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THE ANDREWS-STANLEY PARTITION FUNCTION AND $p(n)$: CONGRUENCES

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Abstract. R. Stanley formulated a partition function $t(n)$ which counts the number of partitions π for which the number of odd parts of π is congruent to the number of odd parts in the conjugate partition π' (mod 4). In light of G. E. Andrews' work on this subject, it is natural to ask for relationships between $t(n)$ and the usual partition function $p(n)$. In particular, Andrews showed that the (mod 5) Ramanujan congruence for $p(n)$ also holds for $t(n)$. In this paper we extend his observation by showing that there are infinitely many arithmetic progressions $An + B$ such that for all $n \geq 0$,

$$t(An+B) \equiv p(An+B) \equiv 0 \pmod{l^j}$$

whenever $l \geq 5$ is prime and $j \geq 1$.

1. Introduction and statement of results

Let n be a nonnegative integer. Recall that $p(n)$ counts the number of partitions π of n, and $p(0)$ is defined to be 1. For a partition π, let $\mathcal{O}(\pi)$ be the number of odd parts in π. Let π' denote the conjugate partition, which is obtained by reading the columns (instead of the rows) of the Ferrers diagram for π [And98]. Stanley's partition function $t(n)$ counts the number of partitions π of n for which $\mathcal{O}(\pi) \equiv \mathcal{O}(\pi') \pmod 4$.

The generating function for $p(n)$ is known to equal the following infinite product (throughout let $q := e^{2\pi i z}$ and $q_h := e^{\frac{2\pi i z}{h}}$)

$$F(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}. \tag{1.1}$$

Notice that $F(q)$ converges absolutely for all $z \in \mathcal{H}$, the upper half of the complex plane. In [And04], Andrews proves that the generating function for $t(n)$ can also be written as an infinite product

$$G(q) := \sum_{n=0}^{\infty} t(n)q^n = \frac{F(q)F(q^4)^5F(q^{32})^2}{F(q^2)^2F(q^{16})^5}. \tag{1.2}$$

Andrews proved in [And04] that for all $n \geq 0$,

$$t(5n+4) \equiv 0 \pmod 5, \tag{1.3}$$

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showing that the classical (mod 5) Ramanujan congruence for $p(n)$ also holds for $t(n)$. He did this by proving a certain partition identity using q-series, which he said "cries out for a combinatorial proof." Yee, Sills, and Boulet have all proven this identity combinatorially; see [Yee04], [Sil04], and [Bou06]. In addition, Berkovich and Garvan [BG06] have proven the congruence (1.3) combinatorially by deriving statistics related to the Andrews-Garvan crank and the 5-core crank which divide $t(5n+4)$ into 5 equinumerous classes. This extends the famous work of Garvan, Kim, and Stanton [GKS90].

It is natural to ask whether the congruence modulo 5 is just one of many congruences shared by $t(n)$ and $p(n)$. In celebrated work of Ono and Ahlgren it has been shown that there are vastly many congruences for $p(n)$ beyond the Ramanujan ones (see [Ono00], [Ahl00], and [AO01]). Here we show that the (mod 5) congruence $t(n)$ and $p(n)$ share is not an isolated example.

Theorem 1.1. *Let $l \geq 5$, $l \neq 13$, be prime and let j be a positive integer. There are infinitely many (non-nested) arithmetic progressions $An+B$ such that for all $n \geq 0$ we have*

$$t(An+B) \equiv p(An+B) \equiv 0 \pmod{l^j}.$$

Remark. In fact, Theorem 1.1 follows from a more precise statement (see Theorem 3.4).

In Section 2, we will recall some basic facts about integral and half-integral weight modular forms, the Shimura correspondence, and a theorem of Serre used in the proof of Theorem 1.1. In Section 3, we will give the proof of Theorem 1.1.

2. Preliminaries for proof of Theorem 1.1

First we fix some notation. Suppose $w \in \frac{1}{2}\mathbb{Z}$, N is a positive integer (which is divisible by 4 if $w \notin \mathbb{Z}$), and χ is a Dirichlet character (mod N). Let $M_w(\Gamma_0(N), \chi)$ (resp. $S_w(\Gamma_0(N), \chi)$) be the usual space of holomorphic modular forms (resp. cusp forms) on the congruence subgroup $\Gamma_0(N)$, with Nebentypus character χ. There are several operators which act on spaces of modular forms. The first we will discuss are Hecke operators. The results in this section and more information on modular forms can be found in [Ono04].

2.1. Hecke operators.

We will define Hecke operators for both integer and half-integer weight modular forms. We begin with the integer weight case.

Definition 2.1. With the notation above, we let $f(z) := \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$, where k is an integer and $p \nmid N$ is prime. The action of the Hecke operator T_p is defined by

$$f(z)|T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p)p^{k-1}a(n/p)\right) q^n,$$

where $a(n/p) = 0$ when $p \nmid n$.

We note that the half-integer weight case is somewhat different.

Definition 2.2. Let $f(z) := \sum_{n=0}^{\infty} a(n)q^n \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ where λ is an integer and $p \nmid 4N$ is prime. The action of the half-integral Hecke operator $T(p^2)$

is defined by

$$f(z)|T(p^2) := \sum_{n=0}^{\infty} \left(a(p^2n) + \chi(p) \left(\frac{(-1)^\lambda}{p} \right) \left(\frac{n}{p} \right) p^{\lambda-1} a(n) \right.$$
$$\left. + \chi(p^2) \left(\frac{(-1)^\lambda}{p^2} \right) p^{2\lambda-1} a(n/p^2) \right) q^n,$$

where $a(n/p^2) = 0$ when $p^2 \nmid n$.

These operators are useful due to the following.

Proposition 2.3. *If $f(z) \in M_k(\Gamma_0(N), \chi)$, where $k \in \mathbb{Z}$, then*

$$f(z)|T_p \in M_k(\Gamma_0(N), \chi).$$

Similarly, if $f(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$, where $\lambda \in \mathbb{Z}$, then

$$f(z)|T(p^2) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi).$$

Furthermore, both T_p and $T(p^2)$ take cusp forms to cusp forms.

2.2. **Other operators.** We define two other operators U and V which act on formal power series. If d is a positive integer, then we define the U-operator $U(d)$ by

$$\left(\sum_{n=-\infty}^{\infty} c(n) q^n \right) \Bigg| U(d) := \sum_{n=-\infty}^{\infty} c(dn) q^n \tag{2.1}$$

and the V-operator $V(d)$ by

$$\left(\sum_{n=-\infty}^{\infty} c(n) q^n \right) \Bigg| V(d) := \sum_{n=-\infty}^{\infty} c(n) q^{dn}. \tag{2.2}$$

These two operators also act on spaces of modular forms. First we consider the integer weight case.

Proposition 2.4. *Suppose $f(z) \in M_k(\Gamma_0(N), \chi)$ where $k \in \mathbb{Z}$.*

(1) *If d is a positive integer and $d|N$, then*

$$f(z)|U(d) \in M_k(\Gamma_0(N), \chi).$$

(2) *If d is any positive integer, then*

$$f(z)|V(d) \in M_k(\Gamma_0(Nd), \chi).$$

Moreover, both $U(d)$ and $V(d)$ take cusp forms to cusp forms.

The half-integer weight case is slightly more complicated.

Proposition 2.5. *Suppose $f(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ where $\lambda \in \mathbb{Z}$.*

(1) *If d is a positive integer and $d|N$, then*

$$f(z)|U(d) \in M_{\lambda+\frac{1}{2}} \left((\Gamma_0(4N), \left(\frac{4d}{\bullet} \right) \chi \right).$$

(2) *If d is any positive integer, then*

$$f(z)|V(d) \in M_{\lambda+\frac{1}{2}} \left((\Gamma_0(4Nd), \left(\frac{4d}{\bullet} \right) \chi \right).$$

Furthermore, again for the half-integral case both $U(d)$ and $V(d)$ take cusp forms to cusp forms.

2.3. **Shimura correspondence.** Here we recall the famous "Shimura correspondences" [Shi73] which give a means of mapping half-integer weight cusp forms to even integer weight modular forms.

Definition 2.6. Let $f(z) = \sum_{n=1}^{\infty} c(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$, with $\lambda \geq 1$, and let t be a square-free integer. Define the Dirichlet character ψ_t by $\psi_t(n)=\chi(n)\left(\frac{-1}{n}\right)^{\lambda}\left(\frac{t}{n}\right)$. If we define complex numbers $A_t(n)$ by

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s-\lambda+1, \psi_t) \cdot \sum_{n=1}^{\infty} \frac{c(tn^2)}{n^s},$$

then

$$S_t(f(z)) := \sum_{n=1}^{\infty} A_t(n)q^n$$

is a modular form in $M_{2\lambda}(\Gamma_0(2N), \chi^2)$. Furthermore, if $\lambda \geq 2$, then $S_t(f(z))$ is a cusp form. When $\lambda = 1$ there are conditions (here omitted) which guarantee $S_t(f(z))$ is a cusp form.

From the definition above, it is not hard to show that the Shimura correspondences commute with the Hecke operators in the following way.

Proposition 2.7. *Let $f(z) \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ with $\lambda \geq 1$. If t is a square-free integer and $p \nmid 4Nt$ is prime, then*

$$S_t(f(z)|T(p^2)) = S_t(f(z))|T_p.$$

2.4. **A theorem of Serre.** First we recall that the following subgroup of $\Gamma_0(N)$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N} \text{ and } c \equiv 0 \pmod{N} \right\}$$

has the properties

$$M_k(\Gamma_1(N)) = \bigoplus_{\chi} M_k(\Gamma_0(N), \chi), \tag{2.3}$$

$$S_k(\Gamma_1(N)) = \bigoplus_{\chi} S_k(\Gamma_0(N), \chi). \tag{2.4}$$

The following theorem due to Serre (see [Ono00] and [Ser76]) is a crucial component to the proof of Theorem 1.1. The theorem arises from the existence of certain Galois representations with special properties. More details can be found in [Ono04] and [Ser76].

Theorem 2.8. *Let $l \geq 5$ be prime and $k \in \mathbb{Z}$. A positive proportion of the primes $p \equiv -1 \pmod{N}$ have the property that*

$$f(z)|T_p \equiv 0 \pmod{l}$$

for every $f(z)$ that is the reduction modulo l of a cusp form in $S_k(\Gamma_1(N)) \cap \mathbb{Z}[[q]]$.

3. Proof of Theorem 1.1

We will see that using the works of Serre and Shimura, the proof of Theorem 1.1 boils down to the existence of a half-integral weight cusp form satisfying certain key properties. This is given in the following theorem. We make the following notation for ease of exposition:

$$G_l(z) := \sum_{\substack{n\geq 0 \\ ln\equiv -1 \pmod{24}}} t\left(\frac{ln+1}{24}\right) q^n.$$

Theorem 3.1. *Let $j \geq 1$ be a positive integer and $l \geq 5$ prime, $l \neq 13$. For an appropriately chosen positive integer K, there is a cusp form*

$$g_{l,j}(z) \in S_{\frac{Kl^{j+1}-Kl^{j-1}-1}{2}}\left(\Gamma_1(2304l^2)\right)$$

having integer coefficients such that

$$G_l(z) \equiv g_{l,j}(z) \pmod{l^j}.$$

3.1. **Deduction of Theorem 1.1 from Theorem 3.1.** Before we prove Theorem 3.1, we will show how Theorem 1.1 is proved from it. We use techniques of Ono and Ahlgren previously used on $p(n)$ (see [Ono00] and [Ahl00]) and extend them to work on $t(n)$ and $p(n)$ simultaneously.

Consider the Shimura lift of $g_{l,j}(z)$ to an integer weight cusp form with integer coefficients. We have that

$$S_t(g_{l,j}) \in S_{Kl^{j+1}-Kl^{j-1}-2}(\Gamma_1(2304l^2)) \cap \mathbb{Z}[[q]]. \tag{3.1}$$

By Theorem 2.8, there are infinitely many primes $p \equiv -1 \pmod{2304l^2}$ such that every reduction $\pmod{l^j}$ of every form in the space $S_{Kl^{j+1}-Kl^{j-1}-2}(\Gamma_1(2304l^2)) \cap \mathbb{Z}[[q]]$ gets annihilated $\pmod{l^j}$ by the (integer weight) Hecke operator T_p. In particular, by equation (3.1) and Proposition 2.7, it follows that there are infinitely many primes $p \equiv -1 \pmod{2304l^2}$ such that for any square-free integer t,

$$S_t(g_{l,j}(z)|T(p^2)) = S_t(g_{l,j}(z))|T_p \equiv 0 \pmod{l^j}.$$

Thus, in particular,

$$g_{l,j}(z)|T(p^2) \equiv 0 \pmod{l^j}. \tag{3.2}$$

If we write $g_{l,j}(z) = \sum_{n=1}^{\infty} c(n)q^n$ and let $\lambda_{l,j} = (Kl^{j+1} - Kl^{j-1} - 2)/2$, then Definition 2.2 and (3.2) say that

$$\sum_{n=1}^{\infty}\left(c(p^2n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p}\right)\left(\frac{n}{p}\right)p^{\lambda_{l,j}-1}c(n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p^2}\right)p^{2\lambda_{l,j}-1}c\left(\frac{n}{p^2}\right)\right)q^n \equiv 0 \pmod{l^j}.$$

Thus for each of the infinitely many primes p for which equation (3.2) holds, we have

$$c(p^2n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p}\right)\left(\frac{n}{p}\right)p^{\lambda_{l,j}-1}c(n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p^2}\right)p^{2\lambda_{l,j}-1}c\left(\frac{n}{p^2}\right) \equiv 0 \pmod{l^j}$$

for all positive integers n. Replacing n by np, the middle term vanishes to give us that

$$c(p^3n) + \left(\frac{(-1)^{\lambda_{l,j}}}{p^2}\right) p^{2\lambda_{l,j}-1} c\left(\frac{n}{p}\right) \equiv 0 \pmod{l^j}. \tag{3.3}$$

We restrict our attention further by only considering n which are not divisible by p. For these n, $c(n/p)$ is defined to be 0, so equation (3.3) becomes

$$c(p^3n) \equiv 0 \pmod{l^j}.$$

Combining this with Theorem 3.1 we obtain the following proposition.

Proposition 3.2. *A positive proportion of the primes $p \equiv -1 \pmod{2304l^2}$ have the property that*

$$t\left(\frac{p^3ln+1}{24}\right) \equiv 0 \pmod{l^j}$$

for all n, where $ln \equiv -1 \pmod{24}$ and $(n,p)=1$.

The work of Ono and Ahlgren intersects nicely with this analysis of $t(n)$. Define $F_l(z)$ by

$$F_l(z) := \sum_{\substack{n\geq 0 \\ ln\equiv -1 \pmod{24}}} p\left(\frac{ln+1}{24}\right) q^n.$$

In [Ahl00] (see Theorem 1), Ahlgren proves the following theorem.

Theorem 3.3. *Let $l \geq 5$ be prime and $j \geq 1$ be a positive integer. There exists a cusp form*

$$f_{l,j}(z) \in S_{\frac{l^j - l^{j-1} - 1}{2}}\left(\Gamma_0(576l), \left(\frac{12}{\bullet}\right)\right)$$

such that

$$f_{l,j}(z) \equiv F_l(z) \pmod{l^j}.$$

Notice that both $\eta(z)^{l^2}/\eta(l^2z) \equiv 1 \pmod{l}$ and $\eta(z)^l/\eta(lz) \equiv 1 \pmod{l}$. By induction we can argue that for any $j \geq 1$,

$$\frac{\eta(z)^{l^{j+1}}}{\eta(l^2z)^{l^{j-1}}} \equiv 1 \pmod{l^j} \text{ and } \frac{\eta(z)^{l^j}}{\eta(lz)^{l^{j-1}}} \equiv 1 \pmod{l^j}. \tag{3.4}$$

Define

$$\tilde{f}_{l,j}(z) := f_{l,j}(z) \cdot \left(\frac{\eta(z)^{l^{j+1}}}{\eta(l^2z)^{l^{j-1}}}\right)^K \cdot \left(\frac{\eta(z)^{l^j}}{\eta(lz)^{l^{j-1}}}\right)^{-1}. \tag{3.5}$$

Then

$$\tilde{f}_{l,j}(z) \equiv f_{l,j}(z) \pmod{l^j}.$$

Since the second and third factors in (3.5) are holomorphic modular forms on $\Gamma_0(l^2)$, we see that

$$\tilde{f}_{l,j}(z) \in S_{\frac{Kl^{j+1} - Kl^{j-1} - 1}{2}}\left(\Gamma_1(576l^2)\right).$$

We can realize $\tilde{f}_{l,j}(z)$ as a cusp form on the group $\Gamma_1(2304l^2)$. Serre's Theorem gives a statement about every reduction $\pmod{l^j}$ of a cusp form in

$$S_{Kl^{j+1} - Kl^{j-1} - 2}(\Gamma_1(2304l^2)) \cap \mathbb{Z}[[q]];$$

thus we can apply Theorem 2.8 to $S_t(f_{l,j}(z))$ and $S_t(g_{l,j}(z))$ simultaneously. We conclude the following:

Theorem 3.4. *Let $l \geq 5$, $l \neq 13$, be prime. A positive proportion of the primes $p \equiv -1 \pmod{2304 l^2}$ have the property that*

$$t\left(\frac{p^3 ln+1}{24}\right) \equiv p\left(\frac{p^3 ln+1}{24}\right) \equiv 0 \pmod{l^j}$$

for all positive integers n, where $(p, n) = 1$.

Theorem 1.1 follows easily from Theorem 3.4.

3.2. **Preliminaries for the proof of Theorem 3.1.** Now it only remains to prove Theorem 3.1. Recall Dedekind's eta-function:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n). \tag{3.6}$$

From equations (1.1) and (1.2) we deduce that

$$g(z) = \sum_{n=0}^{\infty} t(n) q^{n-\frac{1}{24}} = \frac{\eta(2z)^2 \eta(16z)^5}{\eta(z)\eta(4z)^5\eta(32z)^2}.$$

For ease of notation, we make the following abbreviations. Let δ_l be the positive integer

$$\delta_l := \frac{l^2-1}{24},$$

and define $1 \leq \beta_l \leq l-1$ such that

$$24\beta_l \equiv 1 \pmod{l}.$$

Thus we have that

$$\eta(lz)^l \cdot g(z) = \prod_{n=1}^{\infty}(1-q^{ln})^l \sum_{n=0}^{\infty} t(n) q^{n+\delta_l}.$$

Using the definitions of the U- and V-operators, we obtain

$$\left[\frac{\eta(lz)^l \cdot g(z) | U(l)}{\eta(z)^l}\right] \Bigg| V(24) = \sum_{n=0}^{\infty} t(ln+\beta_l) q^{24n+\frac{24\beta_l - 1}{l}}. \tag{3.7}$$

Letting $k = 24n + (24\beta_l - 1)/l$, we see that

$$ln + \beta_l = \frac{lk+1}{24}.$$

Thus we can show that in fact

$$\left[\frac{\eta(lz)^l \cdot g(z) | U(l)}{\eta(z)^l}\right] \Bigg| V(24) = G_l(z). \tag{3.8}$$

Write $E_j(z) = \frac{\eta(z)^{l^2}}{\eta(l^2 z)}$. It is clear from the definitions of the U- and V-operators that they preserve congruences. Thus in light of (3.4), we have that for any $K \in \mathbb{N}$

$$g_{l,j}(z) := \left[\frac{(\eta(lz)^l \cdot g(z)) | U(l) \cdot E_j(z)^K}{\eta(z)^l}\right] \Bigg| V(24) \equiv G_l(z) \pmod{l^j}.$$

By properties of eta-quotients and the U-, V-operators, we see that $g_{l,j}(z)$ has integer coefficients and that $g_{l,j}(z)$ is modular over the group $\Gamma_0(2304 l^2)$ with weight $\frac{Kl^{j+1} - Kl^{j-1} - 1}{2}$. We wish to show that $g_{l,j}(z)$ vanishes at all the cusps of $\Gamma_0(2304 l^2)$.

3.3. **Proof of Theorem 3.1.** A complete set of representatives for the cusps of $\Gamma_0(N)$, where N is a positive integer, is given by [Mar96]:

(3.9)
$$\left\{\frac{a_c}{c} \in \mathbb{Q} : c \mid N\,,\, 1 \leq a_c \leq N\,,\, \gcd(a_c, N) = 1\,,\, a_c \text{ distinct modulo } \gcd\left(c, \frac{N}{c}\right)\right\}.$$

We recall the definition of the slash operator [Kob93]. If $f(z)$ is a function on the upper half-plane, $\lambda \in \frac{1}{2}\mathbb{Z}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, then

$$f(z) \mid_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (ad - bc)^{\frac{\lambda}{2}} \cdot (cz + d)^{-\lambda} \cdot f\left(\frac{az + b}{cz + d}\right). \tag{3.10}$$

Moreover, let $\gamma_{\frac{a}{c}}$ be the matrix in $SL_2(\mathbb{Z})$ that takes ∞ to $\frac{a}{c}$. We know from [Mar96] that the expansion of a modular form $f(z)$ of weight $\lambda \in \mathbb{R}$ at the cusp $\frac{a}{c}$ is of the form

$$f(z)|_\lambda\, \gamma_{\frac{a}{c}} = k \cdot q^\alpha + \cdots$$

for some nonzero constant k and $\alpha \in \mathbb{Q}$. Thus α is the order of vanishing of $f(z)$ at the cusp $\frac{a}{c}$. We are interested in the expansion of $g_{l,j}(z)$ at the cusps of $\Gamma_0(32l)$. However, $V(24)$ cannot introduce poles so it suffices to consider the Laurent series of

$$\left[\frac{(\eta(lz)^l \cdot g(z))|U(l) \cdot E_j(z)^K}{\eta(z)^l}\right].$$

It is a fact [Mar96] that for all $\gamma \in SL_2(\mathbb{Z})$,

$$\eta(z)^l|_{\frac{l}{2}}\gamma = k \cdot q^{\frac{l}{24}} + \cdots .$$

Since $(\eta(lz)^l \cdot g(z))|U(l) \cdot E_j(z)^K$ is modular on $\Gamma_0(32l^2)$, all that remains to be shown is that $(\eta(lz)^l \cdot g(z))|U(l) \cdot E_j(z)^K$ has order of vanishing $> l/24$ at all cusps of $\Gamma_0(32l^2)$.

It is easy to compute ([Mar96]) that

$$\mathrm{ord}_{\frac{a}{c}} \mathrm{E_j(z)^K} = \begin{cases} 0 & \textit{if } l^2 \mid c, \\ Kl^{j-1}\delta_l & \textit{if } l \parallel c, \\ Kl^{j-3}(l^2+1)\delta_l & \textit{if } l \nmid c. \end{cases}$$

So we choose K large enough such that it kills all poles of $(\eta(lz)^l \cdot g(z))|U(l)$ at cusps $\frac{a}{c}$ with $l^2 \nmid c$.

For convenience, let $h(z) := \eta(lz)^l \cdot g(z)$. Also, let

$$A := \begin{pmatrix} a & b \\ cl^2 & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

correspond to the cusp a/cl^2 of $\Gamma_0(2304l^2)$.

We will use the following proposition which can be obtained easily from the definition of the U-operator.

Proposition 3.5. *If l is a prime positive integer and $P(z) = \sum_{n\geq 1} c(n)q^n$, then*

$$P(z)|U(l) = \frac{1}{l}\sum_{j=0}^{l-1} P\left(\frac{z+j}{l}\right).$$

By Proposition 3.5,

$$h(z)|U(l)|_{\frac{l-1}{2}}A = l^{\frac{l-5}{4}}\sum_{v=0}^{l-1} h(z)|_{\frac{l-1}{2}}\begin{pmatrix} 1 & v \\ 0 & l \end{pmatrix} A.$$

Define for each v, an integer w_v such that $2304|w_v$ and

$$w_v \equiv (a+vcl^2)^{-1}\cdot(b+vd) \pmod{l}.$$

Also, define

$$\gamma_v := \begin{pmatrix} a+vcl^2 & \frac{b+vd-aw_v-w_vvcl^2}{l} \\ cl^3 & d-w_vcl^2 \end{pmatrix}.$$

A calculation shows that $\gamma_v \in SL_2(\mathbb{Z})$ and that

$$\begin{pmatrix} 1 & v \\ 0 & l \end{pmatrix}\cdot A = \gamma_v \cdot \begin{pmatrix} 1 & w_v \\ 0 & l \end{pmatrix}. \tag{3.11}$$

Using (3.11), we can obtain

$$h(z)|U(l)|_{\frac{l-1}{2}}A = l^{\frac{l-5}{4}}\sum_{v=0}^{l-1} h(z)|_{\frac{l-1}{2}}\gamma_0 \cdot \begin{pmatrix} 1 & w_v \\ 0 & l \end{pmatrix}. \tag{3.12}$$

We note that this type of argument can be found in greater generality in [Tre06].

We can calculate $h(z)|_{\frac{l-1}{2}}\gamma_0$ directly (see [Mar96]) to obtain

$$h(z)|_{\frac{l-1}{2}}\gamma_0 = \sum_{n\geq n_c} a_c(n)q_{h_c}^n,$$

where

$$n_c = \begin{cases} \delta_l\cdot h_c & \textit{for } c=1,2,8,32, \\ \delta_l\cdot h_c - 1 & \textit{for } c=4, \\ \delta_l\cdot h_c + 2 & \textit{for } c=16, \end{cases}$$

and

$$h_c = \begin{cases} 32 & \textit{if } c=1 \\ 8 & \textit{if } c=2 \\ 2 & \textit{if } c=4 \\ 1 & \textit{if } c=8,16,32. \end{cases}$$

Plugging back into (3.12) we get that

$$h(z)|U(l)|_{\frac{l-1}{2}}A = l^{-1}\sum_{n\geq n_c} a_c(n)e^{\frac{2\pi i z n}{h_c l}}\cdot\left(\sum_{v=0}^{l-1} e^{\frac{2\pi i w_v n}{h_c l}}\right).$$

One can show that w_v runs through the residue classes $\pmod{l}$ as v does. Since $2304|w_v$ we have $h_c|w_v$ for all c. Since $(h_c,l)=1$, w_v/h_c runs through the residue classes $\pmod{l}$ as v does. Hence,

$$h(z)|U(l)|_{\frac{l-1}{2}}A = l^{-1}\sum_{n\geq n_c} a_c(n)q_{h_c}^{\frac{n}{l}}\cdot\left(\sum_{v=0}^{l-1} e^{\frac{2\pi i v n}{l}}\right) = \sum_{n\geq \frac{n_c}{l}} a(ln)q_{h_c}^n. \tag{3.13}$$

We must work through each case to show that if $h(z)|U(l)|_{\frac{l-1}{2}}A = \star q^\alpha + \cdots$, then $\alpha > \frac{l}{24}$. Since the calculations for the cases are very similar, we will only show the details of the trickiest case here, when $c=4$.

When $c = 4$, (3.13) yields

$$h(z)|U(l)|_{\frac{l-1}{2}}A = \sum_{n \geq \frac{2\delta_l - 1}{l}} a_c(ln)q^{\frac{n}{2}} = \star q^{\alpha} + \cdots.$$

Thus $\alpha = \frac{m}{2}$, where m is an integer satisfying $m \geq (2\delta_l - 1)/l$. Then choose the least integer $x \geq 0$ such that

$$lm = 2\delta_l - 1 + x = \frac{2l^2 - 26 + 24x}{24} \equiv 0 \pmod{l}.$$

Note that if we do not allow $l = 13$, then we must have $x \geq 2$ in order for the congruence to hold. In this case,

$$\alpha = \frac{1}{2}\left(\frac{l^2 - 13 + 12x}{24l}\right) > \frac{l}{24}.$$

The other cases are similar, but do not require the exclusion of any primes.

Remark. In [Tre06], Treneer proves general results such as equation (3.12) by relating eta-quotients and their images under the U-operator to cusp forms.

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