the Weil representation, which is defined by

$$\begin{aligned} \rho_L(T)(\mathbf{e}_\gamma) &:= e(Q(\gamma))\mathbf{e}_\gamma, \\ \rho_L(S)(\mathbf{e}_\gamma) &:= \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e(-\langle \gamma, \delta \rangle)\mathbf{e}_\delta, \end{aligned}$$

where $e(z) := e^{2\pi iz}$ and $(X, Y) := Q(X + Y) - Q(X) - Q(Y)$ is the associated bilinear form. We denote by $\bar{\rho}_L$ the dual representation of ρ_L .

Let $k \in \frac{1}{2}\mathbb{Z}$. A holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is called a *weakly holomorphic modular form of weight k and type ρ_L for the group $\mathrm{Mp}_2(\mathbb{Z})$* if it satisfies:

- (i) $f(M\tau) = \phi(\tau)^{2k} \rho_L(M, \phi) f(\tau)$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$;
- (ii) f is meromorphic at the cusp ∞ .

Here condition (ii) means that f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n \gg -\infty}} c_f(\gamma, n) e(n\tau) \mathbf{e}_\gamma.$$

The space of these $\mathbb{C}[L'/L]$ -valued weakly holomorphic modular forms is denoted by $M_{k, \rho_L}^!$. Similarly we can define the space $M_{k, \bar{\rho}_L}^!$ of $\mathbb{C}[L'/L]$ -valued weakly holomorphic modular forms of type $\bar{\rho}_L$. The subspace of $\mathbb{C}[L'/L]$ -valued holomorphic modular forms (resp. cusp forms) of weight k and type ρ_L is denoted by M_{k, ρ_L} (resp. S_{k, ρ_L}).

A smooth function $f : \mathbb{H} \rightarrow \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form of weight k and type ρ_L for the group $\mathrm{Mp}_2(\mathbb{Z})$* if it satisfies the following conditions:

- (i) $f(M\tau) = \phi(\tau)^{2k} \rho_L(M, \phi) f(\tau)$ for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$.
- (ii) $\Delta_k f = 0$.
- (iii) There is a Fourier polynomial $P_f(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ -\infty \ll n \leq 0}} c_f^+(\gamma, n) e(n\tau) \mathbf{e}_\gamma$ such that $f(\tau) = P_f(\tau) + O(e^{-\varepsilon y})$ as $y \rightarrow \infty$ for some $\varepsilon > 0$.

We denote by H_{k, ρ_L} the space of these $\mathbb{C}[L'/L]$ -valued harmonic weak Maass forms. This space is denoted by $H_{k, L}^+$ in [8]. We have $M_{k, \rho_L}^! \subset H_{k, \rho_L}$. Similarly we define the space $H_{k, \bar{\rho}_L}$. In particular $f \in H_{k, \rho_L}$ has a unique decomposition $f = f^+ + f^-$,

where

$$\begin{aligned} f^+(\tau) &= \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n \gg -\infty}} c_f^+(\gamma, n) e(n\tau) \mathfrak{e}_\gamma, \\ f^-(\tau) &= \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n < 0}} c_f^-(\gamma, n) \Gamma(1-k, 4\pi|n|y) e(n\tau) \mathfrak{e}_\gamma. \end{aligned}$$

2.3. Zagier duality for weakly holomorphic modular forms. We begin with the following rather simple observation.

Lemma 2.1. *Let $k, k' \in \frac{1}{2}\mathbb{Z}$ with $k + k' \in \mathbb{Z}$, and*

$$\begin{aligned} f &= \sum_{\gamma \in L'/L} f_\gamma \mathfrak{e}_\gamma \in M_{k, \rho_L}^!, \\ g &= \sum_{\gamma \in L'/L} g_\gamma \mathfrak{e}_\gamma \in M_{k', \bar{\rho}_L}^!. \end{aligned}$$

Then $\sum_{\gamma \in L'/L} f_\gamma g_\gamma \in M_{k+k'}^!(\mathrm{SL}_2(\mathbb{Z}))$, i.e. a weakly holomorphic elliptic modular form of weight $k + k'$.

Proof. Since $\sum_{\gamma \in L'/L} f_\gamma(\tau) g_\gamma(\tau)$ is holomorphic on $\mathbb{H}$ and meromorphic at the cusp ∞ , it suffices to verify the modular transformation property. First note that

$$\sum_{\gamma \in L'/L} f_\gamma g_\gamma = \left\langle \sum_{\gamma \in L'/L} f_\gamma \mathfrak{e}_\gamma, \sum_{\gamma \in L'/L} \overline{g_\gamma} \mathfrak{e}_\gamma \right\rangle = \langle f, \bar{g} \rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual dot product.

Now we find for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ that

$$\begin{aligned} \left(\sum_{\gamma \in L'/L} f_\gamma g_\gamma \right) \Big|_{k+k'} M &= (c\tau + d)^{-(k+k')} \sum_{\gamma \in L'/L} f_\gamma(M\tau) g_\gamma(M\tau) \\ &= \langle (c\tau + d)^{-k} f(M\tau), \overline{(c\tau + d)^{-k'} g(M\tau)} \rangle \\ &= \langle \rho_L(\tilde{M}) f(\tau), \rho_L(\tilde{M}) \overline{g(\tau)} \rangle. \end{aligned}$$

Since ρ_L is unitary we have

$$\left(\sum_{\gamma \in L'/L} f_\gamma g_\gamma \right) \Big|_{k+k'} M = \langle f(\tau), \overline{g(\tau)} \rangle = \sum_{\gamma \in L'/L} f_\gamma g_\gamma. \quad \square$$

Lemma 2.2. *Let $k, k' \in \frac{1}{2}\mathbb{Z}$ with $k + k' = 2$. We denote the Fourier expansions of $f \in M_{k, \rho_L}^!$ and $g \in M_{k', \bar{\rho}_L}^!$ by*

$$\begin{aligned} f(\tau) &= \sum_{\gamma \in L'/L} \sum_m c_f(\gamma, m) q^m \mathfrak{e}_\gamma, \\ g(\tau) &= \sum_{\gamma \in L'/L} \sum_n c_g(\gamma, n) q^n \mathfrak{e}_\gamma. \end{aligned}$$

Then

$$\sum_{\gamma \in L'/L} \sum_{m+n=0} c_f(\gamma, m) c_g(\gamma, n) = 0.$$

Proof. By Lemma 2.1, $\sum_{\gamma \in L'/L} f_\gamma g_\gamma$ is a weakly holomorphic elliptic modular form of weight 2. Thus by the residue theorem we immediately obtain the assertion. $\square$

- Remark 4.* (1) In Lemmas 2.1 and 2.2 the Weil representation ρ_L can be replaced by arbitrary unitary representation because we only used the unitary property of the Weil representation.
- (2) Lemma 2.2 is indeed a special case of Bruinier and Funke's result (see Proposition 3.2 below). This can be used to show Zagier duality for weakly holomorphic modular forms of integral or half integral weight. For example one can easily infer [21, Theorem 4] and [20, Theorem 1] from Lemma 2.2 and the results [14, Theorems 5.1, 5.4] and [7, Theorem 5] concerning the isomorphism between the spaces of scalar and vector valued modular forms.

3. THE RESULTS OF BRUINIER, FUNKE, ONO, AND RHOADES

Assume that $k \leq 0$ is an integer. Recall that there is an antilinear differential operator

$$\xi_k : H_{k, \rho_L} \longrightarrow S_{2-k, \bar{\rho}_L}$$

defined by

$$\xi_k(f)(\tau) := y^{k-2} \overline{L_k f(\tau)},$$

where $L_k := -2iy^2 \frac{\partial}{\partial \bar{\tau}}$ is the Maass lowering operator (see [8, 9]). The Maass raising operator is defined by $R_k := 2i \frac{\partial}{\partial \tau} + ky^{-1}$. By [8, Corollary 3.8] the following sequence is exact:

$$0 \longrightarrow M_{k, \rho_L}^! \longrightarrow H_{k, \rho_L} \xrightarrow{\xi_k} S_{2-k, \bar{\rho}_L} \longrightarrow 0.$$

Proposition 3.1 ([8, Proposition 3.11]). *For every Fourier polynomial of the form*

$$P(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n < 0}} c^+(\gamma, n) e(n\tau) \mathfrak{e}_\gamma$$

with $c^+(\gamma, n) = (-1)^{k + \frac{b^- - b^+}{2}} c^+(-\gamma, n)$, there exists an $f \in H_{k, \rho_L}$ with principal part $P_f(\tau) = P(\tau) + \mathfrak{c}$ for some T -invariant constant $\mathfrak{c} \in \mathbb{C}[L'/L]$. The function f is uniquely determined if $k < 0$.

Using the Petersson scalar product, we can obtain a bilinear pairing between $M_{2-k, \bar{\rho}_L}$ and H_{k, ρ_L} defined by

$$\{g, f\} = (g, \xi_k(f)) := \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \langle g, \xi_k(f) \rangle y^{2-k} \frac{dx dy}{y^2},$$

where $g \in M_{2-k, \bar{\rho}_L}$ and $f \in H_{k, \rho_L}$.

Proposition 3.2 ([8, Proposition 3.5]). *If $g \in M_{2-k, \bar{\rho}_L}$ and $f \in H_{k, \rho_L}$, then*

$$\{g, f\} = \sum_{\gamma \in L'/L} \sum_{n \leq 0} c_f^+(\gamma, n) c_g(\gamma, -n).$$

As pointed out in Remark 4, one can easily infer Zagier duality for weakly holomorphic modular forms from Proposition 3.2. But this is insufficient for our purpose because we are going to deal with harmonic weak Maass forms. To this end we need to define the regularized bilinear pairing $\{\cdot, \cdot\}^{reg}$.

Following Borchers [1, Section 6], we define the regularized bilinear pairing $\{g, f\}^{reg}$ between $M_{2-k, \bar{\rho}_L}^!$ and H_{k, ρ_L} as the constant term in the Laurent expansion at $s = 0$ of the meromorphic continuation in s of the function

$$\lim_{t \rightarrow \infty} \int_{\mathcal{F}_t} \langle g, \xi_k(f) \rangle y^{2-k-s} \frac{dx dy}{y^2},$$

where

$$\mathcal{F}_t = \{\tau \in \mathbb{H} \mid |\tau| \geq 1, |x| \leq \frac{1}{2}, \text{ and } y \leq t\}$$

denotes the truncated fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

Proposition 3.3. *Let $f \in H_{k,\rho_L}$ and $g \in M_{2-k,\bar{\rho}_L}^!$. Suppose that g has vanishing constant terms, i.e. $c_g(\gamma, 0) = 0$ for all $\gamma \in L'/L$. Then*

$$\{g, f\}^{reg} = \sum_{\gamma \in L'/L} \sum_{m+n=0} c_f^+(\gamma, m) c_g(\gamma, n).$$

Proof. According to [9, Remark 8] we have

$$\{g, f\}^{reg} = \lim_{t \rightarrow \infty} \int_{\mathcal{F}_t} \langle g, \xi_k(f) \rangle y^{2-k} \frac{dx dy}{y^2}.$$

Now one can apply the same argument as in the proof of [8, Proposition 3.5]. $\square$

Recall the differential operator

$$D := \frac{1}{2\pi i} \frac{d}{d\tau}.$$

From Bol's identity [9, Lemma 2.1] one finds

$$D^{1-k} = \frac{1}{(-4\pi)^{1-k}} R_k^{1-k}.$$

Proposition 3.4. *If $h \in H_{k,\rho_L}$, then*

$$D^{1-k} h \in M_{2-k,\rho_L}^!.$$

Moreover, we have

$$D^{1-k} h = D^{1-k} h^+ = \sum_{\gamma, n} c_h^+(\gamma, n) n^{1-k} e(n\tau) \mathfrak{e}_\gamma.$$

Proof. Note that $R_k^{1-k} h$ satisfies the transformation behavior for vector valued modular forms of weight $2-k$ and type ρ_L . The remaining assertions can be derived by the same argument as in the proof of [9, Theorem 1.1]. $\square$

Proposition 3.5. *If $f \in H_{k,\rho_L}$ and $h \in H_{k,\bar{\rho}_L}$, then*

$$\{D^{1-k} h, f\}^{reg} = 0.$$

Proof. Note that $\xi_k(f) \in S_{2-k,\bar{\rho}_L}$ is a cusp form. Now the assertion follows from [9, Corollary 4.2]. $\square$

Combining Propositions 3.3-3.5 we have the following.

Theorem 3.6. *Let $k \leq 0$ be an integer, $f \in H_{k,\rho_L}$, and $g \in D^{1-k}(H_{k,\bar{\rho}_L}) \subset M_{2-k,\bar{\rho}_L}^!$. Then*

$$\sum_{\gamma \in L'/L} \sum_{m+n=0} c_f^+(\gamma, m) c_g(\gamma, n) = 0.$$

4. PROOF OF THEOREM 1.2

Let $\alpha, \beta \in L'/L$ and $m \in \mathbb{Z} - Q(\alpha)$, $n \in \mathbb{Z} + Q(\beta)$ with $m, n > 0$. Then by Proposition 3.1 there exist $f_{\alpha, m} \in H_{k, \rho_L}$ and $h_{\beta, n} \in H_{k, \bar{\rho}_L}$ such that

$$f_{\alpha, m}(\tau) = f_{\alpha, m}^-(\tau) + q^{-m} \mathbf{e}_{\alpha} + (-1)^{k + \frac{b^+ - b^-}{2}} q^{-m} \mathbf{e}_{-\alpha} + \sum_{\gamma \in L'/L} \sum_{\substack{l \in \mathbb{Z} + Q(\gamma) \\ l \geq 0}} c_{f_{\alpha, m}}^+(\gamma, l) q^l \mathbf{e}_{\gamma},$$

$$h_{\beta, n}(\tau) = h_{\beta, n}^-(\tau) + q^{-n} \mathbf{e}_{\beta} + (-1)^{k + \frac{b^+ - b^-}{2}} q^{-n} \mathbf{e}_{-\beta} + \sum_{\gamma \in L'/L} \sum_{\substack{l \in \mathbb{Z} - Q(\gamma) \\ l \geq 0}} c_{h_{\beta, n}}^+(\gamma, l) q^l \mathbf{e}_{\gamma}.$$

By Proposition 3.4 the $g_{\beta, n} := (-n)^{k-1} D^{1-k} h_{\beta, n} \in M_{2-k, \bar{\rho}_L}^!$ has a Fourier expansion of the form

$$g_{\beta, n}(\tau) = q^{-n} \mathbf{e}_{\beta} + (-1)^{k + \frac{b^+ - b^-}{2}} q^{-n} \mathbf{e}_{-\beta} + \sum_{\gamma \in L'/L} O(q^{\nu}) \mathbf{e}_{\gamma}$$

for some $\nu > 0$. Now Theorem 1.2 follows from Theorem 3.6.

5. PROOF OF PROPOSITION 1.4

By the same argument as in [7] one can see that F satisfies the desired modular transformation property. To check $\Delta_k F = 0$ we recall from [7, Proposition 2] that

$$F = i^{\frac{b^+ - b^-}{2}} p^{\frac{k-1}{2}} \sum_{M \in \Gamma_0(p) \backslash \mathrm{SL}_2(\mathbb{Z})} (\rho_L(M)^{-1} \mathbf{e}_0) f|_k W_p|_k M,$$

where $W_p := \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$ is the Fricke involution. Since Δ_k commutes with the Petersson slash operator (see [19]), we have $\Delta_k F = 0$. For the converse we recall from [7, Lemma 1] that the inverse isomorphism $F \mapsto f$ is given by

$$f = \frac{i^{\frac{b^+ - b^-}{2}}}{2} p^{\frac{1-k}{2}} F_0|_k W_p.$$

Hence $\Delta_k f$ vanishes.

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