

HOLONOMIES AND THE SLOPE INEQUALITY OF LEFSCHETZ FIBRATIONS

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ABSTRACT. In this paper, we consider two necessary conditions: the irreducibility of the holonomy group and the slope inequality for which a Lefschetz fibration over a closed orientable surface is a holomorphic fibration. We show that these two conditions are independent in the sense that neither one implies the other.

1. INTRODUCTION

In this paper, we will consider relations between two necessary conditions for which a Lefschetz fibration $f : X \rightarrow \hat{S}$ over a closed orientable surface \hat{S} whose general fiber is a surface of genus $g \geq 2$ is a holomorphic fibration.

One of the conditions is the *slope inequality*, which comes from a complex geometrical point of view. This says that the slope of the fibration of a relatively minimal holomorphic fibration of genus g on a non-singular algebraic surface is at least $4 - 4/g$. (Cf. [11]. See also §2.2.)

The other condition is *irreducibility* of the holonomy group, which comes from a topological point of view. It is known that the image of the holonomy of a surface bundle $f : X_0 \rightarrow S$ is irreducible if the bundle is non-trivial and holomorphic, where $X_0 = f^{-1}(S)$ and $S \subset \hat{S}$ is the complement of the union of mutually disjoint disk neighborhoods of the critical values of f . (Cf. [9]. See also §2.3.)

The aim of this paper is to show that neither of the above conditions implies the other. More precisely, we first show that an example given by H. Endo, M. Korkmaz, D. Kotschick, B. Ozbagci, and A. Stipsicz in [2] satisfies the slope inequality but has a reducible holonomy group. Secondly, for every sufficiently big rational number p/q , we construct a Lefschetz fibration whose holonomy group is irreducible but its slope is equal to $-p/q$. Indeed, it was expected that there might exist a lower bound of slopes for every surface bundle over a surface (K. Konno, oral communication). However, our example gives a negative answer. We also show

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that one of the two conditions is not a sufficient condition for a fibration to admit a complex structure.

2. PRELIMINARIES

2.1. Holonomy representations of surface bundles. Let h and r be positive integers with $2h - 2 + r > 0$. Let S be a compact orientable surface of genus h with r boundary components. Let Σ_g be a closed surface of genus g and let $\text{Mod}(g)$ be the mapping class group of Σ_g . Let $f : X_0 \rightarrow S$ be a surface bundle with fiber Σ_g . Fix a base point $b_0 \in \text{Int}(S)$ and a diffeomorphism $\varphi_{b_0} : \Sigma_g \rightarrow f^{-1}(b_0)$.

Now we define the holonomy $\rho : \pi_1(S, b_0) \rightarrow \text{Mod}(g)$ of the fibration as follows (cf. §2 of [6]):

Let $l : [0, 1] \rightarrow S$ be a loop with $l(0) = l(1) = b_0$. Since the pull-back l^*f is a trivial bundle over $[0, 1]$, there is a map $\varphi : [0, 1] \times \Sigma_g \rightarrow X_0$ such that

- (1) $f(\varphi(t, p)) = l(t)$ for all $(t, p) \in [0, 1] \times \Sigma_g$,
- (2) the map $\varphi_t : \Sigma_g \rightarrow F_t = f^{-1}(l(t))$ defined by $\varphi_t(p) = \varphi(t, p)$ is an orientation-preserving diffeomorphism, and
- (3) $\varphi_0 = \varphi_{b_0}$.

We call the image $\rho(\pi_1(S, b_0))$ the *holonomy group*.

Since $F_0 = F_1$, we have a diffeomorphism $\varphi_{b_0}^{-1} \circ \varphi_1 : \Sigma_g \rightarrow \Sigma_g$. The mapping class of $\varphi_{b_0}^{-1} \circ \varphi_1$ is determined by the homotopy class $\{l\}$ of l and is called the monodromy associated with $\{l\}$ and φ_{b_0} . By sending $\{l\}$ to the mapping class of $\varphi_{b_0}^{-1} \circ \varphi_1$ we obtain a map

$$\rho : \pi_1(S, b_0) \rightarrow \text{Mod}(g).$$

The map ρ is a homomorphism since $\text{Mod}(g)$ acts on Σ_g from the right by convention, and it is called a *monodromy* by Y. Matsumoto (see [6]). However, we often employ the terminology “monodromy” to represent a homomorphism from a Fuchsian group to the Teichmüller modular transformation group which commutes the classifying map of the holomorphic family of Riemann surfaces (see [9]). Hence, to avoid confusions, we adopt the terminology “holonomy” to represent ρ above.

2.2. Lefschetz fibrations and the slope inequality. For our purposes, we shall state the slope inequality via the topological point of view in the case for Lefschetz fibrations. (Cf. Notation 4.4 of [3]. Compare [11].)

2.2.1. Lefschetz fibrations. We recall the definition of Lefschetz fibrations (cf. §2 of [6]).

Definition 2.1. Let \hat{S} be a compact surface with genus h . A C^∞ -map $f : X \rightarrow \hat{S}$ is said to be a *Lefschetz fibration* if the following conditions are satisfied:

- (a) There is a finite set of points b_1, b_2, \dots, b_n (called the *critical values* of f) in \hat{S} such that $f : X_0 \rightarrow S$ is a C^∞ -fiber bundle with the fiber diffeomorphic to Σ_g , where $X_0 = f^{-1}(S)$ and $S \subset \hat{S}$ is the complement of the union of the disk neighborhoods of b_i for $i = 1, \dots, n$.
- (b) For each i ($1 \leq i \leq n$), there exists a single point $p_i \in f^{-1}(b_i)$ such that
 - (1) $df_p : T_p(M) \rightarrow T_{b_i}(\hat{S})$ is onto for any $f^{-1}(b_i) - \{p_i\}$;
 - (2) about p_i (resp. b_i), there exist local complex coordinates z_1, z_2 with $z_1(p_i) = z_2(p_i) = 0$ (resp. local complex coordinate ξ with $\xi(b_i) = 0$), so that f is locally written $\xi = f(z_1, z_2) = z_1 z_2$;

- (3) no fiber contains a (-1) -sphere, i.e., a smoothly embedded 2-sphere with self-intersection number -1 .

Two Lefschetz fibrations $f_i : X_i \rightarrow \hat{S}_i$ ($i = 1, 2$) are said to be *equivalent* if there exist diffeomorphisms $H : X_1 \rightarrow X_2$ and $h : \hat{S}_1 \rightarrow \hat{S}_2$ such that $f_2 \circ H = h \circ f_1$. A Lefschetz fibration is, by definition, *holomorphic* if there are complex structures on both X and \hat{S} with holomorphic projection f . A Lefschetz fibration is said to *admit a complex structure* if it is equivalent to a holomorphic Lefschetz fibration.

A Lefschetz fibration $f : X \rightarrow \hat{S}$ defines a surface bundle $f : X_0 \rightarrow S$ where $X_0 = f^{-1}(S)$. Hence, it gives a holonomy as in §2.1. It is known that the holonomy around a critical value is given by a negative Dehn twist along the vanishing cycle (cf. Figure 1). Indeed, this is a somewhat sufficient condition for a homomorphism

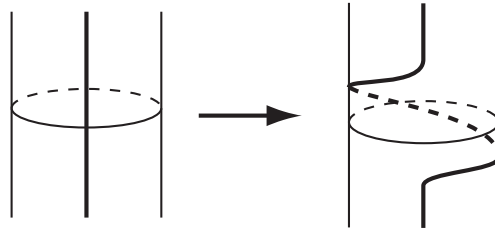


FIGURE 1. A negative Dehn twist

$\pi_1(S) \rightarrow \text{Mod}(g)$ to be a holonomy of a Lefschetz fibration as follows.

Proposition 2.1 (Theorem 2.6 in [6]). *Let \hat{S} be a closed orientable surface and b_1, \dots, b_n points in \hat{S} . Set $S \subset \hat{S}$ as above and let γ_i be a loop on S based on b_0 which surrounds exclusively b_i for $i = 1, \dots, n$. Let $\rho : \pi_1(S, b_0) \rightarrow \text{Mod}(g)$ be a homomorphism. Suppose that all of $\rho(\gamma_i)$ is a negative Dehn twist for a simple closed curve on Σ_g . Then there exists a Lefschetz fibration $f : X \rightarrow \hat{S}$ such that the holonomy is ρ . Moreover the fibration is unique up to the equivalence.*

2.2.2. The slopes of Lefschetz fibrations. Now we define the slope of a Lefschetz fibration.

Definition 2.2. Let $f : X \rightarrow \hat{S}$ be a Lefschetz fibration. Let σ and e be the signature and Euler characteristic of X , respectively. We set $\chi_h = (\sigma + e)/4$ (the *holomorphic Euler characteristic*) and $K^2 = 3\sigma + 2e$. Let $K_f^2 = K^2 - 8(g-1)(h-1)$ and $\chi_f = \chi_h - (g-1)(h-1)$. The *slope* is the quotient K_f^2/χ_f .

By a simple calculation, we see the slope is equal to $4(3\sigma(X) + 2n)/(\sigma(X) + n)$, where n is the number of singular fibers.

Proposition 2.2 ([11]). *Let $f : X \rightarrow \hat{S}$ be a Lefschetz fibration which admits a complex structure. Then the slope inequality*

$$(2.1) \quad \frac{K_f^2}{\chi_f} \geq 4 \left(1 - \frac{1}{g} \right)$$

holds.

It is important that, as we recall in §3, for a Lefschetz fibration $f : X \rightarrow \hat{S}$, the signature of the 4-manifold X , and hence the slope of the fibration $f : X \rightarrow \hat{S}$, is calculated from the holonomy of a surface bundle $f : X_0 \rightarrow S$ and the number of non-dividing vanishing cycles.

2.3. Irreducibility of subgroups of $\text{Mod}(g)$. A system of homotopy classes of mutually disjoint non-trivial and non-peripheral simple closed curves on Σ_g is called an *admissible system of curves* on Σ_g . A subgroup G of the mapping class group $\text{Mod}(g)$ is said to be *reducible* if there is an admissible system C of curves such that any element of G fixes C as a set in the homotopy sense. When G is not reducible, G is called *irreducible* (cf. Definition in §2 of [9]).

In [9], it is shown that the irreducibility of the holonomy group is a necessary condition for a surface bundle to be holomorphic.

Proposition 2.3. *Let $f : X \rightarrow S$ be a non-trivial holomorphic surface bundle with the holonomy $\rho : \pi_1(S, b_0) \rightarrow \text{Mod}(g)$. Then the holonomy group is an infinite and irreducible subgroup of $\text{Mod}(g)$.*

Proposition 2.3 also holds when the fibers admit punctures.

3. CALCULATION OF SIGNATURES: MEYER'S COCYCLE

Let $A, B \in Sp(2g, \mathbb{Z})$. Consider a subspace $V_{A,B}$ of $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$ defined by

$$V_{A,B} = \{(x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid (A^{-1} - I_{2g})x + (B - I_{2g})y = 0\},$$

where I_{2g} is the identity matrix of rank $2g$. Define the inner product on $V_{A,B}$ by

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} = (x_1 + y_1)J(I_{2g} - B)y_2,$$

where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. Set $\tau_h(A, B)$ to be the signature of $(V_{A,B}, \langle \cdot, \cdot \rangle_{A,B})$. We call τ_h the *Meyer cocycle*.

Let $f : X_0 \rightarrow S$ be a surface bundle with fiber Σ_g . Let $\rho : \pi_1(S, b_0) \rightarrow \text{Mod}(g)$ by the holonomy of the fibration. Now we fix a symplectic basis of $H_1(\Sigma_h)$. By composing ρ and the symplectic representation $\Pi : \text{Mod}(g) \rightarrow Sp(2g, \mathbb{Z})$, we get an anti-homomorphism $\chi = \Pi \circ \rho : \pi_1(S, b_0) \rightarrow Sp(2g, \mathbb{Z})$.

Let $\{\alpha_i, \beta_i, \gamma_j\}_{1 \leq i \leq h, 1 \leq j \leq r}$ be a system of generators of $\pi_1(S, b_0)$ satisfying

$$\prod_{i=1}^h [\alpha_i, \beta_i] \prod_{j=1}^r \gamma_j = 1.$$

Let $\kappa_i = [\alpha_i, \beta_i]$ for $i = 1, \dots, h$. The following result is due to Meyer [7] (see also Theorem 13 of [2]).

Proposition 3.1 (Meyer). *Let $f : X_0 \rightarrow S$ be an oriented surface bundle with holonomy representation $\rho : \pi_1(S, b_0) \rightarrow \text{Mod}(g)$. Then the signature $\sigma(X_0)$ of the total space X_0 is given by the formula*

$$(3.1) \quad \begin{aligned} \sigma(X_0) = & \sum_{i=1}^h \tau_h(\chi(\kappa_i), \chi(\beta_i)) - \sum_{i=2}^h \tau_h(\chi(\kappa_1 \cdots \kappa_{i-1}), \chi(\kappa_i)) \\ & - \sum_{j=1}^{r-1} \tau_h(\chi(\kappa_1 \cdots \kappa_h \gamma_1 \cdots \gamma_{j-1}), \chi(\gamma_j)). \end{aligned}$$

4. THE SLOPE INEQUALITY DOES NOT IMPLY IRREDUCIBILITY

Here we show that a Lefschetz fibration given in [2] satisfies the slope inequality but the holonomy group is reducible.

In Proposition 17 of [2], the authors construct a Lefschetz fibration $f : X \rightarrow \hat{S}$ over a surface \hat{S} of genus 3 with eight singular fibers. We assume that the genus of a general fiber is greater than 4. It is shown that the signature of their manifold X is -4 . Hence the slope is 4, which implies that the fibration $f : X \rightarrow \hat{S}$ satisfies the slope inequality (2.1). On the other hand, in the construction, every simple closed curve in Dehn twists generating the holonomy group is contained in a genus 3 subsurface with 2 holes of a general fiber. Therefore, the holonomy group must be reducible.

5. IRREDUCIBILITY DOES NOT IMPLY THE SLOPE INEQUALITY

In this section, we construct a Lefschetz fibration such that the slope is an arbitrarily small rational number but the holonomy group is irreducible.

Fix integers $m \geq 0$ and $g \geq 3$. Let $h = 2m$ and $r = m + 4(2g + 1)$. Let \hat{S} be a closed orientable surface of genus h . Let S be the complement of the union of disks of centers b_1, \dots, b_r in \hat{S} . We construct a family of Lefschetz fibrations $f_{g,m} : X_{g,m} \rightarrow \hat{S}$ with general fiber Σ_g such that the corresponding holonomies are irreducible but the slopes of the total spaces are small.

5.1. The holonomies. Let $\{\{\mathfrak{X}_i\}_{i=1}^{2g+1}, \mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{B}_1, \mathfrak{B}_2\}$ be a system of simple closed curves on Σ_g taken as in Figure 2. Let τ_i and τ_α be the mapping

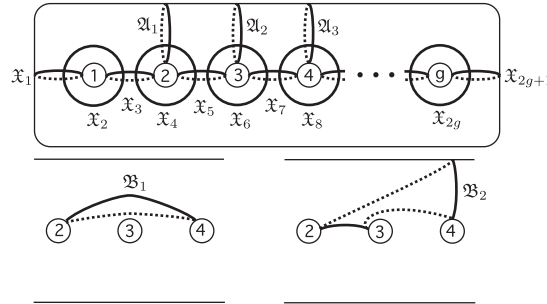


FIGURE 2. Curves on Σ_g : The numbers in circles enumerate handles.

classes of negative Dehn twists along \mathfrak{X}_i for $i = 1, \dots, 2g + 1$ and along $\alpha \in \{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{B}_1, \mathfrak{B}_2\}$, respectively. We define

$$\begin{aligned} \phi_1 &= \tau_{\mathfrak{A}_3}, \\ \phi_2 &= \tau_8 \tau_{\mathfrak{B}_2} \tau_{\mathfrak{A}_3} \tau_8, \\ \phi_3 &= \tau_4 \tau_{\mathfrak{B}_1} \tau_6 \tau_7 \tau_{\mathfrak{A}_2} \tau_6 \tau_5 \tau_4, \\ \phi_4 &= \tau_5^{-1} \tau_{\mathfrak{A}_3}, \\ \phi_5 &= \tau_{\mathfrak{A}_1}. \end{aligned}$$

Then from Proposition 14 of [2], the mapping classes ϕ_i above satisfy

$$(5.1) \quad [\phi_1, \phi_2][\phi_3, \phi_4]\phi_5 = 1.$$

Let $\alpha_1, \beta_1, \dots, \alpha_h, \beta_h, \gamma_1, \dots, \gamma_r$ be a canonical system of simple closed curves generating $\pi_1(S)$ with

$$(5.2) \quad \prod_{i=1}^m [\alpha_{2i-1}, \beta_{2i-1}] [\alpha_{2i}, \beta_{2i}] \gamma_i \prod_{j=m+1}^r \gamma_j = 1,$$

so that each γ_i represents a simple closed curve around b_i . Hence, we can define a homomorphism ρ from $\pi_1(S)$ to $\text{Mod}(\Sigma_g)$ by setting $\rho(\alpha_{2i-1})$, $\rho(\beta_{2i-1})$, $\rho(\alpha_{2i})$, $\rho(\beta_{2i})$, and $\rho(\gamma_i)$ as ϕ_1 , ϕ_2 , ϕ_3 , ϕ_4 , and ϕ_5 respectively for $i = 1, \dots, m$, and

$$\begin{aligned} \rho(\gamma_{m+j}) &= \tau_k \quad (1 \leq j \leq 2g+1, 4g+3 \leq j \leq 6g+3), \\ \rho(\gamma_{m+j}) &= \tau_{2g+2-k} \quad (2g+2 \leq j \leq 4g+2, 6g+4 \leq j \leq 8g+4), \end{aligned}$$

where $k \equiv j \pmod{2g+1}$ with $1 \leq k \leq 2g+1$. Indeed, by the hyperelliptic relation, it is known that

$$(\tau_1 \tau_2 \cdots \tau_{2g} \tau_{2g+1}^2 \tau_{2g} \cdots \tau_2 \tau_1)^2 = 1,$$

and the homomorphism ρ given above is well-defined (see [3]).

Since $\rho(\gamma_i)$ ($i = 1, \dots, r$) are all negative Dehn twists, by Proposition 2.1, we get a Lefschetz fibration $f_{g,m} : X_{g,m} \rightarrow \hat{S}$.

5.2. The signatures and the slopes. The purpose of this section is to show the following.

Theorem 5.1. *The signature of $X_{g,m}$ is equal to $-m - 4(g+1)$.*

Proof. Since each vanishing cycle is non-dividing, the signature of $X_{g,m}$ is equal to $X'_{g,m} = f^{-1}(S)$ (see [3]). By definition, it follows that $\chi(\kappa_{2i-1}) = \chi(\kappa_1)$, $\chi(\kappa_{2i}) = \chi(\kappa_2)$, and $\chi(\kappa_{2i-1}\kappa_{2i}) = \chi(\gamma_i^{-1}) = \chi(\gamma_1^{-1})$ for $i = 1, \dots, m$. Hence by a straightforward calculation, one can see that

$$\begin{aligned} \tau_h(\chi(\kappa_{2i-1}), \chi(\beta_{2i-1})) &= \tau_h(\chi(\kappa_1), \chi(\beta_1)) = 0, \\ \tau_h(\chi(\kappa_{2i}), \chi(\beta_{2i})) &= \tau_h(\chi(\kappa_2), \chi(\beta_2)) = -1, \\ \tau_h(\chi(\kappa_1 \cdots \kappa_{2i-1}), \chi(\kappa_{2i})) &= \tau_h(\chi(\gamma_1^{-i+1} \kappa_1), \chi(\kappa_2)) = -1, \\ \tau_h(\chi(\kappa_1 \cdots \kappa_{2i-2}), \chi(\kappa_{2i-1})) &= \tau_h(\chi(\gamma_1^{-i+1}), \chi(\kappa_1)) = 0 \end{aligned}$$

for $i = 1, \dots, m$. By (5.1), $\prod_{i=1}^m \kappa_{2i-1} \kappa_{2i} \prod_{i=1}^m \gamma_i = 1$. Hence, we get

$$\tau_h(\chi(\kappa_1 \cdots \kappa_{2m} \gamma_1 \cdots \gamma_{j-1}), \chi(\gamma_j)) = \tau_h(\chi(\gamma_1^{-m+j-1}), \chi(\gamma_1)) = 1$$

for $j = 1, \dots, m$. Since the product $\prod_{j=m+1}^r \rho(\gamma_j)$ is the hyperelliptic relation in $\text{Mod}(g)$, we have

$$\begin{aligned} &\sum_{j=m+1}^r \tau_h(\chi(\kappa_1 \cdots \kappa_{2m} \gamma_1 \cdots \gamma_{j-1}), \chi(\gamma_j)) \\ &= \sum_{j=1}^{4(2g+1)} \tau_h(\chi(\gamma_{m+1} \cdots \gamma_{m+j-1}), \chi(\gamma_{m+j})) \\ &= 4(g+1) \end{aligned}$$

(see [3]). Thus, by Proposition 3.1, we get

$$\begin{aligned}
 \sigma(X_{g,m}) &= \sigma(X'_{g,m}) \\
 &= \sum_{i=1}^h \tau_h(\chi(\kappa_i), \chi(\beta_i)) - \sum_{i=2}^h \tau_h(\chi(\kappa_1 \cdots \kappa_{i-1}), \chi(\kappa_i)) \\
 &\quad - \sum_{j=1}^{r-1} \tau_h(\chi(\kappa_1 \cdots \kappa_h \gamma_1 \cdots \gamma_{j-1}), \chi(\gamma_j)) \\
 &= m(0 + (-1)) - (m-1) \cdot (0 + (-1)) - ((m-1) \cdot 1 + 4(g+1)) \\
 &= -m - 4(g+1).
 \end{aligned}$$

□

Corollary 1. *The slope of $X_{g,m}$ is $-(m - 4g + 4)/g$ if $g > 0$. In particular, for any $p/q \in \mathbb{Q}$ with $p + 4q \geq 1$ and $q \geq 1$, there is a Lefschetz fibration $f : X \rightarrow \hat{S}$ with the irreducible monodromy such that the slope of X is equal to $-p/q$.*

Proof. The first statement is obtained from a simple calculation. For the second statement, consider the equation $(m - 4g + 4)/g = p/q$. Then there is a non-zero integer a such that $m - 4g + 4 = ap$ and $g = aq$. This means that $m = a(p + 4q) - 4$ and $g = aq$. Thus, when p and q satisfy the conditions $p + 4q \geq 1$ and $q \geq 1$, take $a \geq 1$ such that $a(p + 4q) - 4 \geq 1$ and set $m = a(p + 4q) - 4$ and $g = aq$. Then the Lefschetz fibration $f_{g,m} : X_{g,m} \rightarrow \hat{S}$ has the slope $-p/q$. Moreover, from Theorem 3.1 of [8], we can see that

$$\rho(\gamma_{m+1} \gamma_{m+2}^{-1} \gamma_{m+3} \cdots \gamma_{m+2g-1} \gamma_{m+2g}^{-1} \gamma_{m+2g+1})$$

is a pseudo-Anosov homeomorphism on Σ_g , and hence the holonomy group is irreducible. □

6. CONCLUDING REMARKS

6.1. The holonomy group of the fibration $X_{g,m} \rightarrow \hat{S}$ in §5 contains the Dehn-Lickorish-Humphries generator of the mapping class group which consists of Dehn twists along \mathfrak{X}_i for all $i = 1, \dots, 2g$ and along \mathfrak{A}_1 ([4]). Hence, the holonomy group coincides with the whole of $\text{Mod}(g)$, which also implies the irreducibility of the holonomy of $X_{g,m}$. The authors thank Professor Hisaaki Endo for pointing this out. The authors do not know of the existence of a surface bundle whose holonomy group is irreducible but “small”, for instance, a subgroup of infinite index.

However, the holonomy can be “small” if the fibers are allowed to have punctures. Indeed, let R be a closed Riemann surface of genus $g \geq 2$. Set $X = R \times R \setminus \{\text{diagonal}\}$ and consider the first projection $\pi : X \rightarrow R$. Then the holonomy group is a subgroup of the mapping class group $\text{Mod}(g, 1)$ of $\Sigma_g - \{\text{point}\}$ of infinite index. In fact, it coincides with the kernel $\pi_1(\Sigma_g)$ of Birman’s exact sequence

$$1 \rightarrow \pi_1(\Sigma_g) \rightarrow \text{Mod}(g, 1) \rightarrow \text{Mod}(g) \rightarrow 1$$

(see [5]). Since (X, π, R) is a holomorphic family of Riemann surfaces, the holonomy group is irreducible, as remarked in §2.3.

6.2. We have considered the slope inequality and the irreducibility of the holonomy group each of which is a necessary condition for a Lefschetz fibration to admit a complex structure. However, examples given in §4 and §5 show that neither of them is a sufficient condition. As we mentioned in the introduction, the slope inequality is a complex geometric condition and the irreducibility of the holonomy is a topological condition, so they reflect different aspects of the Lefschetz fibration.

It is optimistically thought that a Lefschetz fibration $f : X \rightarrow \hat{S}$ admits a complex structure when it satisfies the slope inequality (2.1) and its holonomy group is irreducible. However, I. Smith showed that if the genus of \hat{S} is zero, the holonomy group is always irreducible (see Corollary 4.3 of [10]). On the other hand, when the base surface is of genus zero, it is conjectured that the slope inequality also holds for every Lefschetz fibration (by R. Hain; see Question 5.10 of [1] and also see Conjecture 4.12 of [3]).

In any case, it might be expected that a condition other than the slope inequality and the irreducibility of the holonomy group is required for the Lefschetz fibration to admit a complex structure.

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REFERENCES

- [1] J. Amorós, F. Bogomolov, L. Katzarkov and T. Pantev, Symplectic Lefschetz fibrations with arbitrary fundamental groups, *J. Differential Geom.* **54** (2000), no. 3, 489–545. MR1823313 (2002g:57051)
- [2] H. Endo, M. Korkmaz, D. Kotschick, B. Ozbagci, and A. Stipsicz, Commutators, Lefschetz fibrations and the signatures of surface bundles, *Topology* **41** (2002), 961–977. MR1923994 (2003f:57051)
- [3] H. Endo and S. Nagami, Signature of relations in mapping class groups and non-holomorphic Lefschetz fibrations, *Trans. Amer. Math. Soc.* **357** (2004), 3179–3199. MR2135741 (2006g:57051)
- [4] S. P. Humphries, Generators for the mapping class group, *Topology of low-dimensional manifolds*, Lecture Notes in Math. **722** (1979), Springer, Berlin, 44–47. MR547453 (80i:57010)
- [5] I. Kra, On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces, *Acta Math.* **146** (1981), 231–270. MR611385 (82m:32019)
- [6] Y. Matsumoto, Lefschetz fibrations of genus two—a topological approach, *Topology and Teichmüller spaces*, World Sci. Publ. (1995), 123–148. MR1659687 (2000h:14038)
- [7] W. Meyer, Die Signatur von Flächenbündeln, *Math. Ann.* **201** (1973), 239–264. MR0331382 (48:9715)
- [8] R. Penner, Construction of pseudo-Anosov homeomorphism, *Trans. Amer. Math. Soc.* **301** (1988), 179–197. MR930079 (89k:57026)
- [9] H. Shiga, On monodromies of holomorphic families of Riemann surfaces and modular transformations, *Math. Proc. Cambridge Philos. Soc.* **122** (1997), 541–549. MR1466656 (98k:32030)

- [10] I. Smith, Geometric monodromy and the hyperbolic disc, *Quart. J. of Math.* **52** (2001), 217–228. MR1838364 (2002c:57046)
- [11] G. Xiao, Fibered algebraic surfaces with low slope. *Math. Ann.* **276** (1987), 449–466. MR875340 (88a:14046)

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