PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 4, April 2011, Pages 1201–1213
S 0002-9939(2010)10623-8
Article electronically published on September 29, 2010

ŁOJASIEWICZ EXPONENT NEAR THE FIBRE OF A MAPPING

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(Communicated by Ted Chinburg)

Abstract. Let $g : X \to \mathbb{R}^k$ and $f : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$, be continuous semi-algebraic mappings, and $\lambda \in \mathbb{R}^m$. We describe the optimal exponent $\theta =: \mathcal{L}_{\infty,f\to\lambda}(g)$ for which the Łojasiewicz inequality $|g(x)| \geqslant C|x|^\theta$ holds with $C > 0$ as $|x| \to \infty$ and $f(x) \to \lambda$. We prove that there exists a semi-algebraic stratification $\mathbb{R}^m = S_1 \cup \cdots \cup S_j$ such that the function $\lambda \mapsto \mathcal{L}_{\infty,f\to\lambda}(g)$ is constant on each stratum S_i. We apply this result to describe the set of generalized critical values of f.

Introduction

Let M, N, L be finite-dimensional real vector spaces, $X \subset M$ be a closed semi-algebraic set, $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings (see [1]), and let $\lambda \in L$. The aim of this article is to describe the *Łojasiewicz exponent at infinity of g near the fibre* $f^{-1}(\lambda)$, i.e. the supremum of the exponents θ for which the *Łojasiewicz inequality*

$$(\text{Ł}) \qquad |g(x)| \geqslant C|x|^\theta \quad \text{as} \quad x \in X, \quad |x| \to \infty \quad \text{and} \quad f(x) \to \lambda$$

holds with $C > 0$ (cf. [12], [18]), where $|\cdot|$ is a norm. We denote this exponent by $\mathcal{L}_{\infty,f\to\lambda}(g)$ (see Section 1 for details).

We prove that $\mathcal{L}_{\infty,f\to\lambda}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$ for $\lambda \in L$ and that there exists a semi-algebraic stratification $L = S_1 \cup \cdots \cup S_j$ such that the function $\lambda \mapsto \mathcal{L}_{\infty,f\to\lambda}(g)$ is constant on each stratum S_i (Theorem 1.2). If g and f are complex regular mappings, the stratification is complex algebraic (Corollary 1.6). Note that if $\theta = \mathcal{L}_{\infty,f\to\lambda}(g) \in \mathbb{Q}$, then (Ł) holds (Corollary 3.7). The key points in the proofs are Lipschitz stratifications ([13], [14], [20]) and properties of the set of points at which a mapping is not proper ([8]; see also Section 2).

If $f : M \to L$ is a semi-algebraic mapping of class $\mathscr{C}^1$, we define the *Łojasiewicz exponent of df near the fibre* $f^{-1}(\lambda)$ by

$$\mathcal{L}_{\infty,\lambda}(f) = \mathcal{L}_{\infty,f\to\lambda}(\nu(df)),$$

where ν is a function introduced by Rabier [17] (see Section 1). This notion was introduced by Ha [7] in the case of complex polynomial functions in two variables (see also [3], [5]).

Received by the editors May 19, 2009 and, in revised form, April 19, 2010.
2010 *Mathematics Subject Classification.* Primary 14R25; Secondary 58K55, 58K05.
Key words and phrases. Łojasiewicz exponent at infinity, generalized critical values, stratification.
This research was partially supported by the program POLONIUM.

Let us recall that the exponent $\mathcal{L}_{\infty,\lambda}(f)$ is strongly related to the set of bifurcation points of f. Namely, one can define the *set of generalized critical values* of f by

$$K_\infty(f) = \{\lambda \in L : \mathcal{L}_{\infty,\lambda}(f) < -1\}.$$

It is a closed and semi-algebraic set. By Theorem 1.2, the mapping $L \ni \lambda \mapsto \mathcal{L}_{\infty,\lambda}(f)$ has a finite number of values (Corollary 1.5); hence there exists $\alpha > 0$ such that

$$K_\infty(f) = \{\lambda \in L : \mathcal{L}_{\infty,\lambda}(f) < -1 - \alpha\}.$$

If f is of class $\mathscr{C}^2$, then for any $\lambda \in L \setminus K_\infty(f)$ there exist a neighbourhood $U \subset L$ of λ and a compact set $\Delta \subset M$ such that $f : f^{-1}(U) \setminus \Delta \to U$ is a trivial bundle (see [16], [17], [11]; see also [23], [21], [22], [7], [15] for polynomials and polynomial mappings). The smallest set $B \subset L$ such that $L \setminus B$ has the above property is called the *bifurcation set at infinity* of f and is denoted by $B_\infty(f)$. Note that for a complex polynomial f in two variables, $B_\infty(f) = K_\infty(f)$ (see [7], [15]).

Chądzyński and Krasiński ([3], Corollary 4.7) proved that for a complex polynomial f in two variables with $\deg f > 0$ there exists $c_f \in \mathbb{Q}$ with $c_f \geqslant 0$ such that

$$\mathcal{L}_{\infty,\lambda}(f) = c_f \text{ for } \lambda \notin K_\infty(f) \quad \text{and} \quad \mathcal{L}_{\infty,\lambda}(f) < -1 \text{ for } \lambda \in K_\infty(f).$$

They also asked whether $\lambda \mapsto \mathcal{L}_{\infty,\lambda}(f)$ behaves similarly in the general case. Note that in the multi-dimensional case we cannot require $c_f \geqslant 0$. Indeed, for the polynomial $f(z_1, z_2, z_3) = (z_1 z_2 - 1) z_2 z_3$ ([17], Remark 9.1) we have $c_f = -1$ (see [3], Proposition 6.4).

As a corollary from Theorem 1.2 we give a partial answer to the above-mentioned question. Namely, for a nonconstant polynomial $f : \mathbb{C}^n \to \mathbb{C}$ there exist a finite set $S \subset \mathbb{C}$ with $K_\infty(f) \subset S$ and $c_f \geqslant -1$ such that $\mathcal{L}_{\infty,\lambda}(f) = c_f$ for $\lambda \in \mathbb{C} \setminus S$ and $\mathcal{L}_{\infty,\lambda}(f) < c_f$ for $\lambda \in S$ (Corollary 1.7). It is not clear to the authors whether $S = K_\infty(f)$ in Corollary 1.7.

Section 2 has an auxiliary character and contains some results on semi-algebraic mappings, Łojasiewicz exponent and stratifications. In Sections 3 and 4 we prove Theorem 1.2 and Corollary 1.6, respectively.

1. Łojasiewicz exponent near the fibre of a mapping

Let M, N, L be finite-dimensional real vector spaces, $X \subset M$ be a closed set, let $g : X \to N$ and $f : X \to L$, and let $\lambda \in L$.

Definition 1.1. By the *Łojasiewicz exponent at infinity of g near the fibre $f^{-1}(\lambda)$* we mean

$$\mathcal{L}_{\infty,f\to\lambda}(g) := \sup\{\mathcal{L}_\infty(g|f^{-1}(U)) : U \subset L \text{ is a neighbourhood of } \lambda\},$$

where

$$\mathcal{L}_\infty(g|S) := \sup\{\theta \in \mathbb{R} : \exists_{C,R>0}\ \forall_{x\in S}\ (x \geqslant R \Rightarrow |g(x)| \geqslant C|x|^\theta)\}$$

is the *Łojasiewicz exponent at infinity of g on a set $S \subset X$*.

Our main result is

Theorem 1.2. *Let $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings.*

(i) *For any $\lambda \in L$, $\mathcal{L}_{\infty,f\to\lambda}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$.*

(ii) *The function*

$$\vartheta_{g/f} : L \ni \lambda \mapsto \mathcal{L}_{\infty,f\to\lambda}(g)$$

is upper semi-continuous, and there exists a semi-algebraic stratification

$$L = S_1 \cup \cdots \cup S_j \tag{1.1}$$

such that $\vartheta_{g/f}$ *is constant on each stratum* S_i, $i = 1, \ldots, j$.

The proof of Theorem 1.2 is given in Section 3. Theorem 1.2(ii) was proved in [18] for complex polynomials, under the assumption (i).

Now let $f : M \to L$ be a semi-algebraic mapping of class $\mathscr{C}^1$ and let df be the differential of f. Let

$$\nu(df) : M \ni x \mapsto \nu(df(x)) \in \mathbb{R},$$

be the Rabier function, i.e. for $A = df(x) : M \to L$,

$$\nu(A) = \inf_{\|\phi\|=1} \|A^*(\phi)\|,$$

where $A^* : L^* \to M^*$ is the adjoint operator and $\phi \in L^*$. For a semi-algebraic function $f : M \to \mathbb{R}$ (or a complex polynomial) we have $\nu(df) = |\nabla f|$, where ∇f is the gradient of f.

Definition 1.3. The *Łojasiewicz exponent of df near a fibre* $f^{-1}(\lambda)$ is defned to be $\mathcal{L}_{\infty,\lambda}(f) = \mathcal{L}_{\infty,f\to\lambda}(\nu(df))$.

Remark 1.4. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a semi-algebraic mapping of class $\mathscr{C}^1$ and let $\kappa(df) : \mathbb{R}^n \ni x \mapsto \kappa(df(x)) \in \mathbb{R}$ be the Kuo function [10]; i.e., for $A = df(x) = (A_1, \ldots, A_m) : \mathbb{R}^n \to \mathbb{R}^m$,

$$\kappa(A) = \min_{1\leqslant i\leqslant m} \operatorname{dist}(\nabla A_i, \langle \nabla A_j\rangle_{j\neq i}),$$

where $\langle a_j\rangle_{j\neq i}$ is the vector space generated by the vectors $(a_j)_{j\neq i}$. As $\nu(A) \leqslant \kappa(A) \leqslant \sqrt{m}\nu(A)$ ([11], Proposition 2.6), for any $\lambda \in L$ we have

$$\mathcal{L}_{\infty,\lambda}(f) = \mathcal{L}_{\infty,f\to\lambda}(\kappa(df)).$$

An analogous result holds for the Gaffney function [4] (cf. [9], Proposition 2.3).

The function $\nu(df)$ is continuous and semi-algebraic ([11], Proposition 2.4), so Theorem 1.2 implies:

Corollary 1.5. *Let $f : M \to L$ be a semi-algebraic mapping of class $\mathscr{C}^1$. Then $\mathcal{L}_{\infty,\lambda}(f) \in \mathbb{Q} \cup \{-\infty, +\infty\}$ for any $\lambda \in L$, and the function $L \ni \lambda \mapsto \mathcal{L}_{\infty,\lambda}(f)$ is upper semi-continuous and has a finite number of values. In particular, there exists $\alpha > 0$ such that*

$$K_\infty(f) = \{\lambda \in L : \mathcal{L}_{\infty,\lambda}(f) < -1-\alpha\}.$$

In the case of complex regular mappings, from Theorem 1.2 we obtain:

Corollary 1.6. *Let $X \subset \mathbb{C}^n$ be a complex algebraic set, and let $g : X \to \mathbb{C}^m$ and $f : X \to \mathbb{C}^k$ be complex regular mappings. Then there exists a complex algebraic stratification $\mathbb{C}^k = S_1 \cup \cdots \cup S_j$ such that the function*

$$\vartheta_{g/f} : \mathbb{C}^k \ni \lambda \mapsto \mathcal{L}_{\infty,f\to\lambda}(g) \in \mathbb{Q} \cup \{-\infty, +\infty\}$$

is constant on each stratum S_i, $i = 1, \ldots, j$. Moreover, $\vartheta_{g/f}$ is upper semi-continuous.

The proof of the above corollary will be given in Section 4. The crucial fact in the proof is that $\vartheta_{g/f}(\mathbb{C}^n) = \vartheta_{g/f}(\mathbb{R}^{2n})$ and this set is finite (Theorem 1.2).

For a complex polynomial $f : \mathbb{C}^n \to \mathbb{C}$ the set $K_\infty(f)$ is finite (Proposition 2.4; see also [11], Theorem 3.1); hence Corollary 1.6 gives:

Corollary 1.7. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function with $\deg f > 0$. Then there exist a finite set $S \subset \mathbb{C}$ with $K_\infty(f) \subset S$ and a constant $c_f \in \mathbb{Q}$ with $c_f \geqslant -1$ such that $\mathcal{L}_{\infty,\lambda}(f) = c_f$ for $\lambda \in \mathbb{C} \setminus S$, and $\mathcal{L}_{\infty,\lambda}(f) < c_f$ for $\lambda \in S$.*

2. Auxiliary results

In what follows, L, M, N are finite-dimensional real vector spaces. We will use the Euclidean norm $|\cdot|$ in M (or in N, L). For $A \subset M$, let $\varrho(\,\cdot\,, A)$ denote the distance function to A, i.e. $\varrho(x, A) = \inf_{y\in A} |x - y|$ if $A \neq \emptyset$, and $\varrho(x, \emptyset) = 1$.

2.1. **Semi-algebraic mappings.** A subset of M is called semi-algebraic if it is defined by a finite alternative of finite systems of inequalities $P > 0$ or $P \geqslant 0$, where P are polynomials on M (see [1], [2]). A mapping $f : X \to N$, where $X \subset M$, is called *semi-algebraic* if the graph $\Gamma(f)$ of f is a semi-algebraic set. For instance, the distance to a semi-algebraic set is a semi-algebraic function (cf. [2]):

Proposition 2.1. *Let $V \subset M$ be a semi-algebraic set. Then the function $\varrho_V : M \ni x \mapsto \varrho(x, V) \in \mathbb{R}$ is continuous and semi-algebraic.*

Let $X \subset M$ and let $f : X \to N$ be any mapping. We say (cf. [8]) that f is *proper at a point* $y \in N$ if there exists an open neighbourhood U of y such that $f : f^{-1}(U) \to U$ is a proper map. The set of points at which f is not proper is denoted by $\mathfrak{S}_f$. It is obvious that the set $\mathfrak{S}_f$ is closed. It is known that for a complex algebraic set $X \subset \mathbb{C}^n$ and a complex regular mapping $f : X \to \mathbb{C}^m$, the set $\mathfrak{S}_f$ is complex algebraic.

Proposition 2.2. *Let X be a closed semi-algebraic set. If the mapping $f : X \to N$ is semi-algebraic, then the set $\mathfrak{S}_f$ is also semi-algebraic.*

Proof. Since X is a closed set, we have

$$\mathfrak{S}_f = \{y \in N : \forall_{A,\varepsilon>0}\ \exists_{x\in X}\ |x| > A \ \wedge\ |f(x) - y| < \varepsilon\}.$$

Then, by the Tarski-Seidenberg Theorem, we obtain the assertion. □

Let $f : X \to N$ with $X \subset M$. The *degree* of f is defined by

$$\deg f = \inf\{\theta \in \mathbb{R} : \exists_{C,R>0}\ \forall_{x\in X}\ (|x| \geqslant R \ \Rightarrow\ |f(x)| \leqslant C|x|^\theta)\}.$$

Set $\operatorname{supp} f = \{x \in X : f(x) \neq 0\}$.

A curve $\varphi : [r, +\infty) \to M$ is called *meromorphic at* $+\infty$ if φ is the sum of a Laurent series of the form

$$\varphi(t) = a_p t^p + a_{p-1} t^{p-1} + \cdots, \qquad a_i \in M, \quad p \in \mathbb{Z}.$$

In the case of a polynomial function and the Laurent series at infinity, the above degree is the usual degree; that is, $\deg \varphi = p$ if $a_p \neq 0$, and $\deg \varphi = -\infty$ if $\varphi \equiv 0$.

Proposition 2.3. *Let X be a closed semi-algebraic set and let $f : X \to N$ be a semi-algebraic mapping. Then:*

(i) $\deg f \in \mathbb{Q} \cup \{-\infty\}$.

(ii) $\deg f = -\infty$ *if and only if* $\operatorname{supp} f$ *is bounded.*

(iii) *If* $\deg f \in \mathbb{Q}$, *then there exist* $C, R > 0$ *such that*

$$|f(x)| \leqslant C|x|^{\deg f} \qquad \text{for } x \in X, \quad |x| \geqslant R.$$

(iv) *Let* $\beta(f) = \min\{n \in \mathbb{Z} : n > 0,\, n \geqslant \deg f\}$. *Then there exist* $R > 0$ *and* $\alpha < 0$ *such that*

$$|f(x)| \leqslant (1+|x|^2)^{\beta(f)}|x|^{\alpha} \qquad \text{for } x \in X, \quad |x| > R. \tag{2.1}$$

Proof. If $\operatorname{supp} f$ is bounded, then the assertion is obvious. Assume that $\operatorname{supp} f$ is unbounded. Then the set

$$Y = \{(y, f(y)) \in X \times N :\ \forall_{x \in X}\ |x| = |y| \ \Rightarrow\ 2|f(y)| \geqslant |f(x)|\}$$

is unbounded and semi-algebraic. So, by the Curve Selection Lemma at infinity, there exists a curve $\psi = (\varphi, \eta) : [r, +\infty) \to Y$ meromorphic at $+\infty$ such that $\eta = f \circ \varphi$, $\deg \eta \in \mathbb{Z}$, and $\deg \varphi > 0$. Let $\theta = \deg \eta / \deg \varphi$. Then $\theta \in \mathbb{Q}$ and for some $C, D, R > 0$,

$$C|\varphi(t)|^{\theta} \leqslant |f(\varphi(t))| \leqslant D|\varphi(t)|^{\theta}, \qquad t > R. \tag{2.2}$$

The definition of Y now implies that for $x \in X$, $|x| = |\varphi(t)|$, $t > R$,

$$|f(x)| \leqslant |f(\varphi(t))| \leqslant D|\varphi(t)|^{\theta} = D|x|^{\theta}.$$

So, $\deg f \leqslant \theta$. Since, by (2.2), $\deg f \geqslant \theta$, it follows that $\deg f = \theta$. This gives (i), (ii) and (iii). Part (iv) follows immediately from (iii). □

2.2. $\mathscr{C}^1$ **semi-algebraic functions.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a semi-algebraic function of class $\mathscr{C}^1$ in $x = (x_1, \dots, x_n)$. Then the gradient $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) : \mathbb{R}^n \to \mathbb{R}^n$ is a semi-algebraic mapping.

Proposition 2.4. *There exist* $C, \delta, R > 0$ *such that*

$$|f(x)| \geqslant R \ \Rightarrow\ |x|\,|\nabla f(x)| \geqslant C|f(x)|, \tag{2.3}$$

$$|f(x)| \leqslant \delta \ \Rightarrow\ |x|\,|\nabla f(x)| \geqslant C|f(x)|. \tag{2.4}$$

In particular, the set $K_\infty(f)$ *is finite. The assertion also holds for complex polynomials.*

Proof. As in [19] and [6], we use Hörmander's method. To prove (2.3), assume the contrary. Then the semi-algebraic set

$$X = \{(x, y, z, \varepsilon) \in \mathbb{R}^{2n} \times \mathbb{R}^2 : y = \nabla f(x),\ z = f(x),\ |z| \geqslant \varepsilon,\ \varepsilon|y||x| < |z|\}$$

has an accumulation point of the form $(x_0, y_0, z_0, +\infty)$. Thus, by the Curve Selection Lemma at infinity there exists a curve $\psi = (\varphi, \tau, \eta_1, \eta_2) : [r, +\infty) \to X$ meromorphic at infinity such that $\psi(t) \to (x_0, y_0, z_0, +\infty)$ as $t \to +\infty$. Then $\deg \eta_2 > 0$, $\deg \eta_1 > 0$, $\deg \varphi > 0$, and

$$\deg \eta_2 + \deg \tau + \deg \varphi \leqslant \deg \eta_1.$$

On the other hand,

$$\deg \eta_1 = \deg \eta_1' + 1 = \deg (f \circ \varphi)' + 1 \leqslant \deg \tau + \deg \varphi,$$

and we obtain a contradiction. Analogously we prove (2.4) and the assertion in the complex case. □

2.3. **Łojasiewicz exponent.** For three semi-algebraic sets X, Y, $Z \subset M$ such that $X \cap Y \subset Z$, we define a *regular separation exponent of* Y *and* Z *on* X at a point $x_0 \in X \cap Y$ to be any real positive θ such that

$$(\#) \qquad \varrho(x, Y) \geqslant C\varrho(x, Z)^{\theta} \qquad \text{for } x \in X \cap \Omega,$$

where $C > 0$ and Ω is a neighbourhood of x_0. The infimum of all such exponents θ will be denoted by $\mathcal{L}_{x_0}(X; Y, Z)$. By using the method of Lipschitz stratifications ([13], [14]), the following is proved in Theorem 1.5 of [20]:

Proposition 2.5. *Let* X, Y, $Z \subset M$ *be closed semi-algebraic sets such that* $X \cap Y \subset Z$*, and let* $x_0 \in X \cap Y$*.*

(i) *Then* $\mathcal{L}_{x_0}(X; Y, Z) \in \mathbb{Q}$*, and (#) holds for* $\theta = \mathcal{L}_{x_0}(X; Y, Z)$*, some* $C > 0$ *and a neighbourhood* Ω *of* x_0*, provided* $0^0 = 0$*.*

(ii) *If* $x_0 \in \overline{X \setminus Z}$*, then* $\mathcal{L}_{x_0}(X; Y, Z)$ *is attained on an analytic curve, i.e. for any neighbourhood* $\tilde{\Omega}$ *of* x_0 *there exists an analytic curve* $\varphi : [0, r) \to X \cap \tilde{\Omega}$ *such that* $\varphi((0, r)) \subset X \setminus Z$ *and* $\varphi(0) \in X \cap Y$*, and for some constant* $C_1 > 0$*,*

$$C\varrho(\varphi(t), Z)^{\mathcal{L}_{x_0}(X;Y,Z)} \leqslant \varrho(\varphi(t), Y) \leqslant C_1\varrho(\varphi(t), Z)^{\mathcal{L}_{x_0}(X;Y,Z)}, \ t \in [0, r).$$

If $Z = X \cap Y$ and $x_0 \in \overline{X \setminus Y}$, then obviously $\mathcal{L}_{x_0}(X; Y, Z)$ is equal to the *Łojasiewicz exponent* $\mathcal{L}_{x_0}(X, Y)$ *of* X *and* Y *at* x_0, i.e. the optimum exponent θ in the following separation condition:

$$(\mathrm{S}) \qquad \varrho(x, X) + \varrho(x, Y) \geqslant C\varrho(x, X \cap Y)^{\theta} \quad \text{for } x \in \Omega,$$

considered in a neighbourhood $\Omega \subset M$ of x_0 for some constant $C > 0$. Note that Proposition 2.5 also holds in the subanalytic case.

2.4. **Stratification.** By *stratification* of a subset $X \subset M$ we mean a decomposition of X into a locally finite disjoint union

$$(2.5) \qquad X = \bigcup S_{\alpha},$$

where the subsets S_{α} are called *strata*, such that each S_{α} is a connected embedded submanifold of M, and each $(\overline{S_{\alpha}} \setminus S_{\alpha}) \cap X$ is the union of some strata of dimension smaller than $\dim S_{\alpha}$.

The *i-th skeleton* of the stratification (2.5) is

$$X^i = \bigcup_{\dim S_{\alpha} \leqslant i} S_{\alpha}.$$

The stratification (2.5) is called *semi-algebraic* if all the skeletons X^i are semi-algebraic sets (or equivalently if the number of strata is finite and they are all semi-algebraic). The stratification (2.5) of a complex algebraic subset X of a complex linear space M is called *complex algebraic* if all the skeletons X^i are complex algebraic subsets of M and the number of strata is finite.

By Corollaries 2.6 and 2.7 in [20] we have:

Proposition 2.6. *Let* X, Y, $Z \subset M$ *be closed semi-algebraic sets such that* $X \cap Y \subset Z$*. Then there exists a stratification*

$$(2.6) \qquad X \cap Y = S_1 \cup \cdots \cup S_k$$

of $X \cap Y$ *such that the function*

$$(2.7) \qquad X \cap Y \ni x \mapsto \mathcal{L}_x(X; Y, Z)$$

is constant on each stratum S_i. In particular, the function (2.7) *is upper semi-continuous. If additionally $X_1, \dots, X_n \subset X \cap Y$ are semi-algebraic sets, then one can require that the stratification* (2.6) *is compatible with any X_j, i.e. any X_j is a union of some strata S_i.*

3. Proof of Theorem 1.2

Let $X \subset M$ be a closed semi-algebraic set, and let $g : X \to N$ and $f : X \to L$ be continuous semi-algebraic mappings.

The values $\vartheta_{g/f}(\lambda) \in \{-\infty, +\infty\}$ are characterised by the following:

Remark 3.1. (i) By Proposition 2.3 and the definition of $\mathfrak{S}_f$ we have:

$$\vartheta_{g/f}(\lambda) = +\infty \quad \Longleftrightarrow \quad \lambda \in L \setminus \mathfrak{S}_f. \tag{3.1}$$

(ii) Let $h = f|_{g^{-1}(0)}$. From the definition of $\mathcal{L}_{\infty, f\to\lambda}(g)$ we have:

$$\vartheta_{g/f}(\lambda) = -\infty \ \Leftrightarrow\ \lambda \in \mathfrak{S}_h \ \Leftrightarrow\ \mathcal{L}_\infty\left(f - \lambda | g^{-1}(0)\right) < 0. \tag{3.2}$$

Before the proof of Theorems 1.2 we give four lemmas and a proposition. Let $B = \{z \in M : |z| < 1\}$ and let $H : B \to M$ be of the form

$$H(z) = \frac{z}{1 - |z|^2}.$$

Lemma 3.2. *The mapping H is semi-algebraic and invertible with inverse*

$$H^{-1}(x) = \frac{2x}{1 + \sqrt{1 + 4|x|^2}}.$$

Moreover, for any $R > 0$,

$$|H(z)| \geqslant R \quad \Longleftrightarrow \quad \frac{2R}{1 + \sqrt{1 + 4R^2}} \leqslant |z| < 1. \tag{3.3}$$

Proof. H is a semi-algebraic mapping as the restriction of a rational mapping to the semi-algebraic set B. By an easy calculation we obtain (3.3) and the formula for H^{-1}. □

By Lemma 3.2 we may define the following semi-algebraic sets:

$$\begin{aligned} Y &= \{(x, \lambda, \delta) \in X \times L \times \mathbb{R} : |f(x) - \lambda| \leqslant \delta\}, \\ Z_1 &= \{(z, \lambda, \delta) \in B \times L \times \mathbb{R} : (H(z), \lambda, \delta) \in Y\}, \\ Z_2 &= \partial B \times L \times \mathbb{R}, \\ Z &= Z_1 \cup Z_2. \end{aligned}$$

Let $V = g^{-1}(0)$, and let

$$W = \{(z, \lambda, \delta) \in Z_1 : H(z) \in V\}.$$

Define a mapping $F : Z \to \mathbb{R}$ by

$$F(z, \lambda, \delta) = (1 - |z|^2)\varrho((z, \lambda, \delta), W).$$

Since W is a semi-algebraic set, Proposition 2.1 implies that F is a semi-algebraic mapping.

For any $\lambda \in L$, $\delta \geqslant 0$ and $S \subset X$ we set

$$S_{\lambda,\delta} = \{x \in S : |f(x) - \lambda| \leqslant \delta\}.$$

Lemma 3.3. *Let $\lambda_0 \in L$ and $\delta_0 > 0$ be such that the set V_{λ_0,δ_0} is bounded, and suppose $X_{\lambda_0,\delta}$ is unbounded for any $\delta > 0$. Then there exist $C, D, R > 0$ such that for any $(x, \lambda, \delta) \in Y$, where $0 < \delta \leqslant \frac{\delta_0}{2}$ and $|\lambda - \lambda_0| \leqslant \delta$, we have*

$$C|x|^{-1} \leqslant F(H^{-1}(x), \lambda, \delta) \leqslant D|x|^{-1}, \quad x \in X_{\lambda_0,\delta}, \quad |x| \geqslant R. \tag{3.4}$$

Proof. Let $Z^\delta = \{(z, \lambda, \delta) \in Z_1 : |\lambda - \lambda_0| \leqslant \delta\}$. Then $Z^{\delta'} \subset Z^{\delta''}$ if $\delta' \leqslant \delta''$. By the definition of F we have

$$F(z, \lambda, \delta) = |H(z)|^{-1}|z|\varrho((z, \lambda, \delta), W) \quad \text{for} \quad (z, \lambda, \delta) \in Z_1, \quad z \neq 0.$$

Hence, by (3.3), it suffices to prove that for some $c, d, r > 0$, with $r < 1$, and $\delta_1 = \frac{\delta_0}{2}$,

$$c \leqslant |z|\varrho((z, \lambda, \delta), W) \leqslant d \quad \text{for} \quad (z, \lambda, \delta) \in Z^{\delta_1}, \quad r \leqslant |z| < 1. \tag{3.5}$$

Because Z^{δ_1} is bounded, the set $\{|z|\varrho((z, \lambda, \delta), W) : (z, \lambda, \delta) \in Z_{\lambda_0,\delta_1}\}$ is also bounded. Hence the right-hand estimate in (3.5) holds. By (3.3) and the assumptions on V_{λ_0,δ_0} and $X_{\lambda_0,\delta}$, there exists $0 < r < 1$ for which the set W has no accumulation points in $A = \{(z, \lambda, \delta) \in Z^{\delta_1} : r \leqslant |z|\}$. Moreover, A is bounded, so $c = \inf\{|z|\varrho((z, \lambda, \delta), W) : (z, \lambda, \delta) \in A\} > 0$. This gives the left-hand estimate in (3.5). □

Let $X_H = H^{-1}(X) \cup \partial B$ and $V_H = H^{-1}(V)$. Since g and H are semi-algebraic mappings the sets V, X_H, V_H are semi-algebraic. Moreover, X_H is closed and $V_H = (g \circ H)^{-1}(0)$. Define $g_H : X_H \to N$ by

$$g_H(z) = \begin{cases} \dfrac{g \circ H(z)}{(1 + |H(z)|^2)^{\beta(g)}} & \text{for } z \in X_H \cap B, \\ 0 & \text{for } z \in \partial B, \end{cases}$$

where $\beta(g)$ is defined in Proposition 2.3 (iv).

Lemma 3.4. *The mapping g_H is continuous, semi-algebraic and*

$$(g_H)^{-1}(0) = V_H \cup \partial B. \tag{3.6}$$

Proof. By (2.1) in Proposition 2.3, g_H is continuous. Since the mapping g is semi-algebraic, so is $B \ni x \mapsto g \circ H(x)$, and hence also $h : (X_H \cap B) \ni z \mapsto (g(z), (1 + |H(z)|^2)^{\beta(g)}) \in N \times \mathbb{R}$. The graph of g_H is the union of $\partial B \times \{0\}$ and the image of the graph h under the semi-algebraic mapping $M \times N \times (0, +\infty) \ni (z, y, t) \mapsto (z, \frac{1}{t}y) \in M \times N$, so the graph of g_H is semi-algebraic. The equality (3.6) is obvious. □

The set Z is semi-algebraic and X_H is its image under the projection map $Z \ni (z, \lambda, \delta) \mapsto z \in M$. Hence, we may define a semi-algebraic mapping $G : Z \to N$ by

$$G(z, \lambda, \delta) = g_H(z).$$

Let Γ be the graph of the semi-algebraic mapping $(G, F) : Z \to N \times \mathbb{R}$. Since Z is a closed set, so is Γ.

Lemma 3.5. *There exists a stratification*

$$G^{-1}(0) = S_1 \cup \cdots \cup S_j \tag{3.7}$$

such that the function

$$\mathfrak{L} : G^{-1}(0) \ni v \mapsto \mathcal{L}_{(v,0,0)}(\Gamma; Z \times \{0\} \times \mathbb{R}, Z \times N \times \{0\}) \tag{3.8}$$

is constant on each stratum S_i. In particular, the set of values of $\mathfrak{L}$ is a finite subset of $\mathbb{Q}$.

Proof. By (3.6), $G^{-1}(0) = F^{-1}(0)$, so

$$G^{-1}(0) \times \{0\} \times \{0\} = \Gamma \cap (Z \times \{0\} \times \mathbb{R}) \subset Z \times N \times \{0\}.$$

Proposition 2.5 now shows that the values of $\mathfrak{L}$ are rational numbers. Moreover, from Proposition 2.6 we obtain a stratification (3.7) satisfying the assertion. $\square$

Take any $\lambda_0 \in L$ and define

$$l_{\lambda_0}(g) = \max\{\mathfrak{L}(z, \lambda_0, 0) : (z, \lambda_0, 0) \in Z_2\}.$$

By Lemma 3.5, $l_{\lambda_0}(g) \in \mathbb{Q}$.

Proposition 3.6. *Let $\delta_0 > 0$ be such that the set V_{λ_0,δ_0} is bounded, and suppose the set $X_{\lambda_0,\delta}$ is unbounded for any $\delta > 0$. Then*

$$\mathcal{L}_{\infty,f\to\lambda_0}(g) = 2\beta(g) - l_{\lambda_0}(g) \tag{3.9}$$

and for any sufficiently small $0 < \delta \leqslant \frac{\delta_0}{2}$ there exist $C, C', R > 0$ such that

$$|g(x)| \geqslant C|x|^{2\beta(g)-l_{\lambda_0}(g)} \quad \text{for } x \in X_{\lambda_0,\delta}, \quad |x| \geqslant R \tag{3.10}$$

and

$$C'|\varphi(t)|^{2\beta(g)-l_{\lambda_0}(g)} \geqslant |g(\varphi(t))| \geqslant C|\varphi(t)|^{2\beta(g)-l_{\lambda_0}(g)}, t \in [r, +\infty), \tag{3.11}$$

for some curve $\varphi : [r, +\infty) \to X_{\lambda_0,\delta}$ meromorphic at $+\infty$, with $\deg\varphi > 0$.

Proof. Let $E = \{(z, \lambda, \delta) \in Z_2 : \lambda = \lambda_0,\ \delta = 0\}$ and $\alpha = l_{\lambda_0}(g)$. By the definition of $l_{\lambda_0}(g)$, for any $(z, \lambda_0, 0) \in E$ there exist a neighbourhood $\Omega_z \subset M \times L \times \mathbb{R}$ of $(z, \lambda_0, 0)$ and $C_z > 0$ such that

$$|G(y, \lambda, \delta)| \geqslant C_z |F(y, \lambda, \delta)|^{\alpha}, \qquad (y, \lambda, \delta) \in \Omega_z \cap Z.$$

Since the set E is compact, there exists $\tilde{C} > 0$ such that $C_z \geqslant \tilde{C}$ for $(z, \lambda_0, 0) \in E$, and there exist $0 < r_1 < 1$ and $0 < \delta_1 \leqslant \frac{\delta_0}{2}$ such that

$$|G(y, \lambda, \delta_1)| \geqslant \tilde{C}|F(y, \lambda, \delta_1)|^{\alpha}, \quad |\lambda - \lambda_0| \leqslant \delta_1, \quad r_1 \leqslant |y| < 1,$$

where $(y, \lambda, \delta_1) \in Z$. Consequently,

$$\frac{|g(x)|}{(1 + |x|^2)^{\beta(g)}} \geqslant \tilde{C}|F(H^{-1}(x), \lambda_0, \delta_1)|^{\alpha}, \quad x \in X_{\lambda_0,\delta_1}, \quad |x| \geqslant R,$$

where $R > 0$ is the unique solution of the equation $r_1 = \frac{2R}{1+\sqrt{1+4R^2}}$. Together with (3.4) this gives

$$|g(x)| \geqslant \tilde{C}C(1 + |x|^2)^{\beta(g)}|x|^{-\alpha} \quad \text{for } x \in X_{\lambda_0,\delta_1}, \quad |x| \geqslant R.$$

Hence for any $0 < \delta \leqslant \delta_1$, (3.10) follows.

Take any $0 < \delta \leqslant \delta_1$. Let $(z_0, \lambda_0, 0) \in Z_2$ be a point such that $\mathfrak{L}(z_0, \lambda_0, 0) = l_{\lambda_0}(g)$. By the assumption on V_{λ_0,δ_0} we have

$$(z_0, \lambda_0, 0, 0, 0) \in \overline{\Gamma \setminus (Z \times N \times \{0\})}, \qquad (z_0, \lambda_0, 0) \notin \overline{W}, \tag{3.12}$$

and $\mathfrak{L}(z_0, \lambda_0, 0) > 0$. Thus, by Proposition 2.5, for any sufficiently small neighbourhood $\tilde{\Omega}$ of $\omega = (z_0, \lambda_0, 0, G(z_0, \lambda_0, 0), F(z_0, \lambda_0, 0)) = (z_0, \lambda_0, 0, 0, 0)$ there exists an analytic curve

$$\psi = (\psi_1, \psi_2, \psi_3) : [0, r) \to \Gamma \cap \tilde{\Omega},$$

where $\psi_1 : [0,r) \to Z$, $\psi_2 = G \circ \psi_1 : [0,r) \to N$, $\psi_3 = F \circ \psi_1 : [0,r) \to \mathbb{R}$, $\psi((0,r)) \subset \Gamma \setminus (Z \times N \times \{0\})$ and $\psi(0) \in \Gamma \cap (Z \times \{0\} \times \mathbb{R})$, such that for some constant $C_1 > 0$,

$$\varrho(\psi(t), Z \times \{0\} \times \mathbb{R}) \leqslant C_1 \varrho(\psi(t), Z \times M \times \{0\})^\alpha \quad \text{for} \quad t \in [0,r). \tag{3.13}$$

Let $\varphi_1 : [0,r) \to M$, $\varphi_2 : [0,r) \to L$, $\varphi_3 : [0,r) \to \mathbb{R}$, and let $\psi_1 = (\varphi_1, \varphi_2, \varphi_3)$. By the choice of ψ we have $\varphi_1(t) \in B$ for $t \in (0,r)$, and $\varphi(0) \in \partial B$ by (3.12). Hence,

$$|H(\varphi_1(t))| \quad \to \infty \qquad \text{as} \quad t \to 0. \tag{3.14}$$

Since the neighbourhood $\tilde{\Omega}$ of ω can be small, one can assume that $0 \leqslant \varphi_3(t) < \delta$ for $t \in [0,r)$. Then, by the definition of Z, we have $|H(\varphi_1(t)) - \lambda_0| \leqslant \varphi_3(t) < \delta$ for $t \in (0,r)$, and so

$$H(\varphi_1(t)) \in X_{\lambda_0,\delta} \qquad \text{for } t \in (0,r). \tag{3.15}$$

By (3.13),

$$|G(\psi_1(t))| \leqslant C_1 |F(\psi_1(t))|^\alpha \quad \text{for} \quad t \in [0,r].$$

Hence, from (3.4) and (3.14), for some $0 < r_1 < r$,

$$\frac{|g(H(\varphi_1(t)))|}{(1 + |H(\varphi_1(t))|^2)^{\beta(g)}} \leqslant C_1 D^{l_{\lambda_0}(g)} |H(\varphi_1(t))|^{-\alpha}, \quad t \in (0, r_1].$$

Together with (3.14) and (3.15), this gives

$$|g(H(\varphi_1(t)))| \leqslant C' |H(\varphi_1(t))|^{2\beta(g)-\alpha}, \qquad t \in (0, r_1]$$

for some $C' > 0$. Now setting $\varphi(t) = H(\varphi_1(\frac{1}{t}))$ for $t \in [\frac{1}{r_1}, +\infty)$ we obtain (3.11). Finally, (3.11) and (3.10) yield (3.9). □

Proof of Theorem 1.2. Fix $\lambda_0 \in L$. First we prove (i). If for any $\delta > 0$ the set $V_{\lambda_0,\delta}$ is unbounded, then $\mathcal{L}_{\infty, f\to\lambda_0}(g) = -\infty$. If for some $\delta > 0$ the set $X_{\lambda_0,\delta}$ is bounded, then $\mathcal{L}_{\infty, f\to\lambda_0}(g) = +\infty$. The remaining case in (i) follows from the fact that $\beta(g) \in \mathbb{Z}$ (see Proposition 2.3) and from (3.9) in Proposition 3.6.

To prove (ii), we adopt the method of the proof of Theorem 3.2.2 in [18]. By Lemma 3.5, let

$$\vartheta_{g/f}(L) = \{r_1, \dots, r_s\} \subset \mathbb{Q} \cup \{-\infty, +\infty\}, \quad \text{where} \quad r_1 \leqslant \cdots \leqslant r_s.$$

Define $\Lambda_\xi = \{\lambda \in L : \mathcal{L}_{\infty, f\to\lambda}(g) \leqslant \xi\}$ for $\xi \in \overline{\mathbb{R}}$.

Fix r_i. We now prove that the set Λ_{r_i} is closed and semi-algebraic. If $r_i \in \{-\infty, +\infty\}$ this follows from Remark 3.1 and Proposition 2.2. So, let $r_i = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b > 0$. Define

$$T = \{(x,c) \in X \times \mathbb{R} : |g(x)|^b = c|x|^a\},$$

and let $p : T \ni (x,c) \mapsto (f(x), c) \in L \times \mathbb{R}$. Since the mapping p is semi-algebraic, Proposition 2.2 shows that the set $\mathfrak{S}_p$ is also semi-algebraic.

Let $\pi : L \times \mathbb{R} \ni (y,c) \mapsto y \in L$ and observe that

$$\Lambda_{r_i} = \overline{\pi(\mathfrak{S}_p)}. \tag{3.16}$$

Indeed, let $\lambda \in \Lambda_{r_i}$, and let $U \subset L$ be a neighbourhood of λ. Take a neighbourhood $U_1 \subset L$ of λ such that $\overline{U_1} \subset U$. Then, by Proposition 3.6, there exist $C' > 0$ such that the set

$$\{(x,y) \in f^{-1}(U_1) \times N : y = g(x), \ |y|^b \leqslant C' |x|^a\}$$

is unbounded. Since it is semi-algebraic, there exists a curve $\psi = (\varphi, \eta) : [r, +\infty) \to f^{-1}(U_1) \times N$ meromorphic at infinity such that $\deg \varphi > 0$, $\eta = g \circ \varphi$ and

$$|g(\varphi(t)|^b \leqslant C'|\varphi(t)|^a, \qquad t \in [r, +\infty).$$

Then, for some $\lambda' \in \overline{U_1} \subset U$ and $0 \leqslant c \leqslant C'$,

$$f \circ \varphi(t) \to \lambda' \qquad \text{and} \qquad \frac{|g(\varphi(t))|^b}{|\varphi(t)|^a} \to c \qquad \text{as} \quad t \to \infty.$$

Hence, $\lambda' \in \pi(\mathfrak{S}_p) \cap U$, and so $\lambda \in \overline{\pi(\mathfrak{S}_p)}$.

Now let $\lambda \in \overline{\pi(\mathfrak{S}_p)}$. Take any neighbourhood $U \subset L$ of λ, and let $\lambda' \in U$ and $c \in \mathbb{R}$ be such that $(\lambda', c) \in \mathfrak{S}_p$. Then for some sequence $(x_n, c_n) \in T$, where $x_n \in f^{-1}(U)$ and $c_n \in \mathbb{R}$ for $n \in \mathbb{N}$, we have

$$|x_n| \to \infty, \qquad f(x_n) \to \lambda' \qquad \text{and} \qquad c_n \to c \qquad \text{as} \quad n \to \infty.$$

Hence, there exists $C > 0$ such that $|c_n| \leqslant C$ for $n \in \mathbb{N}$, and so

$$|g(x_n)|^b \leqslant C|x_n|^a, \qquad n \in \mathbb{N}.$$

This gives $\mathcal{L}_\infty(g|f^{-1}(U)) \leqslant r_i$, and hence $\mathcal{L}_{\infty, f\to\lambda}(g) \leqslant r_i$. Summing up, $\lambda \in \Lambda_{r_i}$ and (3.16) is proved.

By Proposition 2.2, the set $\mathfrak{S}_p$ is semi-algebraic, so, by (3.16), Λ_{r_i} is closed and semi-algebraic. In particular, the function $\vartheta_{g/f}$ is upper semi-continuous. From the definition of Λ_{r_i} we have $\Lambda_{r_1} \subsetneq \ldots \subsetneq \Lambda_{r_s} = L$. Hence, Λ_ξ is semi-algebraic for any $\xi \in \overline{\mathbb{R}}$. Therefore there exists a semi-algebraic stratification of the form (1.1) compatible with any intersection $X_1 \cap \cdots \cap X_j$, where $X_1, \ldots, X_j \in \{\Lambda_{r_1}, \ldots, \Lambda_{r_s}\}$. Thus, the function $\vartheta_{g/f}$ is constant on each stratum S_i, and Theorem 1.2 is proved. □

Corollary 3.7. *If $\theta = \mathcal{L}_{\infty, f\to\lambda}(g) \in \mathbb{Q}$, then for some $C, C', R, \delta > 0$,*

$$|g(x)| \geqslant C|x|^\theta \qquad \text{for } x \in X,\ |x| \geqslant R,\ |f(x) - \lambda| < \delta, \tag{3.17}$$

$$C'|\varphi(t)|^\theta \geqslant |g(\varphi(t))| \geqslant C|\varphi(t)|^\theta \qquad \text{for } t \in [r, +\infty), \tag{3.18}$$

where $\varphi : [r, +\infty) \to X$ is a curve meromorphic at infinity such that $\deg \varphi > 0$ *and $|f(\varphi(t)) - \lambda| < \delta$ for $t \in [r, +\infty)$.*

Proof. The assertion follows immediately from (3.10), (3.11) and Theorem 1.2. □

4. Proof of Corollary 1.6

Let $(z_1, \ldots, z_n)$, $(y_1, \ldots, y_m)$ be the coordinates of $z \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, respectively.

As in the proof of Theorem 1.2 we now show that for any $\xi \in \mathbb{Q} \cup \{-\infty, +\infty\}$, the set $\Lambda_\xi = \{\lambda \in \mathbb{C}^k : \mathcal{L}_{\infty, f\to\lambda}(g) \leqslant \xi\}$ is complex algebraic. For $\xi \in \{-\infty, +\infty\}$, this is obvious. Fix $\xi = \frac{a}{b}$, where $a, b \in \mathbb{Z}$, $b > 0$, $(a, b) = 1$.

Let $g = (g_1, \ldots, g_m)$. For any $i = 1 \ldots, n$ we define algebraic sets

$$T^i_\xi = \{(z, y, u) \in X \times \mathbb{C}^m \times \mathbb{C} : z_i u = 1,\ g^b_j(z) = y_j z^a_i,\ j = 1, \ldots, m\}$$

if $\xi \geqslant 0$,

$$T^i_\xi = \{(z, y, u) \in X \times \mathbb{C}^m \times \mathbb{C} : z_i u = 1,\ g^b_j(z) z^{-a}_i = y_j,\ j = 1, \ldots, m\}$$

if $\xi < 0$, and mappings

$$p_i : T^i_\xi \ni (z, y, u) \mapsto (f(z), y, u) \in \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C}.$$

Denote by $\mathfrak{S}_i$ the set of points at which p_i is not proper, and

$$A_i = \mathfrak{S}_i \cap \{(\lambda, y, u) \in \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C} : u = 0\}, \qquad i = 1, \ldots, n.$$

Since each $\mathfrak{S}_i$ is algebraic, so is A_i.

Let $\pi : \mathbb{C}^k \times \mathbb{C}^m \times \mathbb{C} \ni (\lambda, y, u) \mapsto \lambda \in \mathbb{C}^k$ and observe that

$$\Lambda_\xi = \bigcup_{i=1}^{n} \overline{\pi(A_i)}. \tag{4.1}$$

Indeed, let $\lambda \in \mathbb{C}^k$ satisfy $\mathcal{L}_{\infty,f\to\lambda}(g) \leqslant \xi$. Take any neighbourhoods $U, W \subset \mathbb{C}^k$ of λ such that $\overline{W} \subset U$. By Corollary 3.7, there exist $C > 0$ and a curve $\varphi = (\varphi_1, \ldots, \varphi_n) : [r, +\infty) \to f^{-1}(W)$ meromorphic at infinity with $\deg \varphi > 0$ such that

$$|g(\varphi(t))| \leqslant C|\varphi(t)|^\xi, \qquad t \in [r, \infty). \tag{4.2}$$

Let $\deg \varphi_i = \deg \varphi$. Then $\deg \varphi_i > 0$. By the definition of φ, there exists $\lambda' \in \overline{W}$ such that

$$f(\varphi(t)) \to \lambda' \qquad \text{as} \quad t \to \infty. \tag{4.3}$$

By (4.2), there exists $y \in \mathbb{C}^m$ such that

$$\eta(t) := \left(\frac{g_1^b(\varphi(t))}{\varphi_i^a(t)}, \ldots, \frac{g_m^b(\varphi(t))}{\varphi_i^a(t)} \right) \to y \quad \text{as} \quad t \to \infty.$$

Since $\deg \varphi_i > 0$, we may assume that $\varphi_i(t) \neq 0$ for $t \in [r, +\infty)$. Putting $u(t) = \frac{1}{\varphi_i(t)}$ for $t \in [r, +\infty)$, we easily see that

$$p_i(\varphi(t), \eta(t), u(t)) \to (\lambda', y, 0) \qquad \text{as} \quad t \to \infty.$$

Hence $(\lambda', y, 0) \in \mathfrak{S}_i$, so $\lambda' \in U \cap \pi(A_i)$, and thus $\lambda \in \overline{\pi(A_i)}$. This gives the inclusion "$\subset$" in (4.1).

We now prove "$\supset$". Let $\lambda \in \overline{\pi(A_i)}$. Take any neighbourhood U of λ. Then there exists $\lambda' \in U \cap \pi(A_i)$, and so $(\lambda', y, 0) \in \mathfrak{S}_i$ for some $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$. The definitions of A_i and T_ξ^i now yield a sequence $x_l = (x_{1,l}, \ldots, x_{n,l}) \in f^{-1}(U)$, $l \in \mathbb{N}$, such that $f(x_l) \to \lambda'$ and

$$|x_{i,l}| \to \infty, \quad \frac{g_j^b(x_l)}{x_{i,l}^a} \to y_j \quad \text{as} \quad l \to \infty, \quad j = 1, \ldots, m.$$

Consequently, there exists $C > |y|$ such that

$$|g(x_l)| \leqslant C|x_l|^\xi \qquad \text{for } l \in \mathbb{N}.$$

Hence, $\mathcal{L}_\infty(g|f^{-1}(U)) \leqslant \xi$. This gives $\mathcal{L}_{\infty,f\to\lambda}(g) \leqslant \xi$, and the inclusion "$\supset$" in (4.1) is proved.

By Theorem 1.2, the set $\vartheta_{g/f}(\mathbb{C}^k) \subset \mathbb{Q} \cup \{-\infty, +\infty\}$ is finite, say $\{r_1, \ldots, r_s\}$ with $r_1 < \cdots < r_s$. By (4.1), the sets Λ_{r_i}, $i = 1, \ldots, s$, are algebraic, and $\Lambda_{r_1} \subsetneq \cdots \subsetneq \Lambda_{r_s} = \mathbb{C}^k$. Then the function $\vartheta_{g/f}$ is upper semi-continuous. Hence the usual complex stratification of $\mathbb{C}^n$ compatible with complex constructible sets $\Lambda_{r_i} \setminus \Lambda_{r_{i-1}}$ is a desired stratification. This ends the proof. $\square$

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