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SETS OF INTEGERS AS SUPERDEGREES AND SUPERCLASS SIZES

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ABSTRACT. Supercharacters have recently been proposed as a sort of stand-in for the characters of p-groups. If $q > 1$ is a prime power, then every set of q-powers that contains 1 is both a set of superdegrees and a set of superclass sizes. Moreover, if r and s are integers that are greater than 1, then there is an algebra with exactly r superdegrees and exactly s superclass sizes. These results are direct analogs of results from the theory of p-groups.

1. INTRODUCTION

In [2], Diaconis and Isaacs introduce supercharacters of algebra groups, which mimic the irreducible characters of p-groups. This article explores the structure of sets of superdegrees and superclass sizes. Specifically, it shows that a set of p-powers is the set of superdegrees of some algebra if and only if it is the set of superclass sizes of some algebra if and only if it contains 1. Also, if r and s are integers that are greater than 1, then there is an algebra with exactly r superdegrees and exactly s superclass sizes. These results are direct analogs of results in [4], [1], and [3], respectively, concerning the theory of p-groups.

Throughout, let F be a finite field of characteristic p, and let $q = |F|$.

If J is a finite-dimensional, nilpotent F-algebra, then the set

$$1 + J = \{1 + x \mid x \in J\}$$

is a group with multiplication defined by

$$(1 + x)(1 + y) = 1 + (x + y + xy).$$

It is easy to see that

$$(1 + x)^{-1} = 1 + \sum_{i=1}^{\infty} (-x)^i,$$

where this expression is well-defined because J is nilpotent. A group of this form is called an algebra group. Note that the nilpotence of the underlying algebra is part of the definition of an algebra group, so this feature is always present when algebra groups are being considered, regardless of whether the word "nilpotent" is explicitly used.

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The group $G = 1 + J$ acts on the right and on the left of J in the expected way; for x and y in J, let

$$x(1 + y) = x + xy$$

and

$$(1 + y)x = x + yx,$$

noting that these actions commute with each other. Write GxG for the two-sided orbit of x. The subset $1 + GxG$ of G is called the superclass of $1 + x$. The second main result, Theorem 3.1, implies that a set S of q-powers is the set of superclass sizes of an F-algebra if and only if $1 \in S$.

Write J^* for the dual space of J. The actions of the previous paragraph induce right and left actions of G on J^*; for $\lambda \in J^*$, $g \in G$, and $x \in J$, let

$$(\lambda \cdot g)(x) = \lambda(xg^{-1})$$

and

$$(g \cdot \lambda)(x) = \lambda(g^{-1}x).$$

Again, note that these actions commute with each other. Write λG and $G\lambda G$ for the right and two-sided orbits, respectively, of λ. The number $|\lambda G|$ is called a superdegree of J. The first main result, Theorem 2.1, implies that a set S of q-powers is the set of superdegrees of an F-algebra if and only if $1 \in S$.

In order to justify the names superdegree and superclass, consider the following. Fix a nontrivial group homomorphism $\widetilde{\ }$ from the additive group of F to the group of nonzero complex numbers. Given any $\lambda \in J^*$, the supercharacter $\chi_\lambda : G \to \mathbb{C}$ is defined by

$$\chi_\lambda(1 + x) = \frac{|\lambda G|}{|G\lambda G|} \sum_{\mu \in G\lambda G} \widetilde{\mu(x)}.$$

As it turns out, χ_λ is a character. Plainly, the degree of the supercharacter χ_λ is $\chi_\lambda(1) = |\lambda G|$. Superclasses are the supercharacter analog of conjugacy classes.

For λ in J^*, define

$$R_\lambda = \{y \in J \mid Jy \subseteq \ker \lambda\}$$

and

$$L_\lambda = \{y \in J \mid yJ \subseteq \ker \lambda\}.$$

Note that R_λ is a left ideal and L_λ is a right ideal. In particular, $1 + R_\lambda$ and $1 + L_\lambda$ are algebra subgroups of G, and, as described in [2], they are the respective stabilizers of λ in G for the right and left actions.

By Theorem 5.10 of [2], the supercharacter χ_λ is irreducible if $R_\lambda + L_\lambda = J$. If this condition holds for all $\lambda \in J^*$, then the supercharacters of J are precisely the irreducible characters of $1 + J$. Thus, the subalgebra $R_\lambda + L_\lambda$ will be referenced in several of the constructions below.

Because the calculations needed to explore the constructions are straightforward, the details are often suppressed. A summary of each is included, however, so the patient reader should be able to reproduce the omitted computations with minimal thought.

2. Superdegrees

Since $1 + R_\lambda$ is the right stabilizer of λ, one sees that

$$|\lambda G| = |G|/|1 + R_\lambda| = |J|/|R_\lambda|.$$

Since R_λ is a subspace of J, it is clear that $|\lambda G|$ is an $|F|$-power.

Suppose S is the set of superdegrees of an F-algebra. By the previous paragraph, the set S must consist of $|F|$-powers. Moreover, the integer 1 is always the degree of the supercharacter corresponding to the functional 0.

Theorem 2.1 demonstrates that these two necessary conditions are also sufficient to guarantee that S is the set of superdegrees of some F-algebra. The analogous irreducible character degree result for p-groups can be found in the theorem of Section 3 of [4], in Theorem 3.1 of [5], and, for odd primes, in Theorem 6 of [7]. As discussed below, Theorem 2.1 generalizes all of these results. Theorem 3.2 in [5] states that if $|S| > 1$ and $2 \le n \le p$, then there is a group of nilpotence class n with set of irreducible character degrees S; the supercharacter analog of this statement is false, since, by Theorem 3.3 of [6], the nilpotence class of J is small whenever $|S|$ is small.

Theorem 2.1. *Write $e_0 = 0$ and let $e_1 \le e_2 \le \cdots \le e_m$ be positive integers. Then there is an $\mathbb{F}_q$-algebra J such that $J^3 = 0$, the supercharacters of J are exactly the irreducible characters of $1+J$, the superdegrees of J are $\{q^{e_i} \mid 0 \le i \le m\}$, and the superclass sizes of J are $\{q^i \mid 0 \le i \le m\}$.*

Proof. Let J be the F-vector space with basis

$$\mathcal{B} = \{x_{i,j} \mid 1 \le i \le m, 1 \le j \le e_i\} \cup \{y_j \mid 1 \le j \le e_m\} \cup \{z_i \mid 1 \le i \le m\}.$$

Write

$$\mathcal{D} = \{\delta_{i,j} \mid 1 \le i \le m, 1 \le j \le e_i\} \cup \{\epsilon_j \mid 1 \le j \le e_m\} \cup \{\gamma_i \mid 1 \le i \le m\}$$

for the basis for J^* dual to $\mathcal{B}$. Define a multiplication on J by setting

$$x_{i,j}y_k = \begin{cases} z_i & \text{if } j = k, \\ 0 & \text{if } j \ne k \end{cases}$$

and letting the product of all other basis elements, including $y_k x_{i,j}$, be 0. Set $G = 1 + J$, and notice that $J^3 = 0$.

Let

$$x = \sum_{i=1}^{m}(\sum_{j=1}^{e_i} a_{i,j}x_{i,j} + c_i z_i) + \sum_{j=1}^{e_m} b_j y_j$$

and

$$\lambda = \sum_{i=1}^{m}(\sum_{j=1}^{e_i} A_{i,j}\delta_{i,j} + C_i\gamma_i) + \sum_{j=1}^{e_m} B_j\epsilon_j$$

for fixed $a_{i,j}, b_j, c_i, A_{i,j}, B_j, C_i \in F$.

Consider R_λ first. Note that $\lambda(x_{k,l}x) = b_l C_k$ and $\lambda(y_l x) = 0 = \lambda(z_k x)$. Thus $x \in R_\lambda$ if and only if $b_l C_k = 0$ whenever $1 \le k \le m$ and $1 \le l \le e_k$. Set

$$K = \max\big(\{0\} \cup \{1 \le k \le m \mid C_k \ne 0\}\big),$$

so $x \in R_\lambda$ if and only if $b_l = 0$ whenever $1 \le l \le e_K$. Then $\{y_l + R_\lambda \mid 1 \le l \le e_K\}$ is a basis for the vector space J/R_λ, and so $|\lambda G| = |J|/|R_\lambda| = q^{e_K}$. Hence, the superdegrees are exactly $\{q^{e_i} \mid 0 \le i \le m\}$.

Next consider the superclasses. Since

$$1 + GxG \subseteq 1 + x + \operatorname{span}\{z_i \mid 1 \le i \le m\},$$

it must be the case that $|1 + GxG| \le q^m$. Considering $x = \sum_{i=1}^{k} x_{i,e_i}$ for each $0 \le k \le m$, one sees that each q^k is achievable.

Finally, calculate $R_\lambda + L_\lambda$. Note that y_k is in L_λ for each $1 \le k \le e_m$. Combining this with the calculation for R_λ above, one sees that the basis $\mathcal{B}$ is in $R_\lambda + L_\lambda$. Thus, by Theorem 5.10 of [2], the supercharacter χ_λ is irreducible. □

Theorem 2.1 deserves two comments. First, it is a generalization of the character theory result, as, in the character theory version, the group may now be taken to be an algebra group. Secondly, Theorem 2.1 does not yield just any algebra with the appropriate superdegrees, but an $\mathbb{F}_q$-algebra. That is, knowing something about the set of prime powers imposes more structure on the resulting algebra.

3. Superclass sizes

For $x \in J$, the notation $Jx = \{zx \mid z \in J\}$ and $xJ = \{xz \mid z \in J\}$ is self-explanatory. Corollary 3.2 of [2] yields that $|1 + GxG| = |Jx + xJ|$. Since $Jx + xJ$ is a subspace of J, it is clear that $|1 + GxG|$ is an $|F|$-power.

Suppose S is the set of superclass sizes of an F-algebra. By the previous paragraph, the set S must consist of $|F|$-powers. Moreover, the integer 1 is always the size of the superclass of the identity element $1 + 0$.

Theorem 3.1 demonstrates that these two conditions are sufficient to guarantee that S is the set of superclass sizes of an F-algebra. The analogous conjugacy class result can be found in the theorem of [1].

Theorem 3.1. *Let $0 = e_0 < e_1 < \cdots < e_m$ be integers. Then there is a commutative $\mathbb{F}_q$-algebra J such that $J^3 = 0$ and the superclass sizes of J are exactly $\{q^{e_i} \mid 0 \le i \le m\}$.*

Proof. Let $\mathcal{G}$ be the set of pairs (i, j) of positive integers such that there exist positive integers u and v satisfying the relations

$$u + v = m + 1,$$
$$i \le e_u,$$

and

$$j \le e_v.$$

Clearly (i, j) is in $\mathcal{G}$ if and only if (j, i) is in $\mathcal{G}$.

Set $\mathcal{H} = \{(i, j) \in \mathcal{G} \mid i \le j\}$, and let J be the F-vector space with basis

$$\mathcal{B} = \{x_i \mid 1 \le i \le e_m\} \cup \{y_{i,j} \mid (i, j) \in \mathcal{H}\}.$$

For all $(i, j) \in \mathcal{H}$, define $y_{j,i} = y_{i,j}$. Define a multiplication by setting

$$x_i x_j = \begin{cases} y_{i,j} & \text{if } (i, j) \in \mathcal{G}, \\ 0 & \text{if } (i, j) \notin \mathcal{G} \end{cases}$$

and letting the product of all other basis elements be 0. Then J is commutative, and $J^3 = 0$. Set $G = 1 + J$.

For each positive integer i satisfying $1 \le i \le e_m$, define

$$t_i = \min\{1 \le t \le m \mid i \le e_t\}.$$

In other words, t_i is the unique integer such that $e_{t_i-1} < i \leq e_{t_i}$. Also, set $t_{e_m+1} = m+1$.

Suppose for the moment that (i,j) is in $\mathcal{G}$, so there exist positive integers u and v such that $u+v = m+1$, such that $i \leq e_u$, and such that $j \leq e_v$. Thus, $t_i \leq u$, and so $m - t_i + 1 \geq m - u + 1 = v$. This implies that $j \leq e_v \leq e_{m-t_i+1}$. Conversely, suppose that i and j are integers such that $1 \leq i \leq e_m$ and $1 \leq j \leq e_{m-t_i+1}$. Setting $u = t_i$ and $v = m - t_i + 1$, it is clear that (i,j) is in $\mathcal{G}$.

The previous paragraph demonstrated that whenever $1 \leq i \leq e_m$ is an integer, then (i,j) is in $\mathcal{G}$ if and only if $1 \leq j \leq e_{m-t_i+1}$.

It now remains only to calculate the superclass sizes, so let

$$x = \sum_{i=1}^{e_m} a_i x_i + \sum_{(i,j)\in\mathcal{H}} b_{i,j} y_{i,j},$$

where the a_i and b_{ij} are elements of $\mathbb{F}_q$. Since

$$|1 + GxG| = |xG| = |xJ| = |x \operatorname{span}\{x_i \mid 1 \leq i \leq e_m\}|,$$

consider $v = \sum_{j=1}^{e_m} c_j x_j$. Setting $l = \min\big(\{i \mid a_i \neq 0\} \cup \{e_m + 1\}\big)$ yields that

$$xv = \sum_{i=l}^{e_m} a_i \sum_{j=1}^{e_{m-t_i+1}} c_j y_{i,j}.$$

Since $e_{m-t_i+1} \leq e_{m-t_k+1}$ whenever $k < i$, the c_j that appear in the equation above are exactly $\{c_j \mid 1 \leq j \leq e_{m-t_l+1}\}$. The elements $y_{l,j}$ are all distinct as j ranges from 1 to e_{m-t_l+1}, and it now follows that $|1 + GxG| = q^{e_{m-t_l+1}}$. Since $1 \leq t_l \leq m+1$ and since x was arbitrary, the superclass sizes all lie in the set $\{q^{e_k} \mid 0 \leq k \leq m\}$. But $t_{e_i} = i$ when $1 \leq i \leq m$, so considering $x = x_{e_{m-k+1}}$ shows that q^{e_k} is achievable for each $1 \leq k \leq m$. The number $q^{e_0} = 1$ is always a superclass size. □

Unlike Theorem 2.1, Theorem 3.1 is not a generalization of the analogous group theory result; in particular, since the algebra constructed in the proof is commutative, the conjugacy class sizes are all 1.

As promised, the following is now clear.

Corollary 3.2. *Let S be a set of integers and $q > 1$ be a prime power. Then the following are equivalent.*

(i) *S is the set of superdegrees for some $\mathbb{F}_q$-algebra.*
(ii) *S is a set of q-powers containing* 1.
(iii) *S is the set of superclass sizes for some $\mathbb{F}_q$-algebra.*

Proof. This is apparent from Theorem 2.1 and Theorem 3.1. □

4. Sizes of sets

This section will address what the sizes of the sets of superdegrees and of superclass sizes can be.

For a group, of course, the set of irreducible character degrees has size 1 if and only if the group is abelian if and only if the set of conjugacy class sizes has size 1. As shown by the next lemma, the analogous statement is true for supercharacters, although the analog of "abelian" is that the algebra has trivial multiplication.

Lemma 4.1. *Let J be a finite, nilpotent algebra. The following are equivalent.*

(i) *The only superdegree of J is* 1.
(ii) $J^2 = 0$.
(iii) *The only superclass size of J is* 1.

Proof. If $\lambda G = 1$, then, since $1 + R_\lambda$ is the right stabilizer of λ, the subalgebra R_λ must be all of J. Then, by the definition of R_λ, the kernel of λ contains J^2. Thus, if every superdegree is 1, then J^2 lies in the kernel of every $\lambda \in J^*$, and so $J^2 = 0$.

It is even easier to check that if $1 + GxG = \{1 + x\}$, then $Jx = 0$. Thus, if every superclass has size 1, it must be that $J^2 = 0$.

Conversely, if the multiplication is trivial, then the actions of G on J and on J^* are trivial, so every superdegree and every superclass size is 1. □

Just as the characters and conjugacy classes of a direct product of groups can be retrieved from the groups being combined, the supercharacters and superclasses of a direct sum of algebras can be retrieved from the summands. In particular, the following holds.

Lemma 4.2. *For $i \in \{1, 2\}$, let R_i and S_i be sets of nonnegative integers. If the $\mathbb{F}_q$-algebras J_i have superdegrees $\{q^r \mid r \in R_i\}$ and superclass sizes $\{q^s \mid s \in S_i\}$, then $J_1 \oplus J_2$ has superdegrees $\{q^{r_1+r_2} \mid r_i \in R_i\}$ and superclass sizes $\{q^{s_1+s_2} \mid s_i \in S_i\}$.*

Proof. Since J_1 and J_2 annihilate each other in $J_1 \oplus J_2$, it is clear that superclasses in the latter algebra are the product of superclasses from the subalgebras.

The superdegree claim is almost as easy, or one can appeal to Lemma 6.5 in [2]. □

Now, consider positive integers r and s. If r and s are to be the respective sizes of the sets of superdegrees and of superclass sizes of some F-algebra, then, by Lemma 4.1, either r and s are either both 1 or both not equal to 1.

Theorem 4.5 shows that this weak condition is sufficient to guarantee that there is an F-algebra with exactly r superdegrees and exactly s superclass sizes. The analogous result for p-groups can be found in the theorem of [3].

Lemma 4.3. *Let m and n be positive integers. Then there is an $\mathbb{F}_q$-algebra J with superdegrees $\{1, q^m\}$ and superclass sizes $\{q^i \mid 0 \le i \le \min\{m, n\}\} \cup \{q^n\}$ such that $J^3 = 0$ and the supercharacters of J are exactly the irreducible characters of $1 + J$.*

Proof. Let J be the F-vector space with basis

$$\mathcal{B} = \{x_{i,j} \mid 1 \le i \le m, 1 \le j \le n\} \cup \{y_i \mid 1 \le i \le m\} \cup \{z_j \mid 1 \le j \le n\}.$$

Write

$$\mathcal{D} = \{\delta_{i,j} \mid 1 \le i \le m, 1 \le j \le n\} \cup \{\epsilon_i \mid 1 \le i \le m\} \cup \{\gamma_j \mid 1 \le j \le n\}$$

for the basis of J^* dual to $\mathcal{B}$. Define a product on J by setting

$$x_{i,j} y_k = \begin{cases} z_j & \text{if } i = k, \\ 0 & \text{if } i \ne k \end{cases}$$

and letting the product of all other basis elements, including $y_k x_{i,j}$, be 0. Clearly, $J^3 = 0$. Set $G = 1 + J$.

Let

$$x = \sum_{i=1}^{m}(\sum_{j=1}^{n} a_{i,j}x_{i,j} + b_iy_i) + \sum_{j=1}^{n} c_jz_j$$

and

$$\lambda = \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} A_{i,j}\delta_{i,j} + B_i\epsilon_i\Big) + \sum_{j=1}^{n} C_j\gamma_j$$

for fixed $a_{i,j}, b_j, c_i, A_{i,j}, B_j, C_i \in F$.

Consider R_λ first. Since $\lambda(x_{k,l}x) = b_kC_l$ and $\lambda(y_kx) = 0 = \lambda(z_lx)$, one observes that $x \in R_\lambda$ if and only if $b_kC_l = 0$ for all $1 \leq k \leq m$ and $1 \leq l \leq n$. If some $C_l \neq 0$, then $R_\lambda = \operatorname{span}\big(\{x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{z_j \mid 1 \leq j \leq n\}\big)$, and $|\lambda G| = |J|/|R_\lambda| = q^m$. If every C_l is 0, then $R_\lambda = J$ and $|\lambda G| = 1$.

Next, consider the superclasses; one may check that

$$1 + GxG = \{1 + x + \sum_{j=1}^{n}\sum_{i=1}^{m}(a_{i,j}d_i + b_ie_{i,j})z_j \mid d_i, e_{i,j} \in F\}.$$

If any $b_k \neq 0$, then the variable $e_{k,j}$ controls the coefficient of z_j for each $1 \leq j \leq n$, and $|1 + GxG| = q^n$. So suppose each b_k is 0; then

$$1 + GxG = \{1 + x + \sum_{j=1}^{n}\sum_{i=1}^{m} a_{i,j}d_iz_j \mid d_i \in F\}.$$

Since there are n coefficients that may vary, namely those of the z_j, and there are m variables in these coefficients, namely the d_i, it must be that $|1+GxG| \leq q^{\min\{m,n\}}$. Considering $x = \sum_{i=1}^{k} x_{i,i}$ for each $0 \leq k \leq \min\{m, n\}$, one sees that each such q^k is achievable.

Finally, consider $R_\lambda + L_\lambda$. As above,

$$\operatorname{span}\big(\{x_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{z_j \mid 1 \leq j \leq n\}\big) \subseteq R_\lambda.$$

Since $\lambda(y_ix_{k,l}) = \lambda(y_iy_k) = \lambda(y_iz_l) = 0$,

$$\operatorname{span}\{y_i \mid 1 \leq i \leq m\} \subseteq L_\lambda,$$

so $R_\lambda + L_\lambda = J$. By Theorem 5.10 in [2], and since λ was arbitrary, every supercharacter is irreducible. □

Lemma 4.4. *Let n be a positive integer. Then there is an $\mathbb{F}_q$-algebra J with superdegrees $\{q^i \mid 0 \leq i \leq n\}$ and superclass sizes $\{1, q^{2n-1}\}$ such that $J^3 = 0$.*

Proof. Let J be the F-vector space with basis

$$\mathcal{B} = \{x_i \mid 1 \leq i \leq n\} \cup \{y_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq n\}.$$

Write

$$\mathcal{D} = \{\delta_i \mid 1 \leq i \leq n\} \cup \{\epsilon_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq n\}$$

for the basis of J^* dual to $\mathcal{B}$. Define a product on J by setting $x_ix_j = y_{i,j}$ and letting the product of all other basis elements be 0. Clearly, $J^3 = 0$. Set $G = 1+J$.

Let

$$x = \sum_{i=1}^{n}(a_ix_i + \sum_{j=1}^{n} b_{i,j}y_{i,j})$$

and

$$\lambda = \sum_{i=1}^{n}(A_i\delta_i + \sum_{j=1}^{n} B_{i,j}\epsilon_{i,j})$$

for fixed $a_i, b_{i,j}, A_i, B_{i,j} \in F$.

First note that $\lambda G = \{\lambda + \sum_{i=1}^{n}\sum_{j=1}^{n} c_j B_{i,j}\delta_i \mid c_j \in F\}$. Since there are n coefficients that may vary, namely those of the δ_i, it must be that $|\lambda G| \le q^n$. Considering $\lambda = \sum_{i=1}^{k}\epsilon_{i,i}$ for each $0 \le k \le n$, one sees that each q^k is achievable.

Finally, consider the superclasses; one may check that

$$1 + GxG = \{1 + x + \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i c_j + d_i a_j)y_{i,j} \mid c_j, d_i \in F\}.$$

If any $a_k \neq 0$, then the variable c_j controls the coefficient of $y_{k,j}$ for each $1 \le j \le n$ and the variable d_i controls the coefficient of $y_{i,k}$ for each $1 \le i \le n$. Since c_k and d_k are not both needed to control the coefficient of $y_{k,k}$, it must be considered if there is an as-yet uncontrolled coefficient that either variable might influence. The only coefficients either variable might affect, though, are those of the $y_{k,j}$ and $y_{j,k}$, and these are already able to vary without limitation. Thus, $|1+GxG| = q^{n+(n-1)}$. If each $a_k = 0$, then $|1 + GxG| = 1$. □

The algebra constructed in the proof of Lemma 4.4 has supercharacters which are not irreducible. Adopting the notation of the proof, the subalgebra $R_{\epsilon_{1,1}} = L_{\epsilon_{1,1}}$ is proper, so, by Theorem 5.10 of [2], the supercharacter $\chi_{\epsilon_{1,1}}$ is not irreducible.

Theorem 4.5. *Let r and s be positive integers that are either both 1 or both not 1. Then there is an $\mathbb{F}_q$-algebra J with exactly r superdegrees and exactly s superclass sizes such that $J^3 = 0$.*

Proof. If $r = s = 1$, the algebra $J = 0$ will work, so it suffices to consider $1 < r, s$.

First, assume that $r \le s$. Taking $m = s - 1$ and

$$e_i = \begin{cases} i & \text{if } 1 \le i \le r-1, \\ r-1 & \text{if } r \le i \le s-1 \end{cases}$$

in Theorem 2.1 yields an algebra J with superdegrees $\{q^i \mid 0 \le i \le r-1\}$ and superclass sizes $\{q^i \mid 0 \le i \le s-1\}$, so it suffices to assume $1 < s < r$.

Next suppose that s is even, and write $s = 2l$. Taking $n = r - l$ in Lemma 4.4 yields an algebra K with superdegrees $\{q^i \mid 0 \le i \le r-l\}$ and superclass sizes $\{1, q^{2r-2l-1}\}$. Taking $m = l-1$ and $e_i = i$ for $0 \le i \le l-1$ in Theorem 2.1 yields an algebra L with both superdegrees and superclass sizes $\{q^i \mid 0 \le i \le l-1\}$. Set $J = K \oplus L$. Lemma 4.2 then says that J has superdegrees $\{q^i \mid 0 \le i \le r-1\}$ and superclass sizes $\{q^i \mid 0 \le i \le l-1\} \cup \{q^i \mid 2r-2l-1 \le i \le 2r-l-2\}$. But $l - 1 < 2r - 2l - 1$, and the latter set has size $2l = s$. So it suffices to assume that s is odd.

Now suppose that $s > 3$, and write $s = 2l+3$. Taking $m = l$ and $n = 2r-2l-3$ in Lemma 4.3 results in an algebra K with respective superdegrees and superclass sizes $\{1, q^l\}$ and $\{q^i \mid 0 \le i \le l\} \cup \{q^{2r-2l-3}\}$. Taking $n = r-l-1$ in Lemma 4.4 yields an algebra L with superdegrees $\{q^i \mid 0 \le i \le r-l-1\}$ and superclass sizes $\{1, q^{2r-2l-3}\}$. Set $J = K \oplus L$. Then J has superdegrees $\{q^i \mid 0 \le i \le r-1\}$ and superclass sizes $\{q^i \mid 0 \le i \le l\} \cup \{q^i \mid 2r-2l-3 \le i \le 2r-l-3\} \cup \{q^{4r-4l-6}\}$. Since

$l < 2r-2l-3$ and $2r-l-3 < 4r-4l-6$, the latter set has size $(l+1)+(l+1)+1 = s$. It suffices to consider $3 = s < r$.

Next assume that r is odd, and write $r = 2l+1$. Taking $n = l$ in Lemma 4.4 yields an algebra K with superdegrees $\{q^i \mid 0 \leq i \leq l\}$ and superclass sizes $\{1, q^{2l-1}\}$. Set $J = K \oplus K$, so J has superdegrees $\{q^i \mid 0 \leq i \leq 2l\}$ and superclass sizes $\{1, q^{2l-1}, q^{4l-2}\}$. The former set has size $2l + 1 = r$, so assume that r is even.

Next suppose that $r \geq 6$, and write $r = 2l+2$. Taking $n = l$ in Lemma 4.4 yields an algebra K with superdegrees $\{q^i \mid 0 \leq i \leq l\}$ and superclass sizes $\{1, q^{2l-1}\}$. Applying Theorem 2.1 to the situation where the size of the field is q^{2l-1}, where $m = 1$, and where $e_1 = 1$ yields an $\mathbb{F}_{q^{2l-1}}$-algebra L with superdegrees $\{1, q^{2l-1}\}$ and superclass sizes $\{1, q^{2l-1}\}$. View $\mathbb{F}_q$ as a subfield of $\mathbb{F}_{q^{2l-1}}$, so the algebra L is an $\mathbb{F}_q$-algebra. Clearly, the superclasses do not depend on the underlying field of the algebra. By Theorem 2.2 in [2], the supercharacters also remain unaffected by changing the field. Set $J = K \oplus L$. Then J has respective superdegrees and superclass sizes $\{q^i \mid 0 \leq i \leq l\} \cup \{q^i \mid 2l-1 \leq i \leq 3l-1\}$ and $\{1, q^{2l-1}, q^{4l-2}\}$. There are $(l+1)+(l+1) = r$ superdegrees, and it remains to consider $r = 4$.

For the last case, apply Theorem 2.1 twice to the situation where $m = 1$. For the first algebra K, take $e_1 = 1$, and for the second algebra L, take $e_1 = 2$. Then $J = K \oplus L$ has superdegrees $\{1, q, q^2, q^3\}$ and superclass sizes $\{1, q, q^2\}$.

Each of the constructions of J above consists either of an algebra from Theorem 2.1, Lemma 4.3, or Lemma 4.4, or it is the direct sum of such algebras. Regardless, one sees that $J^3 = 0$. □

5. Conclusion

It is now natural to wonder if, given any pair of nontrivial sets of q-powers that contain 1, there is an algebra whose superdegrees and superclass sizes are the given sets. Corollary 5.4 confirms that the answer is negative.

Lemma 5.1. *The $\mathbb{F}_q$-algebra J has superclass sizes $\{1, q\}$ if and only if* $\dim J^2 = 1$.

Proof. For $v \in J$, write Fv for the one-dimensional subspace of J spanned by v. Recall that the superclass size of the element $1 + x$ is $|1 + GxG| = |Jx + xJ|$.

First suppose that the superclass sizes are $\{1, q\}$. If $u, v \in J$ are such that $uv \neq 0$, then

$$q = |Fuv| \leq |Jv + vJ| = q,$$

so $Jv + vJ = Fuv$. Similarly, $Ju + uJ = Fuv$.

Since J has a nontrivial superclass size, Lemma 4.1 guarantees that there are $y, z \in J$ such that $yz \neq 0$. It suffices to show that if $w, x \in J$, then $wx \in Fyz$.

If $wy \neq 0$, then

$$wx \in Jw + wJ = Fwy = Jy + yJ = Fyz,$$

as desired, so assume $wy = 0$. Similarly, if $xz \neq 0$, then $wx \in Fyz$, so assume $xz = 0$.

Then $(x+y)z = yz \neq 0$, and $J(x+y) + (x+y)J = F(x+y)z = Fyz$. Hence,

$$wx = w(x+y) \in J(x+y) + (x+y)J = Fyz.$$

Conversely, assume that $\dim J^2 = 1$. But $|Jx + xJ| \leq |J^2| = q$ for all $x \in J$, so the superclass sizes are a subset of $\{1, q\}$. Since $J^2 \neq 0$, it cannot be the case that 1 is the only superclass size. □

Lemma 5.2. *If* $\dim J^2 = 1$*, then* J *has exactly* 2 *superdegrees.*

Proof. Let $\mathcal{B}_1$ be a basis for J^2, and extend this to a basis $\mathcal{B}_1 \cup \mathcal{B}_2$ for J. Let $\mathcal{D}_1 \cup \mathcal{D}_2$ be the basis for J^* dual to $\mathcal{B}_1 \cup \mathcal{B}_2$, where $|\mathcal{B}_i| = |\mathcal{D}_i|$. Let V be the span of $\mathcal{B}_2$, and identify $\mathcal{D}_2$ with $\{\lambda|_V \mid \lambda \in \mathcal{D}_2\}$, so V^* may be viewed as a subspace of J^*. Likewise view $(J^2)^*$ as a subspace of J^*. Note that this process of carrying the vector space decomposition $J = J^2 \oplus V$ to J^* is highly dependent on the bases chosen.

With this setup, the first goal is to check that if $\lambda \in V^* \leq J^*$ and $\mu \in J^*$, then $|(\lambda+\mu)G| = |\mu G|$. This is immediate from the observation that $(\lambda+\mu)\cdot g = \lambda+(\mu\cdot g)$ for all g in G. Second, observe that for all $\mu \in J^*$, $\alpha \in \mathbb{F}_q$, and $g \in G$, one has $(\alpha\mu)\cdot g = \alpha(\mu \cdot g)$, so $|(\alpha\mu)G| = |\mu G|$.

By construction, every $\nu \in J^*$ can be written as $\nu = \lambda + \mu$, where $\lambda \in V^*$ and $\mu \in (J^2)^*$. Since $\dim J^2 = 1$, the space $(J^2)^*$ also has dimension 1. The previous paragraph thus guarantees that the superdegrees are precisely $\{1, |\mu G|\}$, where μ is any element of $(J^2)^*$. Since $J^2 \neq 0$, the superdegree $|\mu G|$ is not 1, and so there are 2 superdegrees. □

Lemma 5.3. *Let* n *be a positive integer. Then there is an* $\mathbb{F}_q$*-algebra* J *with superdegrees* $\{1, q^n\}$ *such that* $\dim J^2 = 1$.

Proof. Let J be the F-vector space with basis

$$\mathcal{B} = \{x_i \mid 1 \leq i \leq n+1\} \cup \{y\}.$$

Write

$$\mathcal{D} = \{\delta_i \mid 1 \leq i \leq n+1\} \cup \{\epsilon\}$$

for the basis of J^* dual to $\mathcal{B}$. Define a product on J by setting

$$x_i x_j = \begin{cases} y & \text{if } i < j, \\ 0 & \text{if } i \geq j \end{cases}$$

and letting the product of all other basis elements be 0.

Clearly, $\dim J^2 = 1$, so, by Lemma 5.2, it suffices to find a superdegree equal to q^n. One may verify that $|\epsilon(1 + J)| = q^n$. □

This now yields a complete characterization of which sets of q-powers are the set of superdegrees of an algebra with superclass sizes $\{1, q\}$. As claimed at the beginning of the section, the sets cannot be arbitrary.

Corollary 5.4. *Let* S *be a set of integers. Then* S *is the set of superdegrees of an* $\mathbb{F}_q$*-algebra with superclass sizes* $\{1, q\}$ *if and only if* $S = \{1, q^n\}$ *for some positive integer* n.

Proof. The forward implication follows from Lemma 5.1 and Lemma 5.2. Lemma 5.3 and Lemma 5.1 give the reverse. □

So, the question at the beginning of the section was overly optimistic. The right question to consider is what restrictions choosing either the set of superdegrees or the set of superclass sizes imposes on the other set.

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