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MOTIVIC INVARIANTS OF ALGEBRAIC TORI

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ABSTRACT. We prove a trace formula and a global form of Denef and Loeser's motivic monodromy conjecture for tamely ramified algebraic tori over a discretely valued field. If the torus has purely additive reduction, the trace formula gives a cohomological interpretation for the number of components of the Néron model.

1. INTRODUCTION

Let R be a complete discrete valuation ring, with quotient field K and residue field k. We assume that k is algebraically closed. We denote by p the characteristic exponent of k, and by $\mathfrak{M}$ the maximal ideal of R. We fix a separable closure K^s of K, and we denote by K^t the tame closure of K in K^s. We denote by I the inertia group $G(K^s/K)$, by $P \subset I$ the wild inertia subgroup, and by $I^t = I/P = G(K^t/K)$ the tame inertia group. We fix a prime $\ell \neq p$ and a topological generator σ of I^t.

We denote by $K_0(Var_k)$ the Grothendieck ring of k-varieties, by $\mathbb{L} = [\mathbb{A}^1_k]$ the class of the affine line in $K_0(Var_k)$, and by $\mathcal{M}_k$ the localization of $K_0(Var_k)$ w.r.t. $\mathbb{L}$. We denote by
$$\chi_{top} : K_0(Var_k)/(\mathbb{L}-1) \to \mathbb{Z}$$
the ℓ-adic Euler characteristic (it is independent of ℓ). See [11, 2.1] for details.

If R has equal characteristic, then we put $K_0^R(Var_k) = K_0(Var_k)$ and $\mathcal{M}_k^R = \mathcal{M}_k$. If R has mixed characteristic, we denote by $K_0^R(Var_k)$ the *modified* Grothendieck ring of k-varieties [15, 3.2]. It is a quotient of $K_0(Var_k)$, obtained by identifying the classes of universally homeomorphic k-varieties. With a slight abuse of notation, we denote the image of $\mathbb{L}$ in $K_0^R(Var_k)$ again by $\mathbb{L}$; we will always clearly indicate in which ring we are working. We denote by $\mathcal{M}_k^R$ the localization of $K_0^R(Var_k)$ with respect to $\mathbb{L}$. The Euler characteristic χ_{top} factors through $K_0^R(Var_k)/(\mathbb{L}-1)$.

Let X be a separated rigid K-variety. A *weak Néron model* for X is a separated smooth formal R-scheme $\mathfrak{U}$, topologically of finite type, endowed with an open immersion of rigid K-varieties $i : \mathfrak{U}_\eta \to X$ such that $i(K) : \mathfrak{U}_\eta(K) \to X(K)$ is a bijection. Here we denote by $\mathfrak{U}_\eta$ the generic fiber of $\mathfrak{U}$. Such a weak Néron model exists iff X admits a smooth quasi-compact open rigid subvariety which contains

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all the K-points of X [3, 3.3]. A weak Néron model is far from being unique in general. Nevertheless, using motivic integration on formal schemes, it was shown in [7, 4.5.3] and [14, 5.11] (see also [15, § 5.3] for an erratum) that the class

$$[\mathfrak{U}_s] \in K_0^R(Var_k)/(\mathbb{L}-1)$$

of the special fiber $\mathfrak{U}_s$ of $\mathfrak{U}$ only depends on X, and not on $\mathfrak{U}$. It is called the *motivic Serre invariant* of X and is denoted by $S(X)$. We consider $S(X)$ as a measure for the set of rational points on X. In particular, $S(X)$ vanishes if $X(K) = \emptyset$ (but the converse implication does not hold). If Y is an algebraic K-variety such that Y^{an} admits a weak Néron model, we put $S(Y) = S(Y^{an})$ (here $(\cdot)^{an}$ is the non-Archimedean GAGA functor). This definition applies in particular when Y is smooth and proper.

We showed in [11, 6.3] that the motivic Serre invariant admits a cohomological interpretation by means of a *trace formula*: if Y is a smooth and proper K-variety, then under a certain tameness condition on Y (in particular if k has characteristic zero), we have

$$\chi_{top}(S(Y)) = \sum_{i\geq 0}(-1)^i Trace(\sigma \mid H^i(Y \times_K K^t, \mathbb{Q}_\ell)). \tag{1.1}$$

An analogous statement holds for rigid varieties [10, 6.4]. It seems plausible that (1.1) holds for any smooth, proper, geometrically connected algebraic K-variety Y such that the wild inertia P acts trivially on the ℓ-adic cohomology of Y and such that $Y(K^t)$ is non-empty. The results in [11, § 7] show that this is true when Y is a curve, and that the assumption $Y(K^t) \neq \emptyset$ cannot be dropped. We proved in [12] the case where Y is an abelian variety.

In this paper we consider the case of a tamely ramified algebraic K-torus G. Its analytification G^{an} does not admit a weak Néron model in general, so the motivic Serre invariant has to be defined in another way. There are two possible definitions. First, G^{an} has a natural quasi-compact open rigid subgroup G^b such that $G^b(K)$ is the maximal subgroup of $G(K)$ which is bounded in G [6, 3.12]. We show in Theorem 3.8 that (1.1) holds for G if the left hand side is replaced by $\chi_{top}(S(G^b))$. Second, if K has characteristic zero, we showed in [11, 5.4] (see also [15, § 5.3] for an erratum) that there exists a unique ring morphism

$$S : K_0(Var_K) \to K_0^R(Var_k)/(\mathbb{L}-1)$$

such that $S([Y]) = S(Y)$ for any smooth and proper K-variety Y. If X is a K-variety such that X^{an} admits a weak Néron model, we have $S([X]) = S(X^{an})$ by [11, 5.4]. We show in Proposition 3.3 that, if K has characteristic zero and G is an algebraic K-torus, we have $S([G]) = S(G^b)$. In fact, the analogous equality holds for any algebraic K-group H for which H^b can be defined.

In [6], we associated a *motivic zeta function* $Z_A(T)$ to every semi-abelian K-variety A. It measures the behaviour of the Néron model of A under tame base change. If G is a tamely ramified algebraic K-torus, then the trace formula gives a cohomological interpretation of $Z_G(T)$ (Proposition 4.3). We prove in Theorem 6.1 that G satisfies a global version of Denef and Loeser's monodromy conjecture for motivic zeta functions of complex hypersurface singularities. More precisely, we show that $Z_G(\mathbb{L}^{-s})$ has a unique pole at $s = c(G)$, where $c(G)$ denotes the base change conductor of G [4]. This pole has order one, and $\exp(2\pi i c(G))$ belongs to $\{-1, 1\}$ and is the unique eigenvalue of σ on $H^g(G \times_K K^t, \mathbb{Q}_\ell)$, with g the dimension of G. We also compute the characteristic polynomial of the σ-action on $H^1(G\times_K K^t,\mathbb{Q}_\ell)$ in terms of the elementary divisors $c_i(G)$ of G (Proposition 3.5). We refer to [6] for analogous results for abelian varieties and for the exact relation with Denef and Loeser's motivic monodromy conjecture.

The assumption that G is tamely ramified is crucial for the arguments in this paper. It would be highly interesting to adapt the results to the wildly ramified case.

2. Preliminaries

We denote by $\mathbb{N}'$ the set of integers $d>0$ prime to p. For each $d\in\mathbb{N}'$, we denote by $K(d)$ the unique degree d extension of K in K^t, by $I(d)$ the subgroup $G(K^s/K(d))$ of I, and by $I^t(d)$ the subgroup $G(K^t/K(d))$ of I^t.

Let G be an algebraic torus over K. We denote by g the dimension of G and by X the character group of G. It is a free $\mathbb{Z}$-module of rank g, endowed with a continuous action of the inertia group $I=G(K^s/K)$. The splitting degree of G is the degree of the minimal extension L of K where G splits. We say that G is tamely ramified if the wild inertia P acts trivially on X. This is equivalent to the property that the splitting degree of G is prime to p, i.e. that G splits over a finite tame extension of K.

For any element γ of I^t, we denote by $P_\gamma(t)$ the characteristic polynomial

$$\det(t\cdot Id-\gamma\,|\,H^1(G\times_K K^t,\mathbb{Q}_\ell))$$

of the action of γ on $H^1(G\times_K K^t,\mathbb{Q}_\ell)$. Since there exists an I^t-equivariant isomorphism

$$H^1(G\times_K K^t,\mathbb{Q}_\ell)\cong(X\otimes_{\mathbb{Z}}\mathbb{Q}_\ell)^P,$$

the polynomial $P_\gamma(t)$ coincides with the characteristic polynomial of the γ-action on $X^P\otimes_{\mathbb{Z}}\mathbb{Q}$. In particular, $P_\gamma(t)$ belongs to $\mathbb{Z}[t]$, and it is a product of cyclotomic polynomials, independent of ℓ.

An algebraic K-group is a group K-scheme of finite type. When we speak of the Néron model of an algebraic K-group H, we mean the Néron model defined in [6, 3.6]. By [6, 3.10], a smooth commutative algebraic K-group H admits a Néron model $\mathcal{H}$ iff it admits a Néron lft-model $\mathcal{H}^{lft}$ in the sense of [2, 10.1.1]. This condition is also equivalent to the property that H does not admit a subgroup of type $\mathbb{G}_{a,K}$ [2, 10.2.2]. The Néron model $\mathcal{H}$ is the maximal quasi-compact open subgroup R-scheme of $\mathcal{H}^{lft}$ [6, 3.7]. We denote by Φ_H the group of connected components of the special fiber $\mathcal{H}_s$ and by Φ'_H the subgroup of elements of order prime to p. The group Φ_H is the torsion part of the component group of $\mathcal{H}^{lft}_s$. We denote by H^b the generic fiber of the formal $\mathfrak{M}$-adic completion of $\mathcal{H}$. Then H^b is a separated smooth quasi-compact rigid K-variety, and $H^b(K)$ is the maximal subgroup of $H(K)$ which is bounded in H [6, 3.12].

We denote by $\mathcal{G}$ the Néron model of the K-torus G. The identity component $\mathcal{G}^o_s$ of $\mathcal{G}_s$ splits canonically into a product $T\times_k U$ with T a k-torus and U a unipotent k-group. The dimensions of T and U are called the toric, resp. unipotent, rank of $\mathcal{G}^o_s$. We say that G has good reduction if U is trivial and that G has purely additive reduction if T is trivial.

For each integer $d>0$, we denote by $\Phi_d(t)\in\mathbb{Z}[t]$ the cyclotomic polynomial whose zeroes are the primitive d-th roots of unity. For every rational number a, we denote by $\tau(a)$ the order of a in the quotient group $\mathbb{Q}/\mathbb{Z}$.

3. The motivic Serre invariant and the trace formula

Lemma 3.1. *Let H be a smooth connected commutative algebraic k-group and consider its Chevalley decomposition*

$$0 \longrightarrow (L = U \times_k T) \longrightarrow H \xrightarrow{\pi} B \longrightarrow 0$$

with U unipotent, T a torus, and B an abelian variety. If we denote by u and t the dimensions of U, resp. T, then

$$[H] = \mathbb{L}^u(\mathbb{L}-1)^t[B]$$

in $K_0(Var_k)$.

Proof. As a k-variety, U is isomorphic to $\mathbb{A}^u_k$ [18, VII, n^o 6]. By the scissor relations in the Grothendieck ring, it suffices to show that π is a Zariski-locally trivial fibration. But π is an L-torsor with respect to the $fppf$ topology, and hence also with respect to the Zariski topology, because L is a successive extension of $\mathbb{G}_m$ and $\mathbb{G}_a$ [8, III.3.7 and III.4.9]. $\square$

Lemma 3.2. *Let H be a smooth commutative algebraic K-group. Assume that H admits a Néron model $\mathcal{H}$, and denote by t the dimension of the maximal torus in $\mathcal{H}^o_s$. If H admits a subgroup $T \cong \mathbb{G}_{m,K}$, then $t > 0$.*

Proof. Denote by $\mathcal{T}$ the Néron model of T. If $t = 0$, then any morphism of group k-schemes $\mathbb{G}_{m,k} \to \mathcal{H}^o_s$ is trivial. Since $\mathcal{T}^o_s \cong \mathbb{G}_{m,k}$, it suffices to show that the natural morphism $f : \mathcal{T}^o_s \to \mathcal{H}^o_s$ is non-trivial. For each integer q prime to p, we have a commutative diagram

$$\begin{array}{ccccc}
{}_q\mathcal{T}^o_s(k) & \xleftarrow{\cong} & {}_q\mathcal{T}^o(R) & \xrightarrow{(*)} & {}_qT(K) \\
\downarrow & & \downarrow & & \downarrow (*) \\
{}_q\mathcal{H}^o_s(k) & \xleftarrow{\cong} & {}_q\mathcal{H}^o(R) & \xrightarrow{(*)} & {}_qH(K),
\end{array}$$

where the lower index ${}_q(\cdot)$ denotes the kernel of multiplication by q. The left horizontal morphisms are bijections by [2, 7.3.3], and all arrows marked by $(*)$ are injective. It follows that the map

$${}_q\mathcal{T}^o_s(k) \longrightarrow {}_q\mathcal{H}^o_s(k)$$

is injective. But $\mathcal{T}^o_s$ is isomorphic to $\mathbb{G}_{m,k}$, so that ${}_q\mathcal{T}^o_s(k)$ is non-trivial. Hence, f is non-trivial. $\square$

Proposition 3.3. *Let H be a smooth commutative algebraic K-group and assume that H admits a Néron model $\mathcal{H}$. Denote by t the dimension of the maximal torus in $\mathcal{H}^o_s$ and by B the abelian quotient in the Chevalley decomposition of $\mathcal{H}^o_s$. Then*

$$S(H^b) = \begin{cases} |\Phi_H| \cdot [B] & \text{if } t = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, if K has characteristic zero, then $S([H]) = S(H^b)$.

Proof. We start by computing $S(H^b)$. The $\mathfrak{M}$-adic completion of $\mathcal{H}$ is a weak Néron model for H^b, so that $S(H^b) = [\mathcal{H}_s]$ in $K^R_0(Var_k)/(\mathbb{L}-1)$. Since k is algebraically closed, every connected component of $\mathcal{H}_s$ is isomorphic to $\mathcal{H}^o_s$. By Lemma 3.1, $[\mathcal{H}_s]$ vanishes in $K^R_0(Var_k)/(\mathbb{L}-1)$ unless $t = 0$, and in the latter case, $[\mathcal{H}_s] = |\Phi_H| \cdot [B]$

in $K_0^R(Var_k)/(\mathbb{L}-1)$. Hence, it suffices to show that $S([H]) = S(H^b)$, assuming that K has characteristic zero.

Assume that H admits a subgroup $T \cong \mathbb{G}_{m,K}$. Consider the short exact sequence of K-groups

$$1 \longrightarrow T \longrightarrow H \xrightarrow{f} H/T \longrightarrow 1.$$

Since T is a split torus, f is a T-torsor in the Zariski topology, and in particular a Zariski-locally trivial fibration with fiber T. This implies that $[H] = [T][H/T]$ in $K_0(Var_K)$. Since the motivic Serre invariant

$$S(\cdot) : K_0(Var_K) \to K_0^R(Var_k)/(\mathbb{L}-1)$$

maps $[\mathbb{G}_{m,K}]$ to zero [11, 5.4], we see that $S([H]) = 0$. On the other hand, $t > 0$ by Lemma 3.2, so that $S(H^b) = 0$ as well.

Now, assume that H does not admit a subgroup of type $\mathbb{G}_{m,K}$. Then the Néron lft-model $\mathcal{H}^{lft}$ is quasi-compact [2, 10.2.1], so that $\mathcal{H} = \mathcal{H}^{lft}$, and $H^b(K) = H(K)$. Hence, the $\mathfrak{M}$-adic completion of $\mathcal{H}$ is a weak Néron model of H^{an}, and we have

$$S([H]) = S(H^b) = [\mathcal{H}_s] \quad \in K_0^R(Var_k)/(\mathbb{L}-1)$$

by [11, 5.4]. □

Now we turn to our algebraic K-torus G.

Lemma 3.4. *Let L be the minimal splitting field of G. The component group Φ_G is killed by the splitting degree $e = [L : K]$ of G, and $|\Phi_G| = |H^1(G(L/K), X)|$.*

Proof. The component group Φ_G is isomorphic to $H^1(I, X)$, by [1, 7.2.2] and [19, 2.18]. The I-action on X factors through $G(L/K)$. Since X is torsion-free, the inflation morphism

$$H^1(G(L/K), X) \to H^1(I, X)$$

is an isomorphism [17, VII, Prop. 4]. The group $H^1(G(L/K), X)$ is killed by e, by [17, VIII, Cor. 1]. □

Proposition 3.5. *The following are equivalent:*

1. *G has purely additive reduction.*
2. *$P_\sigma(1) \neq 0$.*
3. *$P_\sigma(1) = |\Phi'_G|$.*

Proof. The implication (3) ⇒ (2) is trivial. It follows from [9, 1.3] that the torus G has purely additive reduction iff $X^I = 0$. This proves the equivalence of (1) and (2). It remains to show that (1) ⇒ (3). Assume that G has purely additive reduction, denote by L the minimal splitting field of G, and by L' the maximal tame extension of K inside L. By [17, VII, Prop. 4], we have an exact sequence

$$0 \to H^1(G(L'/K), X^P) \to H^1(G(L/K), X) \to H^1(G(L/L'), X).$$

We have $|H^1(G(L/K), X)| = |\Phi_G|$ by Lemma 3.4. Since $H^1(G(L/L'), X)$ is a p-group, and $H^1(G(L'/K), X^P)$ has order prime to p [17, VIII, Cor. 1], we find that

$$|\Phi'_G| = |H^1(G(L'/K), X^P)| = |X^P/(1-\sigma)X^P|,$$

where the last equality follows from [17, VIII, Prop. 6] and the fact that $X^I = 0$.

By [17, III, Prop. 2] we have

$$|X^P/(1-\sigma)X^P| = |\det(1-\sigma \mid X^P \otimes_{\mathbb{Z}} \mathbb{Q})|.$$

The polynomial $P_\sigma(t)$ equals the characteristic polynomial of σ on $X^P \otimes_{\mathbb{Z}} \mathbb{Q}$. It is a product of cyclotomic polynomials that do not vanish at $t = 1$, which easily implies that $P_\sigma(1) > 0$ (see [12, 2.6]). Hence, we find that

$$|\Phi'_G| = P_\sigma(1). \qquad \square$$

Corollary 3.6. *If G has purely additive reduction, then $|\Phi'_G|$ is invariant under isogeny.*

Corollary 3.6 is false without the assumption that G has purely additive reduction, as is shown by the following easy example.

Example 3.7. Assume that k has characteristic zero. Let G_1 and G_2 be the K-tori whose character group is $\mathbb{Z}^2$ with σ-action given by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ resp. } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then G_1 and G_2 are isogenous, but computing $H^1(I, X)$ for both tori we find that $\Phi_{G_1} = \{0\}$ while $\Phi_{G_2} = \mathbb{Z}/2\mathbb{Z}$.

Theorem 3.8 (Trace formula)**.** *Assume that G is tamely ramified. Then $\Phi'_G = \Phi_G$. Moreover, if G has purely additive reduction, then*

$$\chi_{top}(S(G^b)) = |\Phi_G| = \sum_{i \geq 0} (-1)^i Trace(\sigma \mid H^i(G \times_K K^t, \mathbb{Q}_\ell)).$$

Otherwise,

$$\chi_{top}(S(G^b)) = 0 = \sum_{i \geq 0} (-1)^i Trace(\sigma \mid H^i(G \times_K K^t, \mathbb{Q}_\ell)).$$

Proof. Since G is tamely ramified, the splitting degree e of G is prime to p, so $\Phi'_G = \Phi_G$ by Lemma 3.4. By Proposition 3.3, we find that

$$S(G^b) = |\Phi_G| \quad \in K_0^R(Var_k)/(\mathbb{L} - 1)$$

if G has purely additive reduction, and $S(G^b) = 0$ otherwise.

By tameness of G, we have

$$H^i(G \times_K K^t, \mathbb{Q}_\ell) = H^i(G \times_K K^s, \mathbb{Q}_\ell)^P = H^i(G \times_K K^s, \mathbb{Q}_\ell)$$

for each $i \geq 0$, so there exists a canonical I^t-equivariant isomorphism

$$H^i(G \times_K K^t, \mathbb{Q}_\ell) \cong \bigwedge_{\mathbb{Q}_\ell}^{i} H^1(G \times_K K^t, \mathbb{Q}_\ell)$$

for each $i \geq 0$. It follows easily that

$$\sum_{i \geq 0} (-1)^i Trace(\sigma \mid H^i(G \times_K K^t, \mathbb{Q}_\ell)) = P_\sigma(1),$$

and we may conclude the proof by Proposition 3.5. $\square$

Corollary 3.9. *If K has characteristic zero and G is tamely ramified, then*

$$\chi_{top}(S([G])) = \sum_{i \geq 0} (-1)^i Trace(\sigma \mid H^i(G \times_K K^t, \mathbb{Q}_\ell)).$$

For abelian varieties, we proved an analogue of Theorem 3.8 in [12, 2.5 and 2.8].

4. The motivic zeta function of a torus

For any field F and any smooth F-scheme Z of pure dimension, a gauge form on Z is a nowhere vanishing differential form of maximal degree. If Y is a smooth R-scheme of pure relative dimension, ϕ a gauge form on $Y \times_R K$ and C a connected component of the special fiber Y_s, then we denote by $ord_C\omega$ the order of ϕ along C [6, 2.1].

For any $d \in \mathbb{N}'$ we denote by $\mathcal{G}(d)$ the Néron model of $G(d) = G \times_K K(d)$. We denote by ω a distinguished gauge form on G [6, 7.1], i.e. a translation-invariant gauge form such that $ord_{\mathcal{G}_s^o}\omega = 0$. Such a distinguished gauge form always exists, and it is unique up to multiplication with a unit in R [2, 4.2.3]. For any $d \in \mathbb{N}'$ we denote by $\omega(d)$ the pull-back of ω to $G(d)$. We introduce a function

$$ord_G : \mathbb{N}' \to \mathbb{N}$$

by putting $ord_G(d) = -ord_{\mathcal{G}(d)_s^o}\omega(d)$. This function only depends on G and not on ω. The fact that ord_G takes its values in the positive integers follows from [6, 7.4].

The following definition is taken from [6, §8.1].

Definition 4.1. We define the motivic zeta function $Z_G(T)$ of G by

$$Z_G(T) = \sum_{d \in \mathbb{N}'} [\mathcal{G}(d)_s]\mathbb{L}^{ord_G(d)}T^d \quad \in \mathcal{M}_k[[T]].$$

For each $d \in \mathbb{N}'$, the image of $[\mathcal{G}(d)_s]\mathbb{L}^{ord_G(d)}$ in $\mathcal{M}_k^R$ can be interpreted in terms of a motivic integral of the "Haar measure" $\omega(d)$ on the bounded part $G(d)^b$ of $G(d)$: we have

$$[\mathcal{G}(d)_s]\mathbb{L}^{ord_G(d)} = \mathbb{L}^g \cdot \int_{G(d)^b} |\omega(d)| \quad \in \mathcal{M}_k^R.$$

In order to give a more explicit description of $Z_G(T)$, we introduce some notation. For each $d \in \mathbb{N}'$, we put $\phi_G(d) = |\Phi_{G(d)}|$, the number of connected components of $\mathcal{G}(d)_s$. Moreover, we denote by $t_G(d)$ and $u_G(d)$ the toric, resp. unipotent, rank of $\mathcal{G}(d)_s^o$. Of course, we have $t_G(d) + u_G(d) = g$ for every $d \in \mathbb{N}'$.

Proposition 4.2.

$$Z_G(T) = \sum_{d \in \mathbb{N}'} \phi_G(d)(\mathbb{L}-1)^{t_G(d)}\mathbb{L}^{u_A(d)+ord_G(d)}T^d \quad \in \mathcal{M}_k[[T]].$$

Proof. This is immediate from Lemma 3.1. □

The trace formula yields the following cohomological interpretation of $Z_G(T)$. We denote by $\chi_{top}(Z_G(T))$ the series in $\mathbb{Z}[[T]]$ obtained by applying the morphism $\chi_{top} : \mathcal{M}_k \to \mathbb{Z}$ to the coefficients of $Z_G(T)$.

Proposition 4.3. *If G is tamely ramified, then*

$$\begin{aligned}\chi_{top}(Z_G(T)) &= \sum_{d \in \mathbb{N}',\, t_G(d)=0} \phi_G(d)T^d \\ &= \sum_{d \in \mathbb{N}'} \sum_{i \geq 0} (-1)^i Trace(\sigma^d \mid H^i(G \times_K K^t, \mathbb{Q}_\ell))T^d.\end{aligned}$$

Proof. This is immediate from Theorem 3.8 and Proposition 4.2. □

By Proposition 4.2, the motivic zeta function $Z_G(T)$ only depends on the functions ϕ_G, t_G, u_G and ord_G from $\mathbb{N}'$ to $\mathbb{N}$. We will now recall how the first three of them can be computed from the I-module X.

Proposition 4.4. *For each $d \in \mathbb{N}'$ we have*

$$\begin{aligned} \phi_G(d) &= |H^1(I(d), X)|, \\ t_G(d) &= \mathrm{rank}_{\mathbb{Z}} \left(X^{I(d)} \right), \\ u_G(d) &= \mathrm{rank}_{\mathbb{Z}} \left(X/X^{I(d)} \right). \end{aligned}$$

Proof. It obviously suffices to prove the result for $d = 1$, since $G(d)$ is the torus corresponding to the character group X with the action of $I(d)$. The first equality follows from [1, 7.2.2] and [19, 2.12]; the other two follow easily from [9, 1.3]. □

Corollary 4.5. *Assume that G is tamely ramified, and denote by e the splitting degree of G. Then $\phi_G(d)$, $u_G(d)$ and $t_G(d)$ only depend on d mod e. More precisely, if d is an element of $\mathbb{N}'$ and e' is a multiple of e such that $d + e' \in \mathbb{N}'$, then*

$$\begin{aligned} \phi_G(d + e') &= \phi_G(d), \\ u_G(d + e') &= u_G(d), \\ t_G(d + e') &= t_G(d). \end{aligned}$$

Proof. Since G is tamely ramified, we may replace $I(d)$ by $I^t(d)$ in Proposition 4.4. Note that $I^t(d)$ is topologically generated by σ^d, and $I^t(d + e')$ by $\sigma^{d+e'}$. But σ^e acts trivially on X, so the actions of σ^d and $\sigma^{d+e'}$ on X coincide. □

We recall the following definition [4, 2.4].

Definition 4.6. Let L be the minimal splitting field of G. Denote by R' the normalization of R in L, and by $\mathfrak{M}'$ the maximal ideal of R'. Put $e = [L : K]$ and $G' = G \times_K L$, and denote by $\mathcal{G}'$ the Néron model of G'. By the universal property of the Néron model, there exists a unique morphism of R'-schemes

$$\mathcal{G} \times_R R' \to \mathcal{G}'$$

that extends the isomorphism between the generic fibers. We have an isomorphism of R'-modules

$$Lie(\mathcal{G}')/Lie(\mathcal{G} \times_R R') \cong \bigoplus_{i=1}^{v} (R'/(\mathfrak{M}')^{c_i \cdot e})$$

with $c_1 \leq \ldots \leq c_v$ in $(1/e)\mathbb{Z}_{>0}$. The tuple $(c_1, \ldots, c_v)$ is called the tuple of elementary divisors of G, and

$$c(G) := \sum_{i=1}^{v} c_i = \frac{1}{e} \cdot \mathrm{length}_{R'} \left(Lie(\mathcal{G}')/Lie(\mathcal{G} \times_R R') \right)$$

is called the base change conductor of G.

Note that our definition differs slightly from the one in [4, 2.4]. Chai extends the tuple of elementary divisors by adding zeroes to the left until the length of the tuple equals the dimension of G.

We have $c(G) = 0$ iff G has good reduction. If G is tamely ramified, then the elementary divisors of G coincide with the non-zero jumps of Edixhoven's filtration for semi-abelian varieties [6, 4.18]. This comparison result shows in particular that

$0 < c_i < 1$ for all i. We'll now see that the functions ord_G and u_G (and hence $t_G = g - u_G$) can be computed from the elementary divisors of G. For every real number x, we denote by $\lfloor x \rfloor$ the unique integer in the interval $]x-1, x]$.

Proposition 4.7. *Assume that G is tamely ramified. Denote by $(c_1, \ldots, c_v)$ its tuple of elementary divisors, and by $c(G)$ its base change conductor. For any $d \in \mathbb{N}'$ we have*

$$\begin{aligned} ord_G(d) &= \sum_{i=1}^{v} \lfloor c_i \cdot d \rfloor, \\ u_G(d) &= |\{i \in \{1, \ldots, v\} \mid d \cdot c_i \notin \mathbb{Z}\}|. \end{aligned}$$

In particular, if e is the splitting degree of G, then

$$ord_G(d + e') = ord_G(d) + c(G) \cdot e'$$

for each $d \in \mathbb{N}'$ and each multiple e' of e such that $d + e' \in \mathbb{N}'$.

Proof. This follows from [6, 6.2 and 7.5]. □

Corollary 4.8. *If G is tamely ramified, then the unipotent rank $u(G)$ of $\mathcal{G}_s^o$ is equal to the number v of elementary divisors of G.*

5. Elementary divisors and monodromy

Proposition 5.1. *Let G be a tamely ramified algebraic K-torus and denote by $c_1 \leq \ldots \leq c_v$ its elementary divisors. Denote by e the splitting degree of G, and fix a primitive e-th root of unity ξ in an algebraic closure $\mathbb{Q}^a$ of $\mathbb{Q}$. The characteristic polynomial $P_\sigma(t)$ of σ on $H^1(G \times_K K^t, \mathbb{Q}_\ell)$ is given by*

$$P_\sigma(t) = (t-1)^{t_G(1)} \prod_{i=1}^{v} (t - \xi^{e \cdot c_i}) \in \mathbb{Z}[t]. \tag{5.1}$$

If we put, for each integer $d > 1$,

$$\nu_d = |\{i \in \{1, \ldots, v\} \mid \tau(c_i) = d\}|,$$

then the Euler number $\varphi(d)$ divides ν_d, and

$$P_\sigma(t) = (t-1)^{t_G(1)} \prod_{d>1} \Phi_d(t)^{\nu_d/\varphi(d)}.$$

Recall that $\tau(c_i)$ denotes the order of c_i in $\mathbb{Q}/\mathbb{Z}$.

Proof. The second expression for $P_\sigma(t)$ follows immediately from the first. Note that the product over $d > 1$ is finite since ν_d vanishes unless d divides e.

So let us prove that (5.1) holds. Consider the Néron model $\mathcal{G}(e)$ of $G(e) = G \times_K K(e)$, and denote by $\mathfrak{M}'$ the maximal ideal of the normalization $R(e)$ of R in $K(e)$. If we let $G(K(e)/K)$ act on $K(e)$ on the left, then any element θ of $G(K(e)/K)$ acts on the rank one k-vector space $\mathfrak{M}'/(\mathfrak{M}')^2$ by multiplication with an element θ' of $\mu_e(k)$, and the map $\theta \mapsto \theta'$ defines an isomorphism $G(K(e)/K) \cong \mu_e(k)$. We denote by ζ the image of σ in $G(K(e)/K) \cong \mu_e(k)$. Then by [6, 4.8 and 4.17], the characteristic polynomial of the ζ-action on $Lie(\mathcal{G}(e)_s)$ equals

$$Q(t) = (t-1)^{t_G(1)} \prod_{i=1}^{v} (t - \zeta^{e \cdot c_i}) \in k[t].$$

Since $\mathcal{G}(e)$ is a split $R(e)$-torus, the k-vector space $Lie(\mathcal{G}(e)_s)$ is canonically isomorphic to $Hom_{\mathbb{Z}}(X,k)$, so that $Q(t)$ equals the image of $P_\sigma(t)$ under the morphism $\mathbb{Z}[t]\to k[t]$.

We know that $P_\sigma(t)$ is a product of cyclotomic polynomials $\Phi_d(t)$ with d prime to p. It remains to show that the only such product mapping to $Q(t)\in k[t]$ is

$$Q'(t)=(t-1)^{t_G(1)}\prod_{i=1}^{v}(t-\xi^{e\cdot c_i})\in\mathbb{Z}[t].$$

It suffices to prove that for all tuples $(m_1,\ldots,m_r)$ and $(n_1,\ldots,n_s)$ of elements in $\mathbb{N}'$, the function

$$R(t)=\frac{\prod_{i=1}^{r}(t^{m_i}-1)}{\prod_{j=1}^{s}(t^{n_j}-1)}\in\mathbb{Z}\left[t,\frac{1}{t^n-1}\right]_{n\in\mathbb{N}'}$$

maps to one in $k(t)$ iff $R(t)=1$. This is easily seen by looking at the zeroes of $\prod_{i=1}^{r}(t^{m_i}-1)\in k[t]$ and $\prod_{j=1}^{s}(t^{n_j}-1)\in k[t]$ and using the fact that $\mu_n(k)$ is a cyclic group of order n for each $n\in\mathbb{N}'$. □

Corollary 5.2. *If G is tamely ramified and G has purely additive reduction, then*

$$|\Phi_G|=\prod_{d>1}\Phi_d(1)^{\nu_d/\varphi(d)}.$$

Proof. Apply Lemma 3.4 and Proposition 3.5. □

Corollary 5.3. *If G is tamely ramified, then its elementary divisors are invariant under isogeny.*

Corollary 5.4. *If G is tamely ramified, then its base change conductor satisfies*

$$c(G)=\frac{u(G)}{2},$$

where $u(G)$ is the unipotent rank of $\mathcal{G}_s^o$.

Proof. Let $c=(c_1,\ldots,c_v)$ be the tuple of elementary divisors of G. The fact that the right hand side of (5.1) belongs to $\mathbb{Z}[t]$ implies that the map $x\mapsto 1-x$ defines a permutation of c. Therefore,

$$c(G):=\sum_{i=1}^{v}c_i=v/2=u(G)/2,$$

where the last equality follows from Corollary 4.8. □

Remark. Corollary 5.4 is a special case of a much deeper result by Chai, Yu and de Shalit [5, 11.3 and 12.1], stating that for *any* algebraic K-torus G, the base change conductor $c(G)$ equals half of the Artin conductor of the I-action on $V=X\otimes_{\mathbb{Z}}\mathbb{Q}$. If G is tame, then the Artin conductor simply equals the dimension of V/V^I, and this is precisely $u(G)$ by Proposition 4.4. If G is not tame, it is no longer true that its elementary divisors are invariant under isogeny (see for instance [4, 8.5(b)]).

Corollary 5.5. *If G is tamely ramified, then the determinant D of the σ-action on $H^1(G\times_K K^t,\mathbb{Q}_\ell)$ equals $(-1)^{u(G)}$. Likewise, the determinant of the σ-action on $X\otimes_{\mathbb{Z}}\mathbb{Q}$ equals $(-1)^{u(G)}$.*

Proof. If ξ is any primitive e-th root of unity in $\mathbb{Q}^a$, then by Proposition 5.1 we know that $D=\xi^{c(G)e}$. Now the result follows from Corollary 5.4. □

6. The global monodromy property for algebraic tori

The following result is a global version for algebraic tori of Denef and Loeser's motivic monodromy conjecture. For the notion of a pole of a motivic generating series, we refer to [16, 4.7] (it requires some care since $\mathcal{M}_k$ might not be a domain).

Theorem 6.1. *Let G be a tamely ramified algebraic K-torus of dimension g. Denote by e the splitting degree of G, and by $u(G)$ the unipotent rank of the identity component of the special fiber of the Néron model of G. The motivic zeta function $Z_G(T)$ belongs to*

$$\mathcal{M}_k\left[T, \frac{1}{1-\mathbb{L}^{ep}T^{c(G)ep}}\right].$$

It has degree zero if $p=1$ and has strictly negative degree if $p>1$. Moreover, the order of the unique pole $s=c(G)$ of $Z_G(\mathbb{L}^{-s})$ equals one. The cyclotomic polynomial $\Phi_{\tau(c(G))}(t)$ equals $t+(-1)^{u(G)+1}$, and it coincides with the characteristic polynomial $P_\sigma^{(g)}(t)$ of the action of the monodromy operator σ on $H^g(G\times_K K^t,\mathbb{Q}_\ell)$.

Proof. Denote by J the set of integers in $\{1,\ldots,ep\}$ that are prime to p. By Propositions 4.2, 4.4 and 4.7 we can write

$$\begin{aligned} Z_G(T) &= \sum_{i\in J}\left(\phi_G(i)\mathbb{L}^{u_G(i)+ord_G(i)}(\mathbb{L}-1)^{t_G(i)}T^i\sum_{q\geq 0}\mathbb{L}^{c(G)epq}T^{epq}\right)\\ &= \sum_{i\in J}\left(\phi_G(i)\mathbb{L}^{u_G(i)+ord_G(i)}(\mathbb{L}-1)^{t_G(i)}T^i\right)\frac{1}{1-\mathbb{L}^{c(G)ep}T^{ep}}.\end{aligned}$$

So we see that $Z_G(\mathbb{L}^{-s})$ has a unique pole at $s=c(G)$, of order one (to see that the order is one and not zero, specialize the zeta function with respect to the Poincaré polynomial; see [11, § 8]). Since the σ-action on $H^g(G\times_K K^t,\mathbb{Q}_\ell)$ is the determinant of the σ-action on $H^1(G\times_K K^t,\mathbb{Q}_\ell)$, it follows from Corollary 5.5 that

$$P_\sigma^{(g)}(t)=t+(-1)^{u(G)+1}.$$

This polynomial is equal to $\Phi_{\tau(c(G))}(t)$ by Corollary 5.4. □

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References

[1] L. Bégueri. Dualité sur un corps local à corps résiduel algébriquement clos. *Mém. Soc. Math. Fr., Nouv. Sér. 4*, 121 pp., 1980/81. MR615883 (82k:12019)

[2] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*. Ergebnisse der Mathematik und ihrer Grenzgebiete, volume 21, Springer, 1990. MR1045822 (91i:14034)

[3] S. Bosch and K. Schlöter. Néron models in the setting of formal and rigid geometry. *Math. Ann.*, 301(2):339–362, 1995. MR1314591 (96h:14035)

[4] C.L. Chai. Néron models for semiabelian varieties: congruence and change of base field. *Asian J. Math.*, 4(4):715–736, 2000. MR1870655 (2002i:14025)

[5] C.-L. Chai and J.-K. Yu. Congruences of Néron models for tori and the Artin conductor (with an appendix by E. de Shalit). *Ann. Math. (2)*, 154:347–382, 2001. MR1865974 (2003e:11126)

[6] L. Halvard Halle and J. Nicaise. Motivic zeta functions of abelian varieties, and the monodromy conjecture. Preprint, arXiv:0902.3755v4.

[7] F. Loeser and J. Sebag. Motivic integration on smooth rigid varieties and invariants of degenerations. *Duke Math. J.*, 119:315–344, 2003. MR1997948 (2004g:14026)

[8] J. S. Milne. *Étale Cohomology*, volume 33 of Princeton Mathematical Series. Princeton University Press, 1980. MR559531 (81j:14002)

[9] E. Nart and X. Xarles. Additive reduction of algebraic tori. *Arch. Math. (Basel)*, 57:460–466, 1991. MR1129520 (92m:14056)

[10] J. Nicaise. A trace formula for rigid varieties, and motivic Weil generating series for formal schemes. *Math. Ann.*, 343(2):285–349, 2009. MR2461256 (2010b:14043)

[11] J. Nicaise. A trace formula for varieties over a discretely valued field. To appear in *J. Reine Angew. Math.*, arxiv:0805.1323v2.

[12] J. Nicaise. Trace formula for component groups of Néron models. Preprint, arXiv:0901.1809.

[13] J. Nicaise and J. Sebag. The motivic Serre invariant, ramification, and the analytic Milnor fiber. *Invent. Math.*, 168(1):133–173, 2007. MR2285749 (2009c:14040)

[14] J. Nicaise and J. Sebag. Motivic Serre invariants and Weil restriction. *J. Algebra*, 319(4):1585–1610, 2008. MR2383059 (2009e:14041)

[15] J. Nicaise and J. Sebag. A note on motivic integration in mixed characteristic. Preprint, arXiv:0912.4887.

[16] B. Rodrigues and W. Veys. Poles of zeta functions on normal surfaces. *Proc. London Math. Soc.*, 87(3):164–196, 2003. MR1978573 (2004c:14022)

[17] J.-P. Serre. *Corps locaux*. Hermann, Paris, 1962. MR0150130 (27:133)

[18] J.-P. Serre. *Groupes algébriques et corps de classes*. Hermann, Paris, 1959. MR0103191 (21:1973)

[19] X. Xarles. The scheme of connected components of the Néron model of an algebraic torus. *J. Reine Angew. Math.*, 437:167–179, 1993. MR1212256 (94d:14044)

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