VECTOR SPACES OF ENTIRE FUNCTIONS OF UNBOUNDED TYPE

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ABSTRACT. Let E be an infinite dimensional complex Banach space. We prove the existence of an infinitely generated algebra, an infinite dimensional closed subspace and a dense subspace of entire functions on E whose non-zero elements are functions of unbounded type. We also show that the τ_δ topology on the space of all holomorphic functions cannot be obtained as a countable inductive limit of Fréchet spaces.

1. INTRODUCTION AND NOTATION

The results in this article are related to the *lineability* of sets of functions. We recall that a subset X of a vector space is said to be *lineable* if there exists an infinite dimensional vector space W such that $W \subset X \cup \{0\}$. The first theorems in this topic were obtained by Gurariy in the sixties (see [12]), and in the last ten years this line of research has been extensively developed by several authors such as Aron, Bernal González, García Pacheco, Seoane Sepúlveda and others, who have proved the *lineability* of many collections of special mappings: differentiable nowhere monotone functions, everywhere surjective functions, continuous nowhere differentiable functions, etc. (see [5], [6], [7]).

In this paper, we study the *lineability* of the set of unbounded holomorphic functions defined on a Banach space. To begin with, let us explain the main notation used in our work. Throughout the article, E will denote a complex Banach space and U will be an open subset of E . If $x \in E$ and $r > 0$, then $B(x, r)$ represents the open ball in E with center x and radius r . For each $n \in \mathbb{N}$, $\mathcal{P}^n(E)$ denotes the space of all continuous n -homogeneous polynomials from E into $\mathbb{C}$. The symbol $\mathcal{H}(U)$ denotes the space of all holomorphic functions on U . If $f \in \mathcal{H}(U)$ and X is a subset of U , let

$$\|f\|_X = \sup \{|f(x)| : x \in X\}.$$

Let

$$\mathcal{H}_b(E) = \{f \in \mathcal{H}(E) : \|f\|_B < \infty \text{ for every bounded set } B \subset E\}.$$

If E has infinite dimension, then $\mathcal{H}_b(E)$ is always a proper subset of $\mathcal{H}(E)$ (see [11, page 157]); the elements of $\mathcal{H}(E) \setminus \mathcal{H}_b(E)$ are called entire functions of unbounded type.

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The symbol τ_δ denotes the locally convex topology on $\mathcal{H}(U)$ defined by the seminorms p which verify the following property: for each increasing countable open cover $\{V_n\}_{n=1}^\infty$ of U , there exist $C > 0$ and $n \in \mathbb{N}$ such that

$$p(f) \leq C \|f\|_{V_n} \quad \text{for every } f \in \mathcal{H}(U).$$

τ_δ is the bornological topology associated with the compact open topology τ_0 on $\mathcal{H}(U)$; in particular, $\tau_0 \leq \tau_\delta$. We refer the reader to the books of Dineen [11] and Mujica [14] for extensive studies of holomorphic functions defined on Banach spaces and the topologies on $\mathcal{H}(U)$.

2. LINEABILITY OF $\mathcal{H}(E) \setminus \mathcal{H}_b(E)$

In this section we will see how the set $\mathcal{H}(E) \setminus \mathcal{H}_b(E)$ is *algebrable*, *spaceable* and *dense-lineable*. This means that there exist an infinitely generated algebra $\mathcal{A} \subset \mathcal{H}(E)$, an infinite dimensional closed subspace $\mathcal{F} \subset (\mathcal{H}(E), \tau_0)$ and a dense subspace $W \subset (\mathcal{H}(E), \tau_\delta)$ such that

$$\mathcal{A} \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E), \quad \mathcal{F} \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E)$$

and

$$W \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E).$$

To get these results we need a proposition about biorthogonal systems in Banach spaces, which will be our main tool throughout the article.

Proposition 1 (see [9]). *Let E be an infinite dimensional Banach space. Let $x_0 \in E$ and $\varphi_0 \in E'$. Then there are sequences $\{x_k\}_{k=1}^\infty \subset E$ and $\{\varphi_k\}_{k=1}^\infty \subset E'$ with the following properties:*

- a) $\lim_{k \rightarrow \infty} \varphi_k(x) = 0$ for all $x \in E$,
- b) $\|\varphi_k\| = 1$ for all $k \geq 1$,
- c) $\sup_{k \in \mathbb{N}} \|x_k\| < \infty$,
- d) $\varphi_k(x_k) = 1$ for all $k \geq 1$,
- e) if $k, j \in \mathbb{N} \cup \{0\}$ and $k \neq j$, then $\varphi_k(x_j) = 0$.

The proof of this proposition is the same as in [9, page 250] and it is an inductive process with the subspaces $X_1 = \{x \in E : \varphi_0(x) = 0\}$ and $Z_1 = \{\varphi \in E' : \varphi(x_0) = 0\}$ in the first step.

Theorem 2. *If E is an infinite dimensional Banach space, then there exists an infinitely generated algebra $\mathcal{A} \subset \mathcal{H}(E)$ such that $\mathcal{A} \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E)$.*

Proof. Let $\{x_k, \varphi_k\}_{k=1}^\infty$ be a biorthogonal system in E with the properties given by Proposition 1; in this case, we do not need any particular vector x_0 or functional φ_0 . Let $\{a_j\}_{j=1}^\infty$ be a strictly increasing sequence of prime numbers. For every $j \in \mathbb{N}$, the function

$$f_j = \sum_{k=1}^{\infty} (a_j \cdot \varphi_k)^k$$

is entire on E (see [14, Example 5.5]). Let $\mathcal{A}$ be the algebra generated by $\{f_j\}_{j=1}^\infty$. Every $h \in \mathcal{A} \setminus \{0\}$ can be written as

$$h = \sum_{n=1}^N \lambda_n \prod_{j \in J_n} f_j^{p_{n,j}},$$

where $N \in \mathbb{N}$, $\lambda_n \in \mathbb{C} \setminus \{0\}$, J_n is a finite subset of $\mathbb{N}$ and $p_{n,j} \in \mathbb{N}$ for all n, j . We assume that if $J_n = J_m$ and $p_{n,j} = p_{m,j}$ for every $j \in J_n = J_m$, then $n = m$. Let us compute the function h at the points $\{x_k\}_{k=1}^\infty$:

$$\begin{aligned} h(x_k) &= \sum_{n=1}^N \lambda_n \prod_{j \in J_n} (f_j(x_k))^{p_{n,j}} = \sum_{n=1}^N \lambda_n \prod_{j \in J_n} (a_j^k)^{p_{n,j}} \\ &= \sum_{n=1}^N \lambda_n \left(\prod_{j \in J_n} a_j^{p_{n,j}} \right)^k. \end{aligned}$$

Since the numbers $\{a_j\}_{j=1}^\infty$ are prime, the products $\prod_{j \in J_1} a_j^{p_{1,j}}, \dots, \prod_{j \in J_N} a_j^{p_{N,j}}$ are different, so there exists $m \in \{1, \dots, N\}$ such that

$$\prod_{j \in J_m} a_j^{p_{m,j}} > \prod_{j \in J_n} a_j^{p_{n,j}} \quad \text{for all } n \in \{1, \dots, N\} \setminus \{m\}.$$

Hence

$$\frac{h(x_k)}{\left(\prod_{j \in J_m} a_j^{p_{m,j}} \right)^k} = \lambda_m + \sum_{\substack{n=1 \\ n \neq m}}^N \lambda_n \left(\frac{\prod_{j \in J_n} a_j^{p_{n,j}}}{\prod_{j \in J_m} a_j^{p_{m,j}}} \right)^k \xrightarrow{k \rightarrow \infty} \lambda_m \neq 0.$$

Moreover, each a_j is greater than 1, so $\lim_{k \rightarrow \infty} \left(\prod_{j \in J_m} a_j^{p_{m,j}} \right)^k = \infty$ and therefore $\lim_{k \rightarrow \infty} |h(x_k)| = \infty$. Since the sequence $\{x_k\}_{k=1}^\infty$ is bounded, it follows that h is a function of unbounded type.

Finally, we will show that $\{f_j\}_{j=1}^\infty$ is an algebraically independent set. Suppose

$$\sum_{n=1}^N \lambda_n \prod_{j \in J_n} f_j^{p_{n,j}} = 0,$$

where $N \in \mathbb{N}$, $\lambda_n \in \mathbb{C}$, J_n is a finite subset of $\mathbb{N}$ and $p_{n,j} \in \mathbb{N}$ for all n, j . If $\lambda_n \neq 0$ for some n , then it is possible to repeat the same argument used to prove that $h \in \mathcal{H}(E) \setminus \mathcal{H}_b(E)$ and we deduce that

$$0 = \left| \sum_{n=1}^N \lambda_n \prod_{j \in J_n} (f_j(x_k))^{p_{n,j}} \right| \xrightarrow{k \rightarrow \infty} \infty.$$

This is absurd, so $\lambda_n = 0$ for every n . This proves that $\{f_j\}_{j=1}^\infty$ is an algebraically independent system of generators of $\mathcal{A}$ and this algebra is infinitely generated. $\square$

Proposition 3. *Let E be an infinite dimensional Banach space. Let $\{x_k, \varphi_k\}_{k=1}^\infty$ be a biorthogonal system in E such that $\lim_{k \rightarrow \infty} \varphi_k(x) = 0$ for all $x \in E$ and*

$R = \sup_k \|x_k\| < \infty$ (see Proposition 1). Let

$$f = \sum_{k=1}^{\infty} \varphi_k^k.$$

If P is a non-zero continuous polynomial on E , then $P \cdot f \in \mathcal{H}(E) \setminus \mathcal{H}_b(E)$.

Proof. The polynomial can be written as $P = P_m + \dots + P_0$, where $P_k \in \mathcal{P}({}^k E)$ for $0 \leq k \leq m$ and $P_m \neq 0$. Let $w \in E$ such that $\|w\| = 1$ and $P_m(w) \neq 0$. Let L be the symmetric m -linear function associated with P_m . The function

$$t \in \mathbb{R} \mapsto |P_m(w)| t^m - \sum_{k=0}^{m-1} \binom{m}{k} \|L\| \cdot (2R)^{m-k} \cdot t^k - \sum_{k=0}^{m-1} \|P_k\| \cdot (t + 2R)^k$$

is a non-zero polynomial, and hence there exists $t > 0$ such that

$$(2.1) \quad |P_m(w)| t^m - \sum_{k=0}^{m-1} \binom{m}{k} \|L\| \cdot (2R)^{m-k} \cdot t^k - \sum_{k=0}^{m-1} \|P_k\| \cdot (t + 2R)^k \geq |P_m(w)|.$$

We define $z = tw$. Assume that $x \in E$ and $\|x - z\| \leq 2R$. By the Newton Binomial Formula, we have

$$P_m(x) = P_m(z) + \sum_{k=0}^{m-1} \binom{m}{k} L(z^k, (x - z)^{m-k})$$

(see [11, Lemma 1.9]). Then

$$\begin{aligned} |P_m(x)| &\geq |P_m(z)| - \sum_{k=0}^{m-1} \binom{m}{k} \|L\| \cdot \|z\|^k \cdot \|x - z\|^{m-k} \\ &\geq |P_m(w)| t^m - \sum_{k=0}^{m-1} \binom{m}{k} \|L\| \cdot t^k \cdot (2R)^{m-k}. \end{aligned}$$

On the other hand, since $\|x\| \leq \|z\| + 2R = t + 2R$,

$$\sum_{k=0}^{m-1} |P_k(x)| \leq \sum_{k=0}^{m-1} \|P_k\| \cdot \|x\|^k \leq \sum_{k=0}^{m-1} \|P_k\| \cdot (t + 2R)^k.$$

Therefore,

$$\begin{aligned} |P(x)| &\geq |P_m(x)| - \sum_{k=0}^{m-1} |P_k(x)| \\ &\geq |P_m(w)| t^m - \sum_{k=0}^{m-1} \binom{m}{k} \|L\| \cdot t^k \cdot (2R)^{m-k} - \sum_{k=0}^{m-1} \|P_k\| \cdot (t + 2R)^k. \end{aligned}$$

By (2.1), we deduce that $|P(x)| \geq |P_m(w)| > 0$ for all $x \in \overline{B}(z, 2R)$.

The function f is entire on E because $\lim_{k \rightarrow \infty} \varphi_k(x) = 0$ for all $x \in E$ (see [14, Example 5.5]). Let $k_0 \in \mathbb{N}$ such that $|\varphi_k(z)| < \frac{1}{2}$ for all $k \geq k_0$. Since

$$f(2x_k + z) = (2 + \varphi_k(z))^k + \sum_{j=1, j \neq k}^{\infty} (\varphi_j(z))^j,$$

if $k \geq k_0$, then

$$\begin{aligned} |f(2x_k + z)| &\geq |2 + \varphi_k(z)|^k - \sum_{j=1, j \neq k}^{\infty} |\varphi_j(z)|^j \\ &\geq (2 - |\varphi_k(z)|)^k - \sum_{j=1}^{k_0} |\varphi_j(z)|^j - \sum_{j=k_0+1}^{\infty} \frac{1}{2^j} \\ &\geq \left(\frac{3}{2}\right)^k - \sum_{j=1}^{k_0} |\varphi_j(z)|^j - 1. \end{aligned}$$

This implies that $\lim_{k \rightarrow \infty} |f(2x_k + z)| = \infty$. Since $2x_k + z \in \overline{B}(z, 2R)$ for every k ,

$$\lim_{k \rightarrow \infty} |(P \cdot f)(2x_k + z)| \geq |P_m(w)| \cdot \lim_{k \rightarrow \infty} |f(2x_k + z)| = \infty,$$

so $P \cdot f \in \mathcal{H}(E) \setminus \mathcal{H}_b(E)$. $\square$

Theorem 4. *If E is an infinite dimensional Banach space, then there exists an infinite dimensional closed subspace $\mathcal{F} \subset (\mathcal{H}(E), \tau_0)$ such that $\mathcal{F} \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E)$.*

Proof. Let $\{x_k, \varphi_k\}_{k=1}^{\infty}$ be a biorthogonal system in E with the properties given in Proposition 1. Let

$$f = \sum_{k=1}^{\infty} \varphi_k^k \in \mathcal{H}(E).$$

By Proposition 3,

$$\mathcal{F} = \{\psi \cdot f : \psi \in E'\}$$

is a subspace of $\mathcal{H}(E)$ such that $\mathcal{F} \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E)$.

Let us suppose that $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ and

$$\lambda_1 \varphi_1 \cdot f + \dots + \lambda_m \varphi_m \cdot f = 0.$$

If this sum is computed at the points $\{x_1, \dots, x_m\}$, it follows that $\lambda_1 = \dots = \lambda_m = 0$. Therefore, $\{\varphi_k \cdot f\}_{k=1}^{\infty}$ is a linearly independent set and $\mathcal{F}$ has infinite dimension.

Suppose $(\psi_i \cdot f)_i$ is a net in $\mathcal{F}$ which converges to a function $h \in \mathcal{H}(E)$ for the τ_0 topology:

$$h = \lim_i (\psi_i \cdot f) = \lim_i \sum_{k=1}^{\infty} (\psi_i \cdot \varphi_k^k).$$

The mapping

$$g = \sum_{n=0}^{\infty} \frac{\widehat{d^n g}(0)}{n!} \in (\mathcal{H}(E), \tau_0) \mapsto \frac{\widehat{d^2 g}(0)}{2} \in (\mathcal{P}({}^2 E), \tau_0)$$

is continuous (see [11, Proposition 3.22]), so

$$(2.2) \quad \frac{\widehat{d^2 h}(0)}{2} = \lim_i (\psi_i \cdot \varphi_1),$$

where the convergence is uniform on the compact subsets of E .

Let K be a compact subset of E . Let

$$L = \{x + e^{i\theta} y : x, y \in K \cup \{x_1\}, 0 \leq \theta \leq 2\pi\},$$

which is also a compact subset of E . Let $x, y \in K \cup \{x_1\}$. As is shown in [8, Lemma 1],

$$(\psi_i - \psi_j)(x) \cdot \overline{\varphi_1(y)} = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} \cdot (\psi_i - \psi_j)(x + e^{i\theta}y) \cdot \overline{\varphi_1(x + e^{i\theta}y)} d\theta.$$

Then

$$\begin{aligned} |(\psi_i - \psi_j)(x)| \cdot |\varphi_1(y)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |(\psi_i - \psi_j)(x + e^{i\theta}y) \cdot \varphi_1(x + e^{i\theta}y)| d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \|(\psi_i - \psi_j) \cdot \varphi_1\|_L d\theta \\ &= \|(\psi_i - \psi_j) \cdot \varphi_1\|_L. \end{aligned}$$

Therefore,

$$\|\psi_i - \psi_j\|_{K \cup \{x_1\}} \cdot \|\varphi_1\|_{K \cup \{x_1\}} \leq \|(\psi_i - \psi_j) \cdot \varphi_1\|_L.$$

Moreover, $\|\varphi_1\|_{K \cup \{x_1\}} \geq 1$, so

$$\|\psi_i - \psi_j\|_K \leq \|\psi_i - \psi_j\|_{K \cup \{x_1\}} \cdot \|\varphi_1\|_{K \cup \{x_1\}} \leq \|(\psi_i - \psi_j) \cdot \varphi_1\|_L.$$

By (2.2), $(\psi_i \cdot \varphi_1)_i$ is a Cauchy net in $(\mathcal{P}({}^2E), \tau_0)$, so for every $\varepsilon > 0$ there is an index i_0 such that

$$\|(\psi_i - \psi_j) \cdot \varphi_1\|_L \leq \varepsilon$$

for $i, j \geq i_0$. Thus,

$$\|\psi_i - \psi_j\|_K \leq \|(\psi_i - \psi_j) \cdot \varphi_1\|_L \leq \varepsilon$$

for all $i, j \geq i_0$. This shows that $(\psi_i)_i$ is a Cauchy net in (E', τ_0) , and hence there exists $\psi = \lim_i \psi_i \in E'$ (see [14, Proposition 9.13]). Therefore,

$$h = \lim_i (\psi_i \cdot f) = \psi \cdot f \in \mathcal{F}$$

and $\mathcal{F}$ is a closed subspace in $(\mathcal{H}(E), \tau_0)$. $\square$

Theorem 5. *If E is an infinite dimensional Banach space, then there exists a dense subspace $W \subset (\mathcal{H}(E), \tau_\delta)$ such that $W \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E)$.*

Proof. If A is a set, then $|A|$ denotes the cardinality of A . For each $m \in \mathbb{N}$, let $\mathcal{B}_m = \{P_{m,i} : i \in I_m\}$ be an algebraic basis of $\mathcal{P}({}^mE)$ such that $\|P_{m,i}\| = 1$ for all i . The space

$$\bigoplus_{n=0}^m \mathcal{P}({}^nE) = \left\{ \sum_{n=0}^m P_n : P_0 \in \mathcal{P}({}^0E), \dots, P_m \in \mathcal{P}({}^mE) \right\}$$

has the same cardinality as $\mathcal{B}_m$. Indeed, if $n \leq m$, then $\mathcal{P}({}^nE)$ is a complemented subspace of $\mathcal{P}({}^mE)$ (see [4, Proposition 5.3]). Hence $|\mathcal{P}({}^nE)| \leq |\mathcal{P}({}^mE)|$, which implies that

$$\begin{aligned} |\mathcal{B}_m| &\leq \left| \bigoplus_{n=0}^m \mathcal{P}({}^nE) \right| = |\mathcal{P}({}^0E) \times \dots \times \mathcal{P}({}^mE)| \\ &\leq |\mathcal{P}({}^mE)^{m+1}| = |\mathcal{P}({}^mE)|. \end{aligned}$$

Moreover, $|\mathcal{P}({}^m E)| = |\mathcal{B}_m|$ because $\mathcal{P}({}^m E)$ is an infinite dimensional Banach space (see [13]) and hence $|\bigoplus_{n=0}^m \mathcal{P}({}^n E)| = |\mathcal{B}_m|$. Therefore, $\bigoplus_{n=0}^m \mathcal{P}({}^n E)$ can be indexed in the following way:

$$\bigoplus_{n=0}^m \mathcal{P}({}^n E) = \{Q_{m,i} : i \in I_m\}.$$

We define

$$W = \text{span} \{P_{m,i} \cdot f + Q_{m,i} : m \in \mathbb{N}, i \in I_m\},$$

where $f \in \mathcal{H}(E)$ is the function introduced in Proposition 3. Let

$$\sum_{k=1}^n \lambda_k (P_{m_k, i_k} \cdot f + Q_{m_k, i_k}) \in W \setminus \{0\},$$

where $\lambda_1, \dots, \lambda_n \in \mathbb{C} \setminus \{0\}$, $m_1, \dots, m_n \in \mathbb{N}$ and $i_1 \in I_{m_1}, \dots, i_n \in I_{m_n}$. We assume that if $j, k \in \{1, \dots, n\}$, $m_j = m_k$ and $i_j = i_k$, then $j = k$. Since $\bigcup_{m=1}^{\infty} \mathcal{B}_m$ is a linearly independent set, $\sum_{k=1}^n \lambda_k P_{m_k, i_k} \neq 0$, so

$$\sum_{k=1}^n \lambda_k P_{m_k, i_k} \cdot f \in \mathcal{H}(E) \setminus \mathcal{H}_b(E)$$

by Proposition 3. Since $\sum_{k=1}^n \lambda_k Q_{m_k, i_k}$ is a polynomial, it is bounded on bounded subsets, so

$$\sum_{k=1}^n \lambda_k (P_{m_k, i_k} \cdot f + Q_{m_k, i_k}) \in \mathcal{H}(E) \setminus \mathcal{H}_b(E).$$

Therefore, $W \setminus \{0\} \subset \mathcal{H}(E) \setminus \mathcal{H}_b(E)$.

Let us prove that W is dense in $(\mathcal{H}(E), \tau_\delta)$. Let $h \in \mathcal{H}(E)$, p a continuous seminorm in $(\mathcal{H}(E), \tau_\delta)$ and $\varepsilon > 0$. The Taylor series of h at zero converges to h for the τ_δ topology (see [10, Proposition 1.2]), so there is $m \in \mathbb{N}$ such that

$$p \left(\sum_{n=0}^m \frac{\widehat{d^n h}(0)}{n!} - h \right) < \frac{\varepsilon}{2}.$$

If K is a compact subset of E , then there is $t > 0$ such that $tK \subset B(0, 1)$. Hence

$$\begin{aligned} \sup_{i \in I_m} \|P_{m,i} \cdot f\|_K &\leq \frac{1}{t^m} \cdot \sup_{i \in I_m} \|P_{m,i}\|_{tK} \cdot \|f\|_K \\ &\leq \frac{1}{t^m} \cdot \sup_{i \in I_m} \|P_{m,i}\|_{B(0,1)} \cdot \|f\|_K = \frac{\|f\|_K}{t^m} < \infty. \end{aligned}$$

This proves that $\{P_{m,i} \cdot f : i \in I_m\}$ is bounded for the compact open topology, therefore for τ_δ (see [10, Lemma 3.1]). Then there is $s > 0$ such that

$$s \cdot p(P_{m,i} \cdot f) < \frac{\varepsilon}{2} \quad \text{for all } i \in I_m.$$

Let $i \in I_m$ such that

$$Q_{m,i} = \frac{1}{s} \sum_{n=0}^m \frac{\widehat{d^n h}(0)}{n!} \in \bigoplus_{n=0}^m \mathcal{P}({}^n E).$$

Then $s(P_{m,i} \cdot f + Q_{m,i}) \in W$ and

$$\begin{aligned} p(s(P_{m,i} \cdot f + Q_{m,i}) - h) &\leq s \cdot p(P_{m,i} \cdot f) + p(sQ_{m,i} - h) \\ &= s \cdot p(P_{m,i} \cdot f) + p\left(\sum_{n=0}^m \frac{\widehat{d^n h}(0)}{n!} - h\right) < \varepsilon. \end{aligned}$$

This completes the proof. The definition of W is based on the proof of [7, Theorem 2.2]. $\square$

Remark 6. The density of $\mathcal{H}(E) \setminus \mathcal{H}_b(E)$ in $(\mathcal{H}(E), \tau_\delta)$ was proved by Aron in [3].

3. REPRESENTATION OF THE τ_δ TOPOLOGY

Let U be an open subset of a Banach space E . If $I = \{V_n\}_{n=1}^\infty$ is an increasing countable open cover of U , then $\mathcal{H}_I(U)$ denotes the following subspace of $\mathcal{H}(U)$:

$$\mathcal{H}_I(U) = \{f \in \mathcal{H}(U) : \|f\|_{V_n} < \infty \text{ for all } n \in \mathbb{N}\},$$

endowed with the locally convex topology defined by the seminorms

$$f \in \mathcal{H}_I(U) \mapsto \|f\|_{V_n} \quad (n \in \mathbb{N}).$$

It is well known that $(\mathcal{H}(U), \tau_\delta)$ can be obtained as the inductive limit of the Fréchet spaces $\mathcal{H}_I(U)$, where I ranges over the family of all increasing countable open covers of U :

$$(3.1) \quad (\mathcal{H}(U), \tau_\delta) = \varinjlim_I \mathcal{H}_I(U)$$

(see [11, Proposition 3.18]). In [1], Ansemil, Aron and Ponte showed that if E is an infinite dimensional Banach space with a Schauder basis, then there is no sequence $\{I_n\}_{n=1}^\infty$ of covers of E , such that $\mathcal{H}(E) = \bigcup_{n=1}^\infty \mathcal{H}_{I_n}(E)$; hence the inductive limit in (3.1) is not countable. In Theorem 9 below, we will generalize this result to the case of $(\mathcal{H}(U), \tau_\delta)$, where U will be an open subset of an arbitrary Banach space of infinite dimension. Furthermore, we will get more results on *lineability* of sets of holomorphic functions. With this aim we prove the following propositions.

Proposition 7. *Let U be an open subset of a Banach space such that $0 \in U$. Then there are sequences $\{z_k\}_{k=1}^\infty \subset U$ and $\{R_k\}_{k=0}^\infty \subset \mathbb{R}^+$ with the following properties:*

- a) $\{R_k\}_{k=0}^\infty$ is an increasing sequence,
- b) for each k , $z_k \in B(0, R_k)$ and $z_k \notin \overline{B}(0, R_{k-1})$,
- c) $U \subset \bigcup_{k=0}^\infty B(0, R_k)$.

Proof. Let $s = \sup\{\|x\| : x \in U\}$. Since U is open, it is clear that $\|x\| < s$ for all $x \in U$. Let $0 < R_0 < s$. There is a vector $z_1 \in U$ such that $R_0 < \|z_1\| < s$. If $s < \infty$, then there exists $m_1 \in \mathbb{N}$ such that

$$\|z_1\| < s - \frac{1}{m_1} < s,$$

and we define $R_1 = s - \frac{1}{m_1}$. If $s = \infty$, we choose $R_1 \in \mathbb{N}$ large enough so that $\|z_1\| < R_1$. In both cases we have $R_1 < s$, so there is another vector $z_2 \in U$ such that $R_1 < \|z_2\| < s$. If $s < \infty$, then there exists $m_2 \in \mathbb{N}$, $m_2 > m_1$, such that

$$\|z_2\| < s - \frac{1}{m_2} < s,$$

and we define $R_2 = s - \frac{1}{m_2}$. If $s = \infty$, we choose $R_2 \in \mathbb{N}$, $R_2 > R_1$, large enough so that $\|z_2\| < R_2$. The sequences $\{z_k\}_{k=1}^\infty$ and $\{R_k\}_{k=0}^\infty$ are obtained by recurrence. It is clear that $z_k \in B(0, R_k)$, $z_k \notin \overline{B}(0, R_{k-1})$ for each k and $\lim_{k \rightarrow \infty} R_k = s$. If $x \in U$, then $\|x\| < s$, so there is k such that $\|x\| < R_k$, that is, $x \in B(0, R_k)$. This proves that $U \subset \bigcup_{k=0}^\infty B(0, R_k)$. $\square$

The next proposition extends a recent result by Ansemil, Aron and Ponte, who have proved that given two disjoint balls in an infinite dimensional Banach space, there is an entire function which is bounded on the first ball and unbounded on the second ball (see [2]).

Proposition 8. *If $B(0, R)$ and $B(x_0, r)$ are disjoint balls in an infinite dimensional Banach space E and $\varepsilon > 0$, then there exists a sequence $\{f_n\}_{n=1}^\infty$ of entire functions on E with the following properties:*

- a) $\{f_n\}_{n=1}^\infty$ is a linearly independent set,
- b) $\|f_n\|_{B(0, R)} < \varepsilon$ for every n ,
- c) $\|h\|_{B(x_0, r)} = \infty$ for all $h \in \text{span}\{f_n\}_{n=1}^\infty \setminus \{0\}$.
- d) The sequence $\{f_n\}_{n=1}^\infty$ is locally bounded; that is, for every $x \in E$ there is a neighborhood V of x such that $\sup_{n \in \mathbb{N}} \|f_n\|_V < \infty$.

In particular, the set

$$\left\{ h \in \mathcal{H}(E) : \|h\|_{B(0, R)} < \infty \text{ and } \|h\|_{B(x_0, r)} = \infty \right\}$$

is lineable.

Proof. By the Hahn-Banach Theorem, there exists $\varphi_0 \in E'$ such that $\|\varphi_0\| = 1$ and $\varphi_0(x_0) = \|x_0\|$. Let $\{x_k, \varphi_k\}_{k=1}^\infty$ be a biorthogonal system in E with the properties given in Proposition 1. Let $C > \sup_{k \in \mathbb{N}} \|x_k\|$.

Since $x_0 \notin \overline{B}(0, R)$, then $\frac{R}{\|x_0\|} < 1$, so there is $\alpha \in \mathbb{N}$ such that

$$\left(\frac{R}{\|x_0\|} \right)^\alpha < \frac{r}{CR};$$

that is,

$$\frac{C \cdot R^{\alpha+1}}{\|x_0\|^\alpha \cdot r} < 1.$$

Let $\{a_n\}_{n=1}^\infty$ be a strictly increasing sequence of positive numbers such that

$$\frac{C \cdot R^{\alpha+1}}{\|x_0\|^\alpha \cdot r} < \dots < \frac{1}{a_n} < \dots < \frac{1}{a_1} < 1.$$

This means that for all $n \in \mathbb{N}$,

$$(3.2) \quad a_n \frac{C}{\|x_0\|^\alpha \cdot r} \cdot R^{\alpha+1} < 1 < a_n.$$

For each $n \in \mathbb{N}$, the function

$$g_n = \sum_{k=1}^\infty \left(a_n \frac{C}{\|x_0\|^\alpha \cdot r} \cdot \varphi_0^\alpha \cdot \varphi_k \right)^k$$

is entire on E . Indeed, let K be a compact subset of E . Since $\|\varphi_k\| = 1$ for all k , the sequence $\{\varphi_k\}_{k=1}^\infty$ is equicontinuous and since $\lim_{k \rightarrow \infty} \varphi_k(x) = 0$ for each

$x \in E$, also $\lim_{k \rightarrow \infty} \|\varphi_k\|_K = 0$ (see [14, Proposition 9.11]). Then there is $k_0 \in \mathbb{N}$ such that

$$a_n \cdot \frac{C}{\|x_0\|^\alpha \cdot r} \|\varphi_0^\alpha\|_K \cdot \|\varphi_k\|_K \leq \frac{1}{2}$$

for every $k \geq k_0$. Thus,

$$\sum_{k=k_0}^{\infty} \left(a_n \cdot \frac{C}{\|x_0\|^\alpha \cdot r} \|\varphi_0^\alpha\|_K \cdot \|\varphi_k\|_K \right)^k \leq \sum_{k=k_0}^{\infty} \left(\frac{1}{2} \right)^k < \infty.$$

This implies that the series $\sum_{k=1}^{\infty} \left(a_n \cdot \frac{C}{\|x_0\|^\alpha \cdot r} \cdot \varphi_0^\alpha \cdot \varphi_k \right)^k$ converges uniformly on the compact subsets of E , so $g_n \in \mathcal{H}(E)$ (see [14, Proposition 9.13]). The definition of g_n is based on the functions which appear in [1].

Let

$$h = \sum_{n=1}^N \lambda_n g_n \in \text{span} \{g_n\}_{n=1}^{\infty} \setminus \{0\},$$

where every λ_n belongs to $\mathbb{C}$ and $\lambda_N \neq 0$. By (3.2), each function g_n is bounded on $B(0, R)$:

$$\|g_n\|_{B(0, R)} \leq \sum_{k=1}^{\infty} \left(a_n \frac{C}{\|x_0\|^\alpha \cdot r} \cdot R^{\alpha+1} \right)^k < \infty,$$

so $\|h\|_{B(0, R)} < \infty$. Now let us compute the functions g_n and h at $x_0 + \frac{r}{C}x_k \in B(x_0, r)$ with $k \geq 1$:

$$g_n \left(x_0 + \frac{r}{C}x_k \right) = \left(a_n \frac{C}{\|x_0\|^\alpha \cdot r} \cdot \varphi_0(x_0)^\alpha \cdot \varphi_k \left(\frac{r}{C}x_k \right) \right)^k = a_n^k.$$

Therefore,

$$h \left(x_0 + \frac{r}{C}x_k \right) = \sum_{n=1}^N \lambda_n a_n^k.$$

Since $\{a_n\}_{n=1}^{\infty}$ is a strictly increasing sequence, $0 < \frac{a_n}{a_N} < 1$ for $1 \leq n \leq N-1$. Hence

$$\frac{h \left(x_0 + \frac{r}{C}x_k \right)}{a_N^k} = \lambda_N + \sum_{n=1}^{N-1} \lambda_n \left(\frac{a_n}{a_N} \right)^k \xrightarrow{k \rightarrow \infty} \lambda_N \neq 0.$$

As a_N is greater than 1, $\lim_{k \rightarrow \infty} a_N^k = \infty$. Therefore,

$$\|h\|_{B(x_0, r)} \geq \lim_{k \rightarrow \infty} \left| h \left(x_0 + \frac{r}{C}x_k \right) \right| = \infty.$$

The sequence $\{g_n\}_{n=1}^{\infty}$ is linearly independent. Indeed, suppose $\sum_{n=1}^N \lambda_n g_n = 0$, where $\{\lambda_1, \dots, \lambda_N\} \subset \mathbb{C}$. With the same argument used to prove that $\|h\|_{B(x_0, r)} = \infty$, we deduce that

$$0 = \frac{\sum_{n=1}^N \lambda_n g_n \left(x_0 + \frac{r}{C}x_k \right)}{a_N^k} \xrightarrow{k \rightarrow \infty} \lambda_N,$$

so $\lambda_N = 0$ and $\sum_{n=1}^{N-1} \lambda_n g_n = 0$. By iterating this process, we get that $\lambda_n = 0$ for all n ; that is, $\{g_n\}_{n=1}^{\infty}$ is a linearly independent set.

Let K be a compact subset in E . As we have seen, $\lim_{k \rightarrow \infty} \|\varphi_k\|_K = 0$, so there is $k_0 \in \mathbb{N}$ such that

$$\|\varphi_k\|_K \leq \frac{R^{\alpha+1}}{2\|\varphi_0\|_K^\alpha + 1}$$

for every $k > k_0$. By (3.2), we have

$$a_n \frac{C}{\|x_0\|^\alpha \cdot r} < \frac{1}{R^{\alpha+1}}$$

for every n , and thus we deduce the following inequalities:

$$\begin{aligned} \sup_{n \in \mathbb{N}} \|g_n\|_K &\leq \sum_{k=1}^{\infty} \left(\frac{1}{R^{\alpha+1}} \cdot \|\varphi_0\|_K^\alpha \cdot \|\varphi_k\|_K \right)^k \\ &\leq \sum_{k=1}^{k_0} \left(\frac{\|\varphi_0\|_K^\alpha}{R^{\alpha+1}} \cdot \|\varphi_k\|_K \right)^k + \sum_{k>k_0} \left(\frac{\|\varphi_0\|_K^\alpha}{R^{\alpha+1}} \cdot \frac{R^{\alpha+1}}{2\|\varphi_0\|_K^\alpha + 1} \right)^k \\ &\leq \sum_{k=1}^{k_0} \left(\frac{\|\varphi_0\|_K^\alpha}{R^{\alpha+1}} \cdot \|\varphi_k\|_K \right)^k + 1 < \infty. \end{aligned}$$

This proves that $\{g_n\}_{n=1}^\infty$ is bounded for the compact open topology and consequently it is locally bounded (see [14, Proposition 9.15]). To conclude the proof, it suffices to define

$$f_n = \frac{\varepsilon}{\|g_n\|_{B(0,R)} + 1} g_n$$

for each $n \in \mathbb{N}$. □

Theorem 9. *Let U be an open subset of an infinite dimensional Banach space E . For each $k \in \mathbb{N}$, let $I_k = \{V_{k,j}\}_{j=1}^\infty$ denote an increasing countable open cover of U . Then there exists an infinite dimensional vector space W such that*

$$W \setminus \{0\} \subset \mathcal{H}(U) \setminus \bigcup_{k=1}^{\infty} \mathcal{H}_{I_k}(U).$$

In particular, $\mathcal{H}(U) \neq \bigcup_{k=1}^{\infty} \mathcal{H}_{I_k}(U)$, so the inductive limit in (3.1) is not countable.

Proof. Some of our arguments are based on the proof of the main theorem in [1]. Using a translation, we can assume that $0 \in U$. By Proposition 7, there are sequences $\{z_k\}_{k=1}^\infty \subset U$ and $\{R_k\}_{k=0}^\infty \subset \mathbb{R}^+$ such that $\{R_k\}_{k=0}^\infty$ is increasing, $U \subset \bigcup_{k=0}^\infty B(0, R_k)$, $z_k \in B(0, R_k)$ and $z_k \notin \overline{B}(0, R_{k-1})$ for every $k \geq 1$.

The collection $I_1 = \{V_{1,j}\}_{j=1}^\infty$ is an open cover of U , so there is $j_1 \in \mathbb{N}$ such that $z_1 \in V_{1,j_1}$. Then there exists $r_1 > 0$ such that

$$B(z_1, r_1) \subset V_{1,j_1}, \quad B(z_1, r_1) \subset B(0, R_1) \quad \text{and} \quad B(z_1, r_1) \cap B(0, R_0) = \emptyset.$$

By Proposition 8, there exists a linearly independent sequence $\{f_{1,n}\}_{n=1}^\infty \subset \mathcal{H}(E)$ such that

$$\|f_{1,n}\|_{B(0,R_0)} < \frac{1}{2} \quad \text{for each } n$$

and

$$\|h\|_{B(z_1,r_1)} = \infty \quad \text{for all } h \in \text{span} \{f_{1,n}\}_{n=1}^\infty \setminus \{0\}.$$

$I_2 = \{V_{2,j}\}_{j=1}^\infty$ is also an open cover of U , so there is $j_2 \in \mathbb{N}$ such that $z_2 \in V_{2,j_2}$. Since the sequence $\{f_{1,n}\}_{n=1}^\infty$ is locally bounded, there exists $r_2 > 0$ such that

$$B(z_2, r_2) \subset V_{2,j_2}, \quad B(z_2, r_2) \subset B(0, R_2), \quad B(z_2, r_2) \cap B(0, R_1) = \emptyset$$

and

$$\|f_{1,n}\|_{B(z_2, r_2)} < \infty \quad \text{for all } n.$$

Again, there exists a linearly independent sequence $\{f_{2,n}\}_{n=1}^\infty \subset \mathcal{H}(E)$ such that

$$\|f_{2,n}\|_{B(0, R_1)} < \frac{1}{2^2} \quad \text{for each } n$$

and

$$\|h\|_{B(z_2, r_2)} = \infty \quad \text{for all } h \in \text{span}\{f_{2,n}\}_{n=1}^\infty \setminus \{0\}.$$

By iterating this process, we get sequences $\{j_k\}_{k=1}^\infty \subset \mathbb{N}$, $\{r_k\}_{k=1}^\infty \subset \mathbb{R}^+$ and $\{f_{k,n}\}_{k,n=1}^\infty \subset \mathcal{H}(E)$ with the following properties:

- a) $B(z_k, r_k) \subset V_{k,j_k}$,
- b) $B(z_k, r_k) \subset B(0, R_k)$,
- c) $B(z_k, r_k) \cap B(0, R_{k-1}) = \emptyset$,
- d) $\|f_{j,n}\|_{B(z_k, r_k)} < \infty$ if $j < k$ and $n \in \mathbb{N}$,
- e) $\|f_{k,n}\|_{B(0, R_{k-1})} < \frac{1}{2^k}$ for all n ,
- f) $\|h\|_{B(z_k, r_k)} = \infty$ for all $h \in \text{span}\{f_{k,n}\}_{n=1}^\infty \setminus \{0\}$.

Let $n \in \mathbb{N}$. If K is a compact subset of U , then there is $k \in \mathbb{N}$ such that $K \subset B(0, R_k)$ and thus

$$\sum_{j=k+1}^\infty \|f_{j,n}\|_K \leq \sum_{j=k+1}^\infty \|f_{j,n}\|_{B(0, R_k)} \leq \sum_{j=k+1}^\infty \|f_{j,n}\|_{B(0, R_{j-1})} \leq \sum_{j=k+1}^\infty \frac{1}{2^j} < \infty.$$

Therefore, the series $\sum_{j=1}^\infty f_{j,n}$ converges uniformly on the compact subsets of U . Since each $f_{j,n}$ is entire, the function

$$g_n = \sum_{j=1}^\infty f_{j,n}$$

is holomorphic on U (see [14, Proposition 9.13]). Let W be the vector space generated by $\{g_n\}_{n=1}^\infty$.

If $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $\lambda_1 g_1 + \dots + \lambda_n g_n = 0$, then

$$\begin{aligned} 0 &= \|\lambda_1 g_1 + \dots + \lambda_n g_n\|_{B(z_1, r_1)} \\ &\geq \|\lambda_1 f_{1,1} + \dots + \lambda_n f_{1,n}\|_{B(z_1, r_1)} \\ &\quad - \sum_{j=2}^\infty \left(\|\lambda_1 f_{j,1}\|_{B(z_1, r_1)} + \dots + \|\lambda_n f_{j,n}\|_{B(z_1, r_1)} \right) \\ &\geq \|\lambda_1 f_{1,1} + \dots + \lambda_n f_{1,n}\|_{B(z_1, r_1)} \\ &\quad - \sum_{j=2}^\infty \left(\|\lambda_1 f_{j,1}\|_{B(0, R_{j-1})} + \dots + \|\lambda_n f_{j,n}\|_{B(0, R_{j-1})} \right). \end{aligned}$$

We now use property (e):

$$0 \geq \|\lambda_1 f_{1,1} + \cdots + \lambda_n f_{1,n}\|_{B(z_1, r_1)} - \sum_{j=2}^{\infty} \left(\frac{|\lambda_1|}{2^j} + \cdots + \frac{|\lambda_n|}{2^j} \right).$$

This implies that

$$\|\lambda_1 f_{1,1} + \cdots + \lambda_n f_{1,n}\|_{B(z_1, r_1)} \leq (|\lambda_1| + \cdots + |\lambda_n|) \sum_{j=2}^{\infty} \frac{1}{2^j} < \infty.$$

By (f) we deduce that $\lambda_1 f_{1,1} + \cdots + \lambda_n f_{1,n} = 0$. Then $\lambda_1 = \cdots = \lambda_n = 0$, since $f_{1,1}, \dots, f_{1,n}$ are linearly independent. This shows that the functions $\{g_n\}_{n=1}^{\infty}$ are also linearly independent and that W has infinite dimension.

Now we are going to prove that $W \setminus \{0\} \subset \mathcal{H}(U) \setminus \bigcup_{k=1}^{\infty} \mathcal{H}_{I_k}(U)$. Let $h = \lambda_1 g_1 + \cdots + \lambda_n g_n \in W$, where at least $\lambda_n \neq 0$. Let $k \in \mathbb{N}$. Then

$$\begin{aligned} \|h\|_{V_{k,j_k}} &\geq \|h\|_{B(z_k, r_k)} \\ &\geq \|\lambda_1 f_{k,1} + \cdots + \lambda_n f_{k,n}\|_{B(z_k, r_k)} \\ &\quad - \sum_{j=1}^{k-1} \left(\|\lambda_1 f_{j,1}\|_{B(z_k, r_k)} + \cdots + \|\lambda_n f_{j,n}\|_{B(z_k, r_k)} \right) \\ &\quad - \sum_{j=k+1}^{\infty} \left(\|\lambda_1 f_{j,1}\|_{B(z_k, r_k)} + \cdots + \|\lambda_n f_{j,n}\|_{B(z_k, r_k)} \right). \end{aligned}$$

If $j \geq k+1$, then $B(z_k, r_k) \subset B(0, R_{j-1})$ and by (e),

$$\begin{aligned} \|h\|_{V_{k,j_k}} &\geq \|\lambda_1 f_{k,1} + \cdots + \lambda_n f_{k,n}\|_{B(z_k, r_k)} \\ &\quad - \sum_{j=1}^{k-1} \left(\|\lambda_1 f_{j,1}\|_{B(z_k, r_k)} + \cdots + \|\lambda_n f_{j,n}\|_{B(z_k, r_k)} \right) \\ &\quad - (|\lambda_1| + \cdots + |\lambda_n|) \sum_{j=k+1}^{\infty} \frac{1}{2^j}. \end{aligned}$$

Since $f_{k,1}, \dots, f_{k,n}$ are linearly independent, $\lambda_1 f_{k,1} + \cdots + \lambda_n f_{k,n} \neq 0$. By (f),

$$\|\lambda_1 f_{k,1} + \cdots + \lambda_n f_{k,n}\|_{B(z_k, r_k)} = \infty.$$

By (d), if $1 \leq j \leq k-1$, then

$$\|f_{j,1}\|_{B(z_k, r_k)} < \infty, \dots, \|f_{j,n}\|_{B(z_k, r_k)} < \infty.$$

Hence $\|h\|_{V_{k,j_k}} = \infty$ and h does not belong to any of the spaces $\mathcal{H}_{I_k}(U)$. This completes the proof. $\square$

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