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A NON-RESIDUALLY SOLVABLE HYPERLINEAR ONE-RELATOR GROUP

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ABSTRACT. In this short paper, we prove that the group $\langle a, b | a = [a, a^b]\rangle$ is hyperlinear. Unlike the nonresidually finite Baumslag-Solitar groups, this group is not residually solvable.

1. INTRODUCTION

Let Γ denote the one-relator group $\langle a, b | a^{-1}[a, a^b]\rangle$, where $a^b = bab^{-1}$ and $[a, a^b] = a^{-1}(a^b)^{-1}aa^b$. This group was introduced by G. Baumslag in [Baum69] as an example of a noncyclic one-relator group with the property that all of its finite index quotients are cyclic. It follows that the group Γ is not residually finite. Also, Γ is not residually solvable, since a lies in every one of the derived subgroups of Γ. A countable discrete group G is *hyperlinear* if it can be embedded as a subgroup of the unitary group $U(\mathcal{R}^\omega)$ of an ultrapower $\mathcal{R}^\omega$ of the hyperfinite type II_1 factor $\mathcal{R}$ (cf. [Pest08]). Equivalently, G is hyperlinear if the group von Neumann algebra $L(G)$ is embeddable into $\mathcal{R}^\omega$ (cf. [Pest08]). Proposition 4.14 of [Ueda09] establishes that every HNN extension of an $\mathcal{R}^\omega$-embeddable type II_1 factor over a hyperfinite von Neumann subalgebra is also $\mathcal{R}^\omega$-embeddable. We use this fact along with a now standard trick of McCool and Schupp for one-relator groups to prove that the group Γ above is hyperlinear. The main interest in this example is that it is an example of a nonresidually solvable hyperlinear one-relator group, and thus our result sheds a little light on the question of Nate Brown asking whether every one-relator group is hyperlinear. In [Rad00], Radulescu proved that the nonresidually finite Baumslag-Solitar group $\langle a, b | ab^3a^{-1}b^{-2}\rangle$ is hyperlinear. Radulescu's result is shown in [Pest08] to follow more simply from the fact that these Baumslag-Solitar groups are residually solvable, and hence sofic.

2. MAIN RESULT

Theorem 2.1. *The group* $\Gamma = \langle a, b | a^{-1}[a, a^b]\rangle$ *is hyperlinear.*

Proof. We apply a rewriting process due to McCool and Schupp (cf. [McCSch73]). Let $a_0 = a$ and $a_{-1} = bab^{-1}$. Note that the word

$$a^{-1}[a, a^b] = a^{-2}ba^{-1}b^{-1}abab^{-1}$$

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when rewritten in terms of a_0 and a_{-1} becomes

$$a_0^{-2}(a_{-1})^{-1}a_0(a_{-1}).$$

The group $H = \langle a_0, a_{-1} | a_0^{-2}(a_{-1})^{-1}a_0(a_{-1})\rangle$ is amenable, essentially by the Tits alternative. Or, we may appeal to Theorem 1.2 of [CeGrig97] and note that $a_0^{-2}(a_{-1})^{-1}a_0(a_{-1})$ has exponent sum zero on a_{-1} and can be obtained from $(a_{-1})a_0(a_{-1})^{-1}a_0^{-2}$ by inverting a_{-1} and cyclically shifting, and hence H is amenable. We then note that the group Γ is isomorphic to the HNN extension

$$H*_{\varphi} = \langle t, H | t^{-1}a_{-1}t = a_0\rangle.$$

Now, consider the group von Neumann algebra $L(H*_{\varphi})$. By Corollary 3.5 of [Ueda05], this is isomorphic to a reduced HNN extension of the hyperfinite II_1 factor $\mathcal{R}$ over $L(\mathbb{Z})$. Therefore, by Proposition 4.14 of [Ueda09], $L(H*_{\varphi})$ is embeddable into an $\mathcal{R}^{\omega}$, and therefore Γ is hyperlinear. $\square$

Remark 2.2. We wish to thank the referee for pointing out that recently it has been shown that any HNN extension of a sofic group over an amenable subgroup is sofic. Precisely, this is Corollary 3.4 of [DykCol10]. We may, in the above proof, replace Ueda's result by this one and obtain that Γ is, in fact, a sofic group.

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