PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 4, April 2011, Pages 1237–1245
S 0002-9939(2010)10550-6
Article electronically published on August 27, 2010

THE JOHNSON FILTRATION OF THE MCCOOL STABILIZER SUBGROUP OF THE AUTOMORPHISM GROUP OF A FREE GROUP

TAKAO SATOH

(Communicated by Richard A. Wentworth)

Abstract. Let F_n be a free group of rank n with basis $x_1, x_2, \dots, x_n$. We denote by S_n the subgroup of the automorphism group of F_n consisting of automorphisms which fix each of $x_2, \dots, x_n$ and call it the McCool stabilizer subgroup. Let IS_n be a subgroup of S_n consisting of automorphisms which induce the identity on the abelianization of F_n. In this paper, we determine the group structure of the lower central series of IS_n and its graded quotients. Then we show that the Johnson filtration of S_n coincides with the lower central series of IS_n.

1. Introduction

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, x_2, \dots, x_n$ and $F_n = \Gamma_n(1), \Gamma_n(2), \dots$ its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n which induce the identity on the nilpotent quotient group $F_n/\Gamma_n(k+1)$. The group $\mathcal{A}_n(1)$ is called the IA-automorphism group and is also denoted by IA_n. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, called the Johnson filtration of $\mathrm{Aut}\, F_n$. The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 through the remarkable pioneer work by Andreadakis [1], who showed that $\mathcal{A}_n(1), \mathcal{A}_n(2), \dots$ is a descending central series of $\mathcal{A}_n(1)$ and that for each $k \geq 1$ the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. In general, determining the structure of $\mathrm{gr}^k(\mathcal{A}_n)$ plays an important role on the study of the algebraic structure of $\mathrm{Aut}\, F_n$. For $1 \leq k \leq 3$, the rank of $\mathrm{gr}^k(\mathcal{A}_n)$ has been determined. Andreadakis [1] computed the rank of $\mathrm{gr}^1(\mathcal{A}_n)$. Moreover, by independent works of Cohen-Pakianathan [4, 5], Farb [6] and Kawazumi [10], it is known that $\mathrm{gr}^1(\mathcal{A}_n)$ is isomorphic to the abelianization of IA_n. For $k = 2$ and 3, the rank of $\mathrm{gr}^k(\mathcal{A}_n)$ is determined by Pettet [19]

Received by the editors September 6, 2009 and, in revised form, February 19, 2010 and April 20, 2010.
2010 *Mathematics Subject Classification.* Primary 20F28; Secondary 20F14.
Key words and phrases. Automorphism group of a free group, IA-automorphism group, Johnson filtration.

and Satoh [21] respectively. For $k \geq 4$, however, it seems that there are few results for the structure of $\mathrm{gr}^k(\mathcal{A}_n)$.

In the study of the Johnson filtration of $\mathrm{Aut}\,F_n$, it would also be interesting to determine whether $\mathcal{A}_n(1)$, $\mathcal{A}_n(2)$, $\ldots$ coincides with the lower central series $\mathcal{A}'_n(1)$, $\mathcal{A}'_n(2)$, $\ldots$ of $\mathcal{A}_n(1)$ or not. Andreadakis [1] showed that $\mathcal{A}_2(k) = \mathcal{A}'_2(k)$ and $\mathcal{A}_3(3) = \mathcal{A}'_3(3)$. From the results due to Cohen-Pakianathan [4, 5], Farb [6] and Kawazumi [10], we have $\mathcal{A}_n(2) = \mathcal{A}'_n(2)$ for $n \geq 3$. Furthermore, Pettet [19] obtained that $\mathcal{A}'_n(3)$ has finite index in $\mathcal{A}_n(3)$. Now it is conjectured by Andreadakis that $\mathcal{A}_n(k) = \mathcal{A}'_n(k)$ for any $n \geq 3$ and $k \geq 3$.

In this paper, we give an affirmative answer to the problem above for a certain subgroup of $\mathrm{Aut}\,F_n$. Let S_n be the subgroup of $\mathrm{Aut}\,F_n$ consisting of automorphisms which fix each of $x_2, \ldots, x_n$. We call S_n the McCool stabilizer subgroup of $\mathrm{Aut}\,F_n$. Let IS_n be the subgroup of S_n consisting of automorphisms which induce the identity on the abelianization of F_n. The groups S_n and IS_n were first studied by McCool. He [14] gave a finite presentation of S_n and showed that IS_n is not finitely presentable. Furthermore, he [14] also gave an infinite presentation of IS_n. Set $\mathcal{S}_n(k) := \mathcal{A}_n(k) \cap \mathrm{S}_n$ for each $k \geq 0$. Then $\mathcal{S}_n(0) = \mathrm{S}_n$ and $\mathcal{S}_n(1) = \mathrm{IS}_n$. We call a descending central filtration

$$\mathrm{S}_n = \mathcal{S}_n(0) \supset \mathcal{S}_n(1) \supset \mathcal{S}_n(2) \supset \cdots$$

the Johnson filtration of S_n. On the other hand, we also consider the lower central series of IS_n, denoted by $\mathcal{S}'_n(1)$, $\mathcal{S}'_n(2)$, ... Since the Johnson filtration is central, we see $\mathcal{S}'_n(k) \subset \mathcal{S}_n(k)$ in general. The main theorem of the paper is

Theorem 1 (= Theorem 4.1)**.** *For each $k \geq 1$, we have $\mathcal{S}_n(k) = \mathcal{S}'_n(k)$.*

In order to show this, we study the group structure of the lower central series $\mathcal{S}'_n(k)$. Let F be a subgroup of F_n generated by $x_2, \ldots, x_n$. The group F is a free group of rank $n-1$. Let $\Gamma_F(1), \Gamma_F(2), \ldots$ be the lower central series of F. In Section 3, we show that $\mathcal{S}'_n(k)$ is isomorphic to the semidirect product

$$\mathcal{S}'_n(k) \cong \Gamma_F(k+1) \rtimes \Gamma_F(k) \tag{1}$$

of $\Gamma_F(k+1)$ and $\Gamma_F(k)$ for each $k \geq 1$. (See Lemma 3.1.) We remark that McCool [14] showed this in the case where $k = 1$. Set $\mathrm{gr}^k(\mathcal{S}_n) := \mathcal{S}_n(k)/\mathcal{S}_n(k+1)$ and $\mathrm{gr}^k(\mathcal{S}'_n) := \mathcal{S}'_n(k)/\mathcal{S}'_n(k+1)$ for each $k \geq 1$. Then, using the fact above, we obtain

Corollary 1 (= Corollary 4.1)**.** *For each $k \geq 1$,*

$$\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{S}_n)) = \mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{S}'_n)) = r_{n-1}(k+1) + r_{n-1}(k),$$

where

$$r_n(k) := \mathrm{rank}_{\mathbf{Z}}(\Gamma_n(k)/\Gamma_n(k+1)).$$

Here we remark that each of $r_n(k)$ has been determined by Witt [22]. (See Subsection 2.3 for details.)

In general, it is also not determined whether each subgroup $\mathcal{A}_n(k)$ of the Johnson filtration of $\mathrm{Aut}\,F_n$ is finitely genrated or not. It is still a difficult open problem. It is, however, easily seen that for any $n \geq 3$ and $k \geq 2$, the group $\mathcal{S}_n(k)$ is not finitely generated. More precisely, from (1) and Corollary 1, we have

$$H_1(\mathcal{S}_n(k), \mathbf{Z}) \cong \big(\Gamma_F(k+1)\big/[\Gamma_F(k+1), \Gamma_F(k)]\big) \oplus H_1(\Gamma_F(k), \mathbf{Z}).$$

(See Corollary 3.2.)

This paper consists of five sections. In Section 2, we recall the IA-automorphism group and the Johnson filtration of the automorphism group of a free group. In Section 3, we study the McCool stabilizer subgroup. In Section 4, we show that the Johnson filtration of S_n coincides with its lower central series of IS_n.

2. Preliminaries

In this section, we recall the definition and some properties of the IA-automorphism group and of the Johnson homomorphisms of the automorphism group of a free group.

2.1. **Notation.** Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group automorphism group $\mathrm{Aut}\, G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ without any confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.

2.2. **IA-automorphism group.** For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, \ldots, x_n$. We denote the abelianization of F_n by H and its dual group by $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. Let $\rho : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced from the abelianization of F_n. In this paper we identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H as a free abelian group induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the IA-automorphism group of F_n. It is known due to Nielsen [18] that IA_2 coincides with the inner automorphism group $\mathrm{Inn}\, F_2$ of F_2. Namely, IA_2 is a free group of rank 2. However, IA_n for $n \geq 3$ is much larger than the inner automorphism group $\mathrm{Inn}\, F_n$. Indeed, Magnus [12] showed that for any $n \geq 3$, IA_n is finitely generated by automorphisms

$$K_{ij} : \begin{cases} x_i & \mapsto {x_j}^{-1} x_i x_j, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct i, $j \in \{1, 2, \ldots, n\}$ and

$$K_{ijk} : \begin{cases} x_i & \mapsto x_i x_j x_k {x_j}^{-1} {x_k}^{-1}, \\ x_t & \mapsto x_t, \qquad\qquad (t \neq i) \end{cases}$$

for distinct i, j, $k \in \{1, 2, \ldots, n\}$ such that $j > k$.

For any $n \geq 3$, although a generating set of IA_n is obtained as mentioned above, any presentation of IA_n is still not known. For $n = 3$, Krstić and McCool [11] showed that IA_3 is not finitely presentable. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not. Recently, Cohen-Pakianathan [4, 5], Farb [6] and Kawazumi [10] independently showed

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{2}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module.

2.3. **Free Lie algebras.** In this subsection we recall the free Lie algebra. Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \quad k \geq 2.$$

We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the graded quotient of the lower central series of F_n and by $\mathcal{L}_n := \bigoplus_{k\geq 1} \mathcal{L}_n(k)$ the associated graded sum. Since the group $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{L}_n(k)$ for each $k \geq 1$ and since IA_n acts on it trivially, the action of $\mathrm{GL}(n, \mathbf{Z})$ on each $\mathcal{L}_n(k)$ is well-defined. Furthermore, the graded sum $\mathcal{L}_n$ naturally has a graded Lie algebra structure induced from the commutator bracket on F_n and is called the free Lie algebra generated by H. (See [20] for basic material concerning free Lie algebras.) It is classically well known due to Witt [22] that each $\mathcal{L}_n(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$r_n(k) := \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}, \tag{3}$$

where μ is the Möbius function. For example, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathcal{L}_n(k)$ for $1 \leq k \leq 3$ is given by

$$\mathcal{L}_n(1) = H, \quad \mathcal{L}_n(2) = \Lambda^2 H,$$
$$\mathcal{L}_n(3) = (H \otimes_{\mathbf{Z}} \Lambda^2 H) \big/ \langle x \otimes y \wedge z + y \otimes z \wedge x + z \otimes x \wedge y \mid x, y, z \in H \rangle.$$

Next, we consider an embedding of the free Lie algebra into the tensor algebra. Let $T(H)$ be the tensor algebra of H over $\mathbf{Z}$. Then $T(H)$ is the universal enveloping algebra of the free Lie algebra $\mathcal{L}_n$, and the natural map $\iota : \mathcal{L}_n \to T(H)$ defined by

$$[X, Y] \mapsto X \otimes Y - Y \otimes X$$

for X, $Y \in \mathcal{L}_n$ is an injective graded Lie algebra homomorphism. We denote by ι_k the homomorphism of degree k part of ι and consider $\mathcal{L}_n(k)$ as a submodule $H^{\otimes k}$ through ι_k.

Now, we consider a Lie subalgebra of $\mathcal{L}_n$ generated by $x_2, \ldots, x_n$. Let F be a subgroup of F_n generated by $x_2, x_3, \ldots, x_n$. The group F is a free group of rank $n-1$. We denote the lower central series of F by $\Gamma_F(1), \Gamma_F(2), \cdots$ and write $\mathcal{L}_F(k)$ for its graded quotient $\Gamma_F(k)/\Gamma_F(k+1)$ for each $k \geq 1$. Clearly, $\mathcal{L}_F(k) \cong \mathcal{L}_{n-1}(k)$ as an abelian group. The associated graded sum $\mathcal{L}_F := \bigoplus_{k\geq 1} \mathcal{L}_F(k)/\mathcal{L}_F(k+1)$ also has a graded Lie algebra structure. By the elimination theorem of free Lie algebras, we see that $\mathcal{L}_F$ is a direct summand of $\mathcal{L}_n$. (See Proposition 10 in [3].) Hence, in particular, $\mathcal{L}_F(k)$ is a direct summand of $\mathcal{L}_n(k)$ for each $k \geq 1$.

2.4. **Johnson filtration.** In this subsection, we recall the Johnson filtration of $\mathrm{Aut}\, F_n$. For $k \geq 0$, the action of $\mathrm{Aut}\, F_n$ on each nilpotent quotient $F_n/\Gamma_n(k+1)$ induces a homomorphism

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)).$$

We denote the kernel of the homomorphism above by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, with $\mathcal{A}_n(1) = IA_n$. We call it the Johnson filtration of $\mathrm{Aut}\, F_n$. For each $k \geq 1$, the group $\mathrm{Aut}\, F_n$ acts on $\mathcal{A}_n(k)$ by conjugation, and it naturally induces an action of $\mathrm{GL}(n, \mathbf{Z})$ on $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$. The graded sum

$\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k\geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ has a graded Lie algebra structure induced from the commutator bracket on IA_n.

In order to study the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$, the Johnson homomorphisms are used. For each $k \geq 1$, define a homomorphism $\mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \mapsto (x \mapsto x^{-1}x^{\sigma}), \quad x \in H.$$

Then the kernel of this homomorphism is just $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)).$$

The homomorphism τ_k is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$.

Historically, the study of the Johnson homomorphisms was begun in 1980 by D. Johnson [8]. He studied the Johnson homomorphism of a mapping class group of a closed oriented surface and determined the abelianization of the Torelli group. (See [9].) There is a broad range of remarkable results for the Johnson homomorphisms of a mapping class group. (For example, see [7] and [15], [16], [17].)

In general, since τ_k is injective, to determine the image (or equivalently, the cokernel) of τ_k is an important problem in the study of the structure of $\mathrm{gr}^k(\mathcal{A}_n)$. In this paper, we use the Johnson homomorphism only in the proof of Theorem 4.1.

3. McCool stabilizer subgroup

Here we consider the McCool stabilizer subgroup. In the following, we assume $n \geq 3$. Let S_n be the subgroup of $\mathrm{Aut}\, F_n$ consisting of automorphisms which fix each of $x_2, \ldots, x_n$. We call S_n the McCool stabilizer subgroup. We denote the intersection of S_n with IA_n by IS_n. McCool [14] showed that IS_n is finitely generated but not finitely presentable. He [14] also gave an infinite presentation of IS_n.

For any $i \in \{2, \ldots, n\}$, let v_i be the automorphism of F_n which sends x_1 to x_1x_i and fix the other generators x_t. The subgroup V of $\mathrm{Aut}\, F_n$ generated by all v_i is a free group of rank $n-1$. The subgroup W of IA_n generated by all K_{1i} is also a free group of rank $n-1$. Then McCool [14] showed that IS_n is a semidirect product of $[V, V]$ by W. Namely, we have a split group extension

$$1 \to [V, V] \to \mathrm{IS}_n \to W \to 1. \tag{4}$$

Furthermore, in [14] he showed that $[V, V]$ is the normal closure of $\{K_{1ij} \,|\, i > j\}$ in IS_n, and IS_n is generated by K_{1i} and K_{1ij}. Thus, considering a homomorphism $\mathrm{IS}_n \hookrightarrow \mathrm{IA}_n \to \mathrm{IA}^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H$, we see that $H_1(\mathrm{IS}_n, \mathbf{Z})$ is a free abelian group of rank $n(n-1)/2$ with basis $\{K_{1i}, K_{1jk} \,|\, 2 \leq i, j, k \leq n,\ j > k\}$.

In this paper, we consider the Johnson filtration of $\mathrm{Aut}\, F_n$ restricted to S_n. Namely, set $\mathcal{S}_n(k) := \mathcal{A}_n(k) \cap \mathrm{S}_n$ for each $k \geq 0$. Then $\mathcal{S}_n(0) = \mathrm{S}_n$ and $\mathcal{S}_n(1) = \mathrm{IS}_n$. We call a descending central filtration

$$\mathrm{S}_n = \mathcal{S}_n(0) \supset \mathcal{S}_n(1) \supset \mathcal{S}_n(2) \supset \cdots$$

the Johnson filtration of S_n. Set $\mathrm{gr}^k(\mathcal{S}_n) := \mathcal{S}_n(k)/\mathcal{S}_n(k+1)$. We denote by τ_k^S the restriction of the Johnson homomorphism τ_k to $\mathrm{gr}^k(\mathcal{S}_n) \subset \mathrm{gr}^k(\mathcal{A}_n)$ and call it the Johnson homomorphism of S_n.

Let $\mathcal{S}'_n(1) \supset \mathcal{S}'_n(2) \supset \cdots$ be the lower central series of $\mathrm{IS}_n = \mathcal{S}_n(1)$. Clearly, we have $\mathcal{S}'_n(k) \subset \mathcal{S}_n(k)$ for each $k \geq 1$. Here we determine the group structure of $\mathcal{S}'_n(k)$. To do this, we prepare some notation. For any $x \in F$, let v_x be an automorphism of F_n which maps x_1 to $x_1 x$ and fixes the other generators x_t. Then a map $\psi_V : F \to V$ defined by $\psi_V(x) := v_x$ is an isomorphism. Similarly, for any $y \in F$, let w_y be an automorphism of F_n which maps x_1 to $y^{-1}x_1 y$ and fixes the other generators x_t. Then a map $\psi_W : F \to W$ defined by $\psi_W(y) := w_y$ is also an isomorphism. We denote the lower central series of V and W by $\Gamma_V(k)$ and $\Gamma_W(k)$ respectively. Then we have

Lemma 3.1. *For each $k \geq 1$, $\mathcal{S}'_n(k)$ is a semidirect product of $\Gamma_V(k+1)$ and $\Gamma_W(k)$:*

$$\mathcal{S}'_n(k) = \Gamma_V(k+1) \rtimes \Gamma_W(k).$$

Proof. It is easily seen that

- $\Gamma_W(k)$ is a subgroup of $\mathcal{S}'_n(k)$,
- $\Gamma_V(k+1)$ is a normal subgroup of $\mathcal{S}'_n(k)$ and
- $\Gamma_V(k+1) \cap \Gamma_W(k) = 1$

for each $k \geq 1$. We leave proofs of them to the reader as exercises. Hence it suffices to show that $\mathcal{S}'_n(k) = \Gamma_V(k+1)\Gamma_W(k)$. We prove this by induction on k. It is clear for $k = 1$ by (4). Suppose $k \geq 2$. By the inductive hypothesis, we have $\mathcal{S}'_n(k-1) = \Gamma_V(k)\Gamma_W(k-1)$. It suffices to show that for any $\sigma \in \mathcal{S}'_n(k-1)$ and $\sigma' \in \mathcal{S}'_n(1)$, the commutator $[\sigma, \sigma']$ is in $\Gamma_V(k+1)\Gamma_W(k)$ since $\mathcal{S}'_n(k)$ is generated by all elements of type $[\sigma, \sigma']$.

Let $\sigma = vw$ and $\sigma' = v'w'$ for $v \in \Gamma_V(k)$, $v' \in \Gamma_V(2)$, $w \in \Gamma_W(k-1)$ and $w' \in \Gamma_W(1)$. Then we see that

$$\begin{aligned}[\sigma, \sigma'] &= [vw, v'w'] \\ &= v\,(wv'w^{-1})\,(ww'w^{-1}v^{-1}w{w'}^{-1}w^{-1})\,([w,w']{v'}^{-1}[w',w])\,[w,w'].\end{aligned}$$

Since $[w, w'] \in \Gamma_W(k)$, we show that

$$\sigma'' := v\,(wv'w^{-1})\,(ww'w^{-1}v^{-1}w{w'}^{-1}w^{-1})\,([w,w']{v'}^{-1}[w',w]) \in \Gamma_V(k+1).$$

Now, set

$$v = v_x, \quad v' = v_{x'}, \quad w = w_y, \quad w' = w_{y'}$$

for $x \in \Gamma_F(k)$, $x' \in \Gamma_F(2)$, $y \in \Gamma_F(k-1)$ and $y' \in \Gamma_F(1)$. Then we have

$$x_1^{\sigma''} = x_1\,([y,y']{x'}^{-1}[y',y])\,(yy'y^{-1}x^{-1}y{y'}^{-1}y^{-1})\,(yx'y^{-1})x.$$

Therefore it suffices to show that

$$z := ([y,y']{x'}^{-1}[y',y])\,(yy'y^{-1}x^{-1}y{y'}^{-1}y^{-1})\,(yx'y^{-1})x \in \Gamma_F(k+1).$$

Since $x^{\pm 1}$, $[y, y'] \in \Gamma_F(k)$ commutes with x', y and y' modulo $\Gamma_F(k+1)$,

$$z \equiv ([y,y']{x'}^{-1}[y',y])\,(yx'y^{-1}) \equiv {x'}^{-1}\,(yx'y^{-1}) = [{x'}^{-1}, y] \equiv 0$$

modulo $\Gamma_F(k+1)$. Hence we obtain

$$\sigma'' = \psi_V(z) \in \Gamma_V(k+1).$$

This completes the proof of Lemma 3.1. $\square$

From Lemma 3.1, we see that $\sigma \in \mathcal{S}'_n(k)$ if and only if

$$x_1^{\sigma} = y^{-1}x_1 y x$$

for some $x \in \Gamma_F(k+1)$ and $y \in \Gamma_F(k)$. Furthermore, for any $v_x \in \Gamma_V(k+1)$ and $w_y \in \Gamma_W(k)$ for $x \in \Gamma_F(k+1)$ and $y \in \Gamma_F(k)$, we have

$$(5)\quad w_y^{-1} v_x w_y = v_{y^{-1}xy} \in \Gamma_V(k+1) \text{ and } [w_y, v_x] = v_{[x^{-1},y^{-1}]} \in [\Gamma_V(k+1), \Gamma_V(k)].$$

In particular,

$$w_y^{-1} v_x w_y \equiv v_x \pmod{\Gamma_V(k+2)}. \tag{6}$$

Using Lemma 3.1, we can determine the group structure of the graded quotients of the lower central series $\mathcal{S}'_n(k)$. Set $\mathrm{gr}^k(\mathcal{S}'_n) := \mathcal{S}'_n(k)/\mathcal{S}'_n(k+1)$ for each k.

Proposition 3.1. *For each $k \geq 1$, $\mathrm{gr}^k(\mathcal{S}'_n) \cong \mathcal{L}_F(k+1) \oplus \mathcal{L}_F(k)$ as a $\mathbf{Z}$-module.*

Proof. Consider a surjective map $\Gamma_V(k+1) \rtimes \Gamma_W(k) \to \mathcal{L}_F(k+1) \oplus \mathcal{L}_F(k)$ defined by

$$vw \mapsto (\psi_V^{-1}(v), \psi_W^{-1}(w)).$$

Since $\Gamma_W(k)$ acts trivially on $\mathcal{L}_F(k+1)$ from (6), this map is a homomorphism whose kernel is exactly $\Gamma_V(k+2) \rtimes \Gamma_W(k+1)$. Hence we obtain Proposition 3.1. □

As a corollary, we have

Corollary 3.1. *For each $k \geq 1$, $\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{S}'_n)) = r_{n-1}(k+1) + r_{n-1}(k)$.*

Finally, we remark that the abelianization of $\mathcal{S}_n(k)'$ is not finitely generated. More precisely, we have

Corollary 3.2. *For each $n \geq 3$ and $k \geq 2$,*

$$H_1(\mathcal{S}'_n(k), \mathbf{Z}) \cong \big(\Gamma_F(k+1)/[\Gamma_F(k+1), \Gamma_F(k)]\big) \oplus H_1(\Gamma_F(k), \mathbf{Z}).$$

Proof. This equation immediately follows from Lemma 3.1 and (5). □

4. The Johnson filtration of IS_n

In this section, we show that the Johnson filtaration $\mathcal{S}_n(1) \supset \mathcal{S}_n(2) \supset \cdots$ coincides with the lower central series of IS_n.

For each $k \geq 1$, let $\mathcal{E}_k$ be a $\mathbf{Z}$-submodule of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ consisting of all elements of type $x_1^* \otimes [B, x_1]$ where $B \in \mathcal{L}_F(k)$.

Lemma 4.1. *For any $k \geq 1$, $\mathcal{E}_k \cong \mathcal{L}_F(k)$ as an abelian group.*

Proof. Let $f_k : \mathcal{L}_F(k) \to \mathcal{E}_k$ be a homomorphism defined by $f_k(B) := x_1^* \otimes [B, x_1]$ for any $B \in \mathcal{L}_F(k)$. We construct the inverse of f_k as follows. First, using a contraction, we define homomorphisms $\mu^k : H^* \otimes_{\mathbf{Z}} H^{\otimes(k+1)} \to H^{\otimes k}$ by

$$x_i^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_{k+1}} \mapsto -x_i^*(x_{j_1}) \cdot x_{j_2} \otimes \cdots \otimes x_{j_{k+1}}$$

and

$$\Phi^k := \mu^k \circ (id_{H^*} \otimes \iota_n^{k+1}) : H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1) \to H^{\otimes k}.$$

We denote the restriction of Φ^k to $\mathcal{E}_k$ by g_k. Then identifying $\mathcal{L}_F(k)$ with the image of a natural injective homomorphism $\iota_k : \mathcal{L}_F(k) \to H^{\otimes k}$, we obtain a homomorphism

$$g_k : \mathcal{E}_k \to \mathcal{L}_F(k). \tag{7}$$

It is easily seen that g_k is the inverse of f_k. Hence, we obtain the lemma. □

Next, let $\mathcal{T}_k$ be a $\mathbf{Z}$-submudule of $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ consisting of all elements of type $x_1^* \otimes A$ where $A \in \mathcal{L}_F(k+1)$. Clearly, we have $\mathcal{T}_k \cong \mathcal{L}_F(k+1)$ for any $k \geq 1$. Furthermore, from the elimination theorem of free Lie algebras, we see that the sum $\mathcal{E}_k + \mathcal{T}_k$ in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$ as a $\mathbf{Z}$-module is a direct sum $\mathcal{E}_k \oplus \mathcal{T}_k$ for any $k \geq 1$. (See Proposition 10 in [3].) Then we show our main theorem.

Theorem 4.1. *For each $k \geq 1$, we have $\mathcal{S}_n(k) = \mathcal{S}'_n(k)$.*

Proof. It suffices to show $\mathcal{S}_n(k) \subset \mathcal{S}'_n(k)$ for each $k \geq 1$. For any $\sigma \in \mathcal{S}_n(k)$, by the split extension (4), there are some $v \in [V, V]$ and $w \in W$ such that $\sigma = vw$. Set $x_1^v := x_1 x$ and $x_1^w := y^{-1} x_1 y$ for $x, y \in F$. Then $x_1^{-1} x_1^\sigma = [x_1^{-1}, y^{-1}]x \in \Gamma_n(k+1)$.

First, we show $y \in \Gamma_F(k)$ and $x \in \Gamma_F(k+1)$. If $y \notin \Gamma_F(k)$, there is some $l \in \{1, \ldots, k-1\}$ such that $y \in \Gamma_F(l)$ and $y \notin \Gamma_F(l+1)$. Since both $[x_1^{-1}, y^{-1}]$ and $[x_1^{-1}, y^{-1}]x$ belong to $\Gamma_n(l+1)$, so does x. Hence we have

$$\tau_l^S(\sigma) = x_1^* \otimes x_1^{-1} x_1^\sigma = x_1^* \otimes ([x_1^{-1}, y^{-1}]x) = x_1^* \otimes [x_1, y] + x_1^* \otimes x$$

in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(l+1)$. On the other hand, by $[x_1^{-1}, y^{-1}]x \in \Gamma_n(k+1) \subset \Gamma_n(l+2)$, we see $\tau_l^S(\sigma) = 0$. Since $x_1^* \otimes [x_1, y] \in \mathcal{E}_l$ and $x_1^* \otimes x \in \mathcal{T}_l$ and since $\mathcal{E}_l \cap \mathcal{T}_l = 0$ as mentioned above, we have $x_1^* \otimes [x_1, y] = x_1^* \otimes x = 0$ in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(l+1)$. Then, considering the isomorphism $g_l : \mathcal{E}_l \to \mathcal{L}_F(l)$ defined in (7), we obtain $y = 0 \in \mathcal{L}_F(l)$. Hence $y \in \Gamma_F(l+1)$. This is a contradiction. Therefore we obtain $y \in \Gamma_F(k)$ and $x \in \Gamma_F(k+1)$. This shows that $\sigma \in \mathcal{S}'_n(k)$ by the remark after Lemma 3.1. This completes the proof of Theorem 4.1. $\square$

From Theorem 4.1 and Corollary 3.1, we have

Corollary 4.1. *For each $k \geq 1$, $\mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{S}_n)) = r_{n-1}(k+1) + r_{n-1}(k)$.*

Acknowledgments

The author would like to express his sincere gratitude to the referee for helpful comments and correcting many typos and grammatical mistakes. This research is supported by a JSPS Research Fellowship for Young Scientists.

References

[1] S. Andreadakis; On the automorphisms of free groups and free nilpotent groups, Proc. London Math. Soc. (3) 15 (1965), 239-268. MR0188307 (32:5746)

[2] Y. A. Bakhturin; Identities in Lie algebras, Nauka, Moscow 1985; English translation, Identical relations in Lie Algebras, VNU Science Press, Utrecht (1987). MR886063 (88f:17032)

[3] N. Bourbaki; Lie Groups and Lie Algebras, Chapters 1-3, Springer-Verlag (1989). MR979493 (89k:17001)

[4] F. Cohen and J. Pakianathan; On Automorphism Groups of Free Groups, and Their Nilpotent Quotients, preprint.

[5] F. Cohen and J. Pakianathan; On subgroups of the automorphism group of a free group and associated graded Lie algebras, preprint.

[6] B. Farb; Automorphisms of F_n which act trivially on homology, in preparation.

[7] R. Hain; Infinitesimal presentations of the Torelli groups, Journal of the American Mathematical Society 10 (1997), 597-651. MR1431828 (97k:14024)

[8] D. Johnson; An abelian quotient of the mapping class group, Math. Ann. 249 (1980), 225-242. MR579103 (82a:57008)

[9] D. Johnson; The strucure of the Torelli group. III: The abelianization of $\mathcal{I}_g$, Topology 24 (1985), 127-144. MR793179 (87a:57016)

[10] N. Kawazumi; Cohomological aspects of Magnus expansions, preprint, `arXiv:math.GT/0505497`.

[11] S. Krstić, J. McCool; The non-finite presentability in $IA(F_3)$ and $GL_2(\mathbf{Z}[t, t^{-1}])$, Invent. Math. 129 (1997), 595-606. MR1465336 (98h:20053)

[12] W. Magnus; Über n-dimensinale Gittertransformationen, Acta Math. 64 (1935), 353-367. MR1555401

[13] W. Magnus, A. Karras, D. Solitar; Combinatorial group theory, Interscience Publ., New York (1966). MR2109550 (2005h:20052)

[14] J. McCool; Some remarks on IA automorpshims of free groups, Can. J. Math., Vol. XL, no. 5 (1998), 1144-1155. MR973513 (90d:20057)

[15] S. Morita; Abelian quotients of subgroups of the mapping class group of surfaces, Duke Mathematical Journal 70 (1993), 699-726. MR1224104 (94d:57003)

[16] S. Morita; Structure of the mapping class groups of surfaces: a survey and a prospect, Geometry and Topology Monographs, Vol. 2, Geom. Topol. Publ., Coventry (1999), 349-406. MR1734418 (2000j:57039)

[17] S. Morita; Cohomological structure of the mapping class group and beyond, Proc. of Symposia in Pure Math., 74, Amer. Math. Soc., Providence, RI (2006), 329-354. MR2264550 (2007j:20079)

[18] J. Nielsen; Die Isomorphismen der allgemeinen unendlichen Gruppe mit zwei Erzeugenden, Math. Ann. 78 (1964), 385-397. MR1511907

[19] A. Pettet; The Johnson homomorphism and the second cohomology of IA_n, Algebraic and Geometric Topology 5 (2005) 725-740. MR2153110 (2006j:20050)

[20] C. Reutenauer; Free Lie Algebras, London Mathematical Society Monographs, New Series, 7, Oxford University Press, New York (1993). MR1231799 (94j:17002)

[21] T. Satoh; New obstructions for the surjectivity of the Johnson homomorphism of the automorphism group of a free group, Journal of the London Mathematical Society (2) 74 (2006), 341-360. MR2269583 (2007i:20060)

[22] E. Witt; Treue Darstellung Liescher Ringe, Journal für die Reine und Angewandte Mathematik, 177 (1937), 152-160.

Department of Mathematics, Graduate School of Science, Kyoto University, Kitashirakawaoiwake-cho, Sakyo-ku, Kyoto City, 606-8502, Japan

E-mail address: takao@math.kyoto-u.ac.jp