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AN APPLICATION OF AMPLE VECTOR BUNDLES IN REAL ALGEBRAIC GEOMETRY

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Abstract. Let E be an algebraic vector bundle on a compact nonsingular real algebraic set X, and let Z be the zero locus of a "generic" algebraic section of E. We investigate how certain cohomological invariants of X and Z are related. A crucial role in the proof is played by ample vector bundles on a suitable complexification of X.

1. Introduction

Let X be a compact nonsingular real algebraic set (in $\mathbb{R}^n$ or $\mathbb{P}^n(\mathbb{R})$ for some n). A cohomology class in $H^k(X;\mathbb{Z}/2)$ is said to be *algebraic* if the homology class in $H_l(X;\mathbb{Z}/2)$ Poincaré dual to it can be represented by an l-dimensional algebraic subset of X, $l = \dim X - k$ (cf. [10] and [6, 9]). The set of all algebraic cohomology classes in $H^k(X;\mathbb{Z}/2)$ forms a subgroup denoted by $H^k_{\mathrm{alg}}(X;\mathbb{Z}/2)$. The groups $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$ play a fundamental role in real algebraic geometry and have been extensively studied (cf. [2, 3, 4, 5, 6, 7, 15, 16, 18, 20, 24] and [9] for a short survey).

One can also associate with X and any commutative ring R other cohomological invariants of algebraic-geometric significance. A *nonsingular projective complexification* of X is a pair (V, e), where V is a closed nonsingular subscheme of $\mathbb{P}^N_{\mathbb{R}}$ (for some N) and $e\colon X \to V(\mathbb{C})$ is an injective map such that $e(X) = V(\mathbb{R})$, $V(\mathbb{R})$ is Zariski dense in V, and e induces a biregular isomorphism between X and $V(\mathbb{R})$. Here the set $V(\mathbb{R})$ of real points of V is regarded as a subset of the set $V(\mathbb{C})$ of complex points of V. Moreover, $V(\mathbb{R})$ is also viewed as an algebraic subset of $\mathbb{P}^N(\mathbb{R})$. The existence of (V, e) follows from Hironaka's resolution of singularities [13]. As is well known, (V, e) is uniquely determined up to isomorphism over $\mathbb{R}$, provided that $\dim X = 1$. However, if $\dim X \geq 2$, then X admits infinitely many pairwise nonisomorphic projective complexifications, for V can be blown up along a nonsingular center disjoint from $V(\mathbb{R})$. In view of this nonuniqueness, it is remarkable that the subgroup

$$H^k_{\mathbb{C}}(X;R) := e^\star(H^k(V(\mathbb{C});R))$$

of $H^k(X;R)$, where $e^\star\colon H^k(V(\mathbb{C});R) \to H^k(X;R)$ denotes the homomorphism induced by e, does not depend on the choice of (V, e). This is proved in [22] for

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X orientable over R, and in [11] for arbitrary X. Note that in both [22] and [11] the authors use different notation for our $H^k_{\mathbb{C}}(-;R)$. Properties and applications of $H^k_{\mathbb{C}}(-;R)$ are elaborated upon in [11, 17, 22, 23].

The groups $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$ and $H^k_{\mathbb{C}}(-;R)$ are subtle invariants that are, in general, hard to compute. In this paper we make use of ample vector bundles to estimate the size of $H^k_{\mathrm{alg}}(Z;\mathbb{Z}/2)$ and compute $H^k_{\mathbb{C}}(Z;R)$ in terms of $H^k(X,\mathbb{Z}/2)$ and $H^k_{\mathbb{C}}(X;R)$, respectively, for a large class of algebraic subsets Z of X. The ample vector bundles do not appear in the formulation of our result but only in the proof.

An algebraic vector bundle on X is, by definition, isomorphic to an algebraic subbundle of the trivial vector bundle with total space $X \times \mathbb{R}^q$ for some q; cf. [6, Chapter 12] (such an object was called a strongly algebraic vector bundle in the earlier literature [4, 5, 7]). Thus the category of algebraic vector bundles on X is isomorphic to the category of finitely generated projective modules over the ring of regular functions on X.

The zero locus of a section $s\colon X \to E$ of a vector bundle E on X will be denoted by $Z(s)$,

$$Z(s) = \{x \in X \mid s(x) = 0\}.$$

Theorem 1.1. *Let X be a compact irreducible nonsingular real algebraic set and let E be an algebraic vector bundle on X with $d = \dim X - \operatorname{rank} E$ positive. For each algebraic section s of E, there exist algebraic sections $s_1, \dots, s_p$ of E and a proper algebraic subset Σ of $\mathbb{R}^p$ such that $s_1, \dots, s_p$ generate E and for each point $t = (t_1, \dots, t_p)$ in $\mathbb{R}^p \setminus \Sigma$, the algebraic section $\sigma_t = s + t_1 s_1 + \cdots + t_p s_p$ is transverse to the zero section of E, and its zero locus $Z_t = Z(\sigma_t)$ is either empty or else it is a d-dimensional irreducible nonsingular algebraic subset of X satisfying*

$$H^k_{\mathrm{alg}}(Z_t;\mathbb{Z}/2) \subseteq i_t^\star(H^k(X;\mathbb{Z}/2)) \quad \text{for } k < d/2,$$
$$H^k_{\mathbb{C}}(Z_t;R) = i_t^\star(H^k_{\mathbb{C}}(X;R)) \qquad \text{for } k < d,$$

where $i_t\colon Z_t \hookrightarrow X$ is the inclusion map.

We postpone the proof until Section 2 and now derive the following approximation result for smooth (of class $\mathcal{C}^\infty$) submanifolds of X.

Corollary 1.2. *Let X be a compact irreducible nonsingular real algebraic set and let E be an algebraic vector bundle on X with $d = \dim X - \operatorname{rank} E$ positive. Let M be a smooth submanifold of X that is the zero locus of a smooth section of E transverse to the zero section. Then there exists a smooth diffeomorphism $\varphi\colon X \to X$, arbitrarily close in the $\mathcal{C}^\infty$ topology to the identity map, such that $Z := \varphi(M)$ is an irreducible nonsingular algebraic subset of X satisfying*

$$H^k_{\mathrm{alg}}(Z;\mathbb{Z}/2) \subseteq i^\star(H^k(X;\mathbb{Z}/2)) \quad \text{for } k < d/2,$$
$$H^k_{\mathbb{C}}(Z;R) = i^\star(H^k_{\mathbb{C}}(X;R)) \qquad \text{for } k < d,$$

where $i\colon Z \hookrightarrow X$ is the inclusion map.

Proof. Let $w\colon X \to E$ be a smooth section transverse to the zero section and satisfying $Z(w) = M$. There exists an algebraic section $s\colon X \to E$ arbitrarily close in the $\mathcal{C}^\infty$ topology to w (cf. [6, Theorem 12.3.2]). For this section s, let $s_1, \dots, s_p$ and Σ be as in Theorem 1.1. If t in $\mathbb{R}^p \setminus \Sigma$ is close to $0 \in \mathbb{R}^p$, then the algebraic section $\sigma_t = s + t_1 s_1 + \cdots + t_p s_p$ of E is close to w in the $\mathcal{C}^\infty$ topology. According to [1, Theorem 20.2], there exists a smooth diffeomorphism $\varphi\colon X \to X$, close in the $\mathcal{C}^\infty$ topology to the identity map, with $\varphi(M) = Z_t$. It suffices to set $Z = Z_t$. □

In Theorem 1.1 and Corollary 1.2, the cohomology groups of X tell us nothing about the cohomology groups of Z and Z_t. We obtain interesting relationships only after passing to $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$ and $H^k_{\mathbb{C}}(-;R)$. These last two groups have the expected functorial property: If $f\colon X \to Y$ is a regular map between compact nonsingular real algebraic sets, then

$$f^\star(H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(X;\mathbb{Z}/2) \quad \text{and} \quad f^\star(H^k_{\mathbb{C}}(Y;R)) \subseteq H^k_{\mathbb{C}}(X;R)$$

(cf. [10, Section 5] or [2, 5] for the former inclusion and [22, 11] for the latter). In particular, with notation as in Theorem 1.1, the inclusions

$$i_t^\star(H^k_{\mathrm{alg}}(X;\mathbb{Z}/2)) \subseteq H^k_{\mathrm{alg}}(Z_t;\mathbb{Z}/2),$$
$$H^k_{\mathbb{C}}(Z;R) \supseteq i^\star(H^k_{\mathbb{C}}(X;R))$$

are automatically satisfied for all k. An analogous remark remains valid in the context of Corollary 1.2.

Conjecture 1.3. *In Theorem* 1.1 *and Corollary* 1.2*,*

$$H^k_{\mathrm{alg}}(Z_t;\mathbb{Z}/2) = i_t^\star(H^k_{alg}(X;\mathbb{Z}/2)) \quad \textit{for } k < d/2$$

and

$$H^k_{\mathrm{alg}}(Z;\mathbb{Z}/2) = i^\star(H^k_{alg}(X;\mathbb{Z}/2)) \quad \textit{for } k < d/2,$$

respectively.

In view of the functoriality of $H^k_{\mathrm{alg}}(-;\mathbb{Z}/2)$, Conjecture 1.3 is true if

$$H^k_{\mathrm{alg}}(X;\mathbb{Z}/2) = H^k(X;\mathbb{Z}/2) \quad \text{for } k < d/2.$$

Theorem 1.1 and Corollary 1.2 are applicable, and Conjecture 1.3 is true in the following two cases.

Example 1.4. Each topological real vector bundle on $\mathbb{P}^n(\mathbb{R})$ is isomorphic to an algebraic vector bundle, and

$$H^k_{\mathrm{alg}}(\mathbb{P}^n(\mathbb{R});\mathbb{Z}/2) = H^k(\mathbb{P}^n(\mathbb{R});\mathbb{Z}/2) \quad \text{for all } k$$

(cf. [6, Example 12.3.7c, Proposition 11.3.3]).

Example 1.5. Let $\mathbb{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ denote the space of real $n \times n$ matrices. Given a matrix A, we write ${}^{\mathrm{t}}A$ for its transpose. The real Grassmannian $\mathbb{G}_{n,r}(\mathbb{R})$ of r-dimensional vector subspaces of $\mathbb{R}^n$ can be identified with the algebraic subset

$$\{A \in \mathbb{M}_n(\mathbb{R}) \mid A = {}^{\mathrm{t}}A,\ A^2 = A,\ \operatorname{trace} A = r\}$$

of $\mathbb{M}_n(\mathbb{R})$. The universal vector bundle $E_{n,r}$ on $\mathbb{G}_{n,r}(\mathbb{R})$ is algebraic. Moreover, $\mathbb{G}_{n,r}(\mathbb{R})$ is nonsingular and

$$H^k_{\mathrm{alg}}(\mathbb{G}_{n,r}(\mathbb{R});\mathbb{Z}/2) = H^k(\mathbb{G}_{n,r}(\mathbb{R});\mathbb{Z}/2) \quad \text{for all } k.$$

For these facts one can consult [6, pp. 71, 272, 303].

Corollary 1.2 has an interesting application even when E is a trivial vector bundle. First recall that every compact smooth manifold M is diffeomorphic to a nonsingular real algebraic set, called an *algebraic model* of M; cf. [26] or [6, Theorem 14.1.10] (and also [21] for a weaker but influential result). How the groups $H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2)$ vary as Y runs through the class of all algebraic models of M is a challenging problem (cf. [5, 7, 16, 17]).

Corollary 1.6. *Let M be an m-dimensional smooth manifold that is the boundary of a compact smooth parallelizable manifold with boundary. If $m \geq 1$, then M has an irreducible algebraic model Y satisfying*

$$H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2) = 0 \quad \textit{for } 1 \leq k < m/2.$$

Proof. Let $M = \partial Q$, where Q is a compact smooth parallelizable manifold. We may assume that Q is a smooth submanifold of the unit sphere $\mathbb{S}^n$, $n \geq 2\dim Q + 1$. Since the tangent bundle of Q is trivial, the normal bundle of Q in $\mathbb{S}^n$ is stably trivial, and hence trivial being of rank greater than $\dim Q$ (cf. [14, p. 100, Theorem 1.5]). According to [8, Theorem 1.12] there exists a smooth map $f\colon \mathbb{S}^n \to \mathbb{R}^{n-m}$ for which $0 \in \mathbb{R}^{n-m}$ is a regular value and $M = f^{-1}(0)$. Applying Corollary 1.2 to the trivial vector bundle on $\mathbb{S}^n$ with total space $\mathbb{S}^n \times \mathbb{R}^{n-m}$, we obtain an irreducible nonsingular algebraic subset Y of $\mathbb{S}^n$ that is diffeomorphic to M and satisfies

$$H^k_{\mathrm{alg}}(Y;\mathbb{Z}/2) \subseteq i^\star(H^k(\mathbb{S}^n;\mathbb{Z}/2)) \quad \text{for } k < m/2.$$

The proof is complete. □

Let N be a compact smooth stably parallelizable manifold. Then the smooth manifold $M = (N \times \{0\}) \cup (N \times \{1\})$, which is the disjoint union of two copies of N, satisfies the hypothesis of Corollary 1.6. The same is true for $M = N \times \mathbb{S}^1$.

2. Vector bundles on schemes over $\mathbb{R}$

Let V be a closed nonsingular subscheme of $\mathbb{P}^n_{\mathbb{R}}$ (for some n) with $V(\mathbb{R})$ nonempty. We regard $V(\mathbb{R})$ and $V(\mathbb{C})$ as algebraic subsets of $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$, respectively. Each vector bundle $\mathcal{E}$ on V gives rise to algebraic vector bundles $\mathcal{E}(\mathbb{R})$ on $V(\mathbb{R})$ and $\mathcal{E}(\mathbb{C})$ on $V(\mathbb{C})$. For any section $v\colon V \to \mathcal{E}$, the corresponding sections

$$v(\mathbb{R})\colon V(\mathbb{R}) \to \mathcal{E}(\mathbb{R}) \quad \text{and} \quad v(\mathbb{C})\colon V(\mathbb{C}) \to \mathcal{E}(\mathbb{C})$$

are algebraic. We say that the section v is *transverse regular* if the section $v(\mathbb{C})$ is transverse to the zero section of $\mathcal{E}(\mathbb{C})$. The zero scheme of v will be denoted by $Z(v)$. We consider $V_{\mathbb{C}} = V \times_{\mathbb{R}} \mathbb{C}$ as a scheme over $\mathbb{C}$ and denote by $\mathcal{E}_{\mathbb{C}}$ the vector bundle on $V_{\mathbb{C}}$ determined by $\mathcal{E}$. Similarly, we denote by $v_{\mathbb{C}}\colon V_{\mathbb{C}} \to \mathcal{E}_{\mathbb{C}}$ the section determined by v. If $\mathcal{E}$ is generated by the sections $v_1, \dots, v_p$, then $\mathcal{E}_{\mathbb{C}}$ is generated by the sections $v_{1\,\mathbb{C}}, \dots, v_{p\,\mathbb{C}}$.

Proof of Theorem 1.1. We may assume that $X = V(\mathbb{R})$ and $E = \mathcal{E}(\mathbb{R})$, where V is a closed irreducible nonsingular subscheme of $\mathbb{P}^n(\mathbb{R})$ and $\mathcal{E}$ is a vector bundle on V (cf. [15, Lemma 3.4]). There exist an open neighborhood V_0 of X in V and a section $v_0\colon V_0 \to E$ which is an extension of s, that is, $v_0(\mathbb{R})\colon V_0(\mathbb{R}) = X \to \mathcal{E}(\mathbb{R}) = E$ is equal to s. We have

$$V_0 = V \setminus Z(H_1, \dots, H_l),$$

where the H_j are homogeneous polynomials in $\mathbb{R}[T_0, \dots, T_n]$ and $Z(H_1, \dots, H_l)$ is the closed subset of $\mathbb{P}^n_{\mathbb{R}}$ determined by the equations $H_1 = 0, \dots, H_l = 0$. Set $r_j = \deg H_j$, $r = \max\{r_1, \dots, r_l\}$, and

$$H = \sum_{j=1}^{l} (T_0^2 + \cdots + T_n^2)^{r - r_j} H_j^2.$$

Then H is a homogeneous polynomial of degree $2r$ and

$$V(\mathbb{R}) \subseteq V \setminus Z(H) \subseteq V_0,$$

where $Z(H)$ is the closed subset of $\mathbb{P}^n_{\mathbb{R}}$ determined by the equation $H = 0$.

For each integer c, we have the vector bundle $\mathcal{O}(c)$ on $\mathbb{P}^n_{\mathbb{R}}$. Let $h\colon \mathbb{P}^n_{\mathbb{R}} \to \mathcal{O}(2r)$ be the section of $\mathcal{O}(2r)$ determined by the homogeneous polynomial H. Note that

$$Z(h) = Z(H).$$

Let $\mathcal{L} = \mathcal{O}(2r)|_V$ and $u = h|_V$. Then $\mathcal{L}$ is an ample line bundle on V and $u\colon V \to \mathcal{L}$ is a section. By construction,

$$V(\mathbb{R}) \subseteq V \setminus Z(u) = V \setminus Z(H) \subseteq V_0.$$

Given a positive integer m, we set $\mathcal{E}(m) = \mathcal{E} \otimes \mathcal{L}^m$, where $\mathcal{L}^m = \mathcal{L} \otimes \cdots \otimes \mathcal{L}$ is the m-fold tensor product. There exists a positive integer m_0 such that for each integer $m \geq m_0$, the vector bundle $\mathcal{E}(m)$ is generated by global sections (cf. [12, p. 153]) and the section

$$v_0 \otimes u^m\colon V \setminus Z(u) \to \mathcal{E}(m),$$

where $u^m = u \otimes \cdots \otimes u\colon V \to \mathcal{L}^m$, can be extended to a section $w_m\colon V \to \mathcal{E}(m)$ (cf. [12, Lemma 5.14]). We may also assume that $\mathcal{E}(m)$ is an ample vector bundle.

Fix $m \geq m_0$ and set $v = w_m$. Let $v_1, \dots, v_p$ be sections of $\mathcal{E}(m)$ that generate $\mathcal{E}(m)$. Given $t = (t_1, \dots, t_p)$ in $\mathbb{R}^p$, set

$$\tau_t = v + t_1 v_1 + \cdots + t_p v_p \quad \text{and} \quad V_t = Z(\tau_t).$$

Since the sections $v_{1\mathbb{C}}, \dots, v_{p\mathbb{C}}$ generate $\mathcal{E}(m)_{\mathbb{C}}$, there exists a proper algebraic subset Σ of $\mathbb{R}^p$ such that for each point t in $\mathbb{R}^p \setminus \Sigma$, the section τ_t is transverse regular.

Henceforth we fix t in $\mathbb{R}^p \setminus \Sigma$. Then the subscheme V_t of V is nonsingular of dimension d. According to Sommese's theorem (cf. [25, Proposition 1.16] or [19, Theorem 7.1.1]),

$$H^k(V(\mathbb{C}), V_t(\mathbb{C}); \mathbb{Z}) = 0 \quad \text{for } k \leq d.$$

Taking $k = 1$, we obtain that $V_t(\mathbb{C})$ is connected, and hence V_t is irreducible. Moreover, if $j_t\colon V_t \hookrightarrow V$ is the inclusion morphism, then

(1) $\quad j_t(\mathbb{C})^\star\colon H^k(V(\mathbb{C}); R) \to H^k(V_t(\mathbb{C}); R)$ is an isomorphism for $k < d$.

Suppose that the algebraic subset $V_t(\mathbb{R})$ of $V(\mathbb{R})$ is nonempty. Then

(2) $\quad V_t(\mathbb{R})$ is irreducible and nonsingular, $\dim V_t(\mathbb{R}) = d$.

We will now deal with $H^k_{\mathrm{alg}}(-; \mathbb{Z}/2)$. According to [18, Theorem 2.1], condition (1) (with $R = \mathbb{Z}$) implies that

(3) $$H^k_{\mathrm{alg}}(V_t(\mathbb{R}); \mathbb{Z}/2) \subseteq j_t(\mathbb{R})^\star(H^k(V(\mathbb{R}); \mathbb{Z}/2)) \quad \text{for } k < d/2.$$

In order to analyze $H^k_{\mathbb{C}}(-; R)$, let $e\colon V(\mathbb{R}) \hookrightarrow V(\mathbb{C})$ and $e_t\colon V_t(\mathbb{R}) \hookrightarrow V_t(\mathbb{C})$ be the inclusion maps. The following diagram is commutative:

$$\begin{array}{ccc} H^k(V(\mathbb{C}); R) & \xrightarrow{j_t(\mathbb{C})^\star} & H^k(V_t(\mathbb{C}); R) \\ {\scriptstyle e^\star}\big\downarrow & & \big\downarrow{\scriptstyle e_t^\star} \\ H^k(V(\mathbb{R}); R) & \xrightarrow{j_t(\mathbb{R})^\star} & H^k(V_t(\mathbb{R}); R) \end{array}$$

Since (V, e) and (V_t, e_t) are nonsingular projective complexifications of $V(\mathbb{R})$ and $V_t(\mathbb{R})$, respectively, condition (1) implies that

(4) $$H^k_{\mathbb{C}}(V_t(\mathbb{R}); R) = j_t(\mathbb{R})^\star(H^k_{\mathbb{C}}(V(\mathbb{R}); R)) \quad \text{for } k < d.$$

We are ready for the final step in the proof. Since the zero locus of the section of $\mathcal{O}(2r)$ determined by the polynomial $(T_0^2 + \cdots + T_n^2)^r$ does not intersect $\mathbb{P}^n(\mathbb{R})$, the algebraic vector bundle $\mathcal{O}(2r)(\mathbb{R})$ on $\mathbb{P}^n(\mathbb{R})$ is algebraically trivial, and hence so is the algebraic vector bundle $\mathcal{L}(\mathbb{R})$ on $V(\mathbb{R})$. Consequently, the algebraic vector bundles $\mathcal{E}(m)(\mathbb{R})$ and $\mathcal{E}(\mathbb{R}) = E$ on $V(\mathbb{R}) = X$ are algebraically isomorphic. If $\Phi\colon \mathcal{E}(m)(\mathbb{R}) \to E$ is an algebraic isomorphism, then $\Phi \circ v(\mathbb{R}) = fs$ for some regular function $f\colon X \to \mathbb{R}$ with $f^{-1}(0) = \emptyset$. Let $\pi\colon \mathcal{E}(m)(\mathbb{R}) \to X$ be the bundle projection and $\Psi = (1/f \circ \pi)\Phi$. Then $\Psi\colon \mathcal{E}(m)(\mathbb{R}) \to E$ is an algebraic isomorphism satisfying $\Psi \circ v(\mathbb{R}) = s$. Since the sections $v_1, \ldots, v_p$ generate $\mathcal{E}(m)$, the algebraic sections $s_1 := \Psi \circ v_1(\mathbb{R}), \ldots, s_p := \Psi \circ v_p(\mathbb{R})$ generate E. We have $\Psi \circ \tau_t(\mathbb{R}) = \sigma_t$, where $\sigma_t = s + t_1 s_1 + \cdots + t_p s_p$. The transverse regularity of τ_t implies that σ_t is transverse to the zero section of E. By construction, $Z(\sigma_t) = Z(\tau_t(\mathbb{R})) = V_t(\mathbb{R})$. The proof is complete in view of (2), (3), and (4). $\square$

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