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NONPOSITIVELY CURVED HERMITIAN METRICS ON PRODUCT MANIFOLDS

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ABSTRACT. In this article, we classify all the Hermitian metrics on a complex product manifold $M = X \times Y$ with nonpositive holomorphic bisectional curvature. It is a generalization of a result by Zheng.

In this article, using a trick as in Tam-Yu [1], we prove the following generalization of the result in Zheng [2].

Theorem 0.1. *Let $M = X \times Y$ with X and Y both compact complex manifolds. Let $\phi_1, \phi_2, \cdots, \phi_r$ be a basis of $H^{1,0}(X)$ and $\psi_1, \psi_2, \cdots, \psi_s$ be a basis of $H^{1,0}(Y)$. Then, for any Hermitian metric h on M with nonpositive holomorphic bisectional curvature,*

$$\omega_h = \pi_1^*\omega_{h_1} + \pi_2^*\omega_{h_2} + \rho + \bar{\rho},$$

where h_1 and h_2 are Hermitian metrics on X and Y with nonpositive holomorphic bisectional curvature respectively, π_1 and π_2 are natural projections from M to X and from M to Y respectively, and

$$\rho = \sqrt{-1}\sum_{k=1}^{r}\sum_{l=1}^{s} a_{kl}\phi_k \wedge \psi_l,$$

where the a_{kl}'s are complex numbers.

Before the proof of Theorem 0.1, we need the following lemma.

Lemma 0.2. *Let X^m and Y^n be two compact complex manifolds. Let $\phi_1, \phi_2, \cdots, \phi_r$ be a basis of $H^{1,0}(X)$ and $\psi_1, \psi_2, \cdots, \psi_s$ be a basis of $H^{1,0}(Y)$. Let*

$$\rho = \rho_{ij}(x, y)dx^i \wedge dy^j$$

be a global holomorphic two form on $X \times Y$, where $(x^1, x^2, \cdots, x^m)$ is a local holomorphic coordinate of X and $(y^1, y^2, \cdots, y^n)$ is a local holomorphic coordinate of Y.

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Then

$$\rho = \sum_{k=1}^{r}\sum_{l=1}^{s} a_{kl}\phi_k \wedge \psi_l, \tag{0.1}$$

where the a_{kl}'s are complex numbers.

Proof. Fix a local holomorphic coordinate $(y^1, y^2, \cdots, y^n)$ of Y. It is clear that

$$\theta_j = \sum_{i=1}^{m} \rho_{ij}(x,y)dx^i$$

is a global homomorphic 1-form on $X \times \{y\}$. Then

$$\theta_j = \sum_{k=1}^{r} b_{kj}(y)\phi_k, \tag{0.2}$$

where the b_{kj}'s are local homomorphic functions on Y.

It is clear that

$$\sum_{j=1}^{n} b_{kj}(y)dy^j$$

is a global holomorphic 1-form on Y for each k. So,

$$\sum_{j=1}^{n} b_{kj}(y)dy^j = \sum_{l=1}^{s} a_{kl}\psi_l, \tag{0.3}$$

where the a_{kl}'s are complex numbers. Therefore

$$\rho = \sum_{j=1}^{n} \theta_j \wedge dy^j = \sum_{k}^{r}\sum_{l=1}^{s} a_{kl}\phi_k \wedge \psi_l. \tag{0.4}$$

□

Proof of Theorem 0.1. Let $(z^{m+1}, \cdots, z^{m+n})$ be a local holomorphic coordinate of Y at q. Then, it is clear that

$$h_{\alpha\bar{\alpha}}(x,q) \tag{0.5}$$

is a positive function on $X \times \{q\}$, where $m+1 \le \alpha \le m+n$.

Let Δ be the complex Laplacian on $X \times \{q\}$ and let $(z^1, z^2, \cdots, z^m)$ be a holomorphic coordinate of X such that

$$h_{i\bar{j}}(x,q) = \delta_{i\bar{j}}$$

with $1 \le i, j \le m$. Then

$$\Delta h_{\alpha\bar{\alpha}}(x,q) = \sum_{i=1}^{m} \partial_i\partial_{\bar{i}} h_{\alpha\bar{\alpha}} = -\sum_{i=1}^{m} R_{\alpha\bar{\alpha}i\bar{i}} + \sum_{i=1}^{m} h^{\bar{b}a}\partial_i h_{\alpha\bar{b}}\partial_{\bar{i}} h_{a\bar{\alpha}} \ge 0, \tag{0.6}$$

with $1 \le a, b \le n+m$. By the maximum principle, $h_{\alpha\bar{\alpha}}(x,q)$ is a constant function. Hence

$$\partial_i h_{\alpha\bar{b}} = 0. \tag{0.7}$$

Interchanging the roles of X and Y in the above, we get

$$\partial_\alpha h_{i\bar{b}} = 0. \tag{0.8}$$

By (0.7), we know that

$$\partial_i h_{\alpha\bar{\beta}} = 0 \tag{0.9}$$

for any $m+1 \leq \alpha, \beta \leq n+m$. So, $h_{\alpha\bar{\beta}}$ is independent of the z^i's. Then, $h_{\alpha\bar{\beta}}$ is a Hermitian metric on Y. It is clear that $h_{\alpha\bar{\beta}}$ as a Hermitian metric on Y is of nonpositive holomorphic bisectional curvature since holomorphic bisectional curvature deceases on complex submanifolds. We denote this metric by h_2.

Similarly, by (0.8), $h_{i\bar{j}}$ is a Hermitian metric on X with nonpositive holomorphic bisectional curvature. We denote it by h_1.

By (0.8) and (0.7), we have

$$\partial_\alpha h_{i\bar{\beta}} = 0 \text{ and } \partial_{\bar{i}} h_{j\bar{\alpha}} = 0.$$

So, the form $h_{i\bar{\alpha}} dz^i \wedge dz^{\bar{\alpha}}$ is a holomorphic two form on $M_1 \times \overline{M_2}$, where $\overline{M_2}$ is the complex conjugate of M_2. By the lemma above, we know that

$$h_{i\bar{\alpha}} dz^i \wedge dz^{\bar{\alpha}} = \sum_{k=1}^{q_1} \sum_{l=1}^{q_2} a_{kl} \phi_k \wedge \bar{\psi}_l. \tag{0.10}$$

Hence, we get the conclusion. □

As in Zheng [2], we have the following consequence of the theorem.

Corollary 0.1.

$$codim_{\mathbb{R}}(\mathcal{H}(M_1) \times \mathcal{H}(M_2), \mathcal{H}(M_1 \times M_2)) = 2h^{1,0}(M_1) \cdot h^{1,0}(M_2),$$

where M_1, M_2 are compact complex manifolds, and suppose that $\mathcal{H}(M_i) \neq \emptyset$ for $i = 1, 2$.

Proof. For any $h \in \mathcal{H}(M_1 \times M_2)$, by the theorem, it has a unique decomposition,

$$\omega_h = \pi_1^* \omega_{h_1} + \pi_2^* \omega_{h_2} + \rho + \bar{\rho},$$

where $\rho = \sqrt{-1} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} a_{ij} \phi_i \wedge \bar{\psi}_j$ with $a_{ij} \in \mathbb{C}$, $h_i \in \mathcal{H}(M_i)$. So, we get a map

$$\mathcal{H}(M_1 \times M_2) \to M(q_1 \times q_2; \mathbb{C}),\ h \mapsto (a_{ij})_{q_1 \times q_2}. \tag{0.11}$$

It is clear $\mathbb{R}^+$-linearly. (Note that $\mathcal{H}(M_1 \times M_2)$ is a convex cone.) So, it induces a linear map of real vector spaces,

$$\Psi : \langle \mathcal{H}(M_1 \times M_2) \rangle_{\mathbb{R}} \to M(q_1 \times q_2; \mathbb{C}).$$

It is clear that

$$\ker \Psi = \langle \mathcal{H}(M_1) \times \mathcal{H}(M_2) \rangle_{\mathbb{R}}. \tag{0.12}$$

Moreover, let $E_{kl} = (a_{ij})$ be such that $a_{ij} = \delta_{ik}\delta_{jl}$. Note that

(0.13)
$$\begin{aligned}
&\pi_1^* \omega_{h_1} + \pi_2^* \omega_{h_2} + \sqrt{-1}(\phi_k + \psi_l) \wedge \overline{(\phi_k + \psi_l)} \\
=&[\pi_1^* \omega_{h_1} + \sqrt{-1} \phi_k \wedge \bar{\phi}_k] + [\pi_1^* \omega_{h_2} + \sqrt{-1} \psi_l \wedge \bar{\psi}_l] + \sqrt{-1} \phi_k \wedge \bar{\psi}_l + \sqrt{-1} \psi_l \wedge \bar{\phi}_k.
\end{aligned}$$

So, E_{kl} is in the image of Ψ. Similarly, $\sqrt{-1} E_{kl}$ is also in the image of Ψ. Therefore, Ψ is surjective. By the dimension theorem in linear algebra, we get the identity. □

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