

HARMONIC FUNCTIONS OF POLYNOMIAL GROWTH ON SINGULAR SPACES WITH NONNEGATIVE RICCI CURVATURE

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ABSTRACT. In the present paper, we will derive the Liouville theorem and the finite dimension theorem for polynomial growth harmonic functions defined on Alexandrov spaces with nonnegative Ricci curvature in the sense of Kuwae-Shioya and Sturm-Lott-Villani.

1. INTRODUCTION

In the 1970s, Yau [26] proved the Liouville theorem for harmonic functions on complete Riemannian manifolds with nonnegative Ricci curvature, i.e. manifolds for which there does not exist any nontrivial, positive harmonic function, then conjectured in [27], [28] that the linear space of polynomial growth harmonic functions with a fixed rate is finite dimensional. In 1997, Colding-Minicozzi provided an affirmative answer in [3] by the uniform Poincaré inequality and later obtained the optimal dimension estimate in [4]. In [12], Li used the mean value inequality to simplify the proof. Moreover, his argument can be applied to other cases such as manifolds with nonnegative Ricci curvature outside some compact subset (cf. [24]).

Hua [6] generalized the Liouville theorem to Alexandrov spaces with nonnegative sectional curvature by the Nash-Moser iteration (cf. [15], [5] or [21]). In the present paper, we generalize the previous results to Alexandrov spaces with nonnegative Ricci curvature in the sense of Kuwae-Shioya and Sturm-Lott-Villani. In fact, we prove the uniform Poincaré inequality on such spaces and derive the mean value inequality for subharmonic functions by the Nash-Moser iteration. The ingredient of the proofs involves global properties of harmonic functions depending only on the volume growth property.

In the framework of Colding-Minicozzi [3], [4] and Li [12], one key point is to consider a class of inner products on the subspace of polynomial growth harmonic functions. On Riemannian manifolds, by means of the unique continuation property of the solution to elliptic partial differential equations with smooth coefficients, the integration over B_R , i.e. $\langle u, v \rangle_R = \int_{B_R} uv$, is naturally an inner product for any R . However it is still open for Alexandrov spaces endowed with intrinsic metrics of low regularity (DC-differential structures, BV-Riemannian structures; cf. [17]). In order to overcome the difficulty, we shall derive the key Lemma 3.4 which states

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that the above integrations are inner products for sufficiently large R in Alexandrov spaces.

Throughout the present paper, we denote by X an n -dimensional Alexandrov space with $\text{Sec} \geq -\kappa$ ($\kappa > 0$) and satisfying the infinitesimal Bishop-Gromov volume comparison $\text{BG}(0)$, i.e. $\text{Ric} \geq 0$ in the generalized sense; see Definition 2.2 and Remark 2.3 below.

Our main results are as follows:

Theorem 1.1 (Liouville Theorem). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. Then it does not admit any nonconstant, positive, harmonic function.*

For some fixed $p \in X$, we denote the distance function by $r(x) = d(p, x)$ and set the space of polynomial growth harmonic functions with degree less than or equal to d as

$$H^d(X) = \{u \mid u \text{ is harmonic, } |u(x)| \leq C(r^d(x) + 1)\}.$$

Theorem 1.2 (Finite Dimension Theorem). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. Then $\dim H^d(X) \leq C(n)d^{n-1}$, where the constant $C(n)$ depends only on the dimension n .*

2. PRELIMINARIES

Definition 2.1 (Alexandrov Space; cf. [2] or [1]). A length space is called an Alexandrov space with sectional curvature bounded from below by κ ($\kappa \in \mathbb{R}$) if the Toponogov triangle comparison property holds locally on such a space with respect to the model space of constant curvature κ .

There is a natural extension map by geodesics on the Alexandrov space X . For fixed $p \in X$, $0 < t \leq 1$, the extension map is denoted by $\Phi_{p,t} : X \supset W_{p,t} \rightarrow X$, where $x \in W_{p,t}$ if and only if there exist some $y \in X$ and a minimal geodesic py such that $x \in py$ and $d(p, x) : d(p, y) = t : 1$; then we define $\Phi_{p,t}(x) = y$. By the Alexandrov convexity such a y is unique and the map is well defined.

The following is one of the definitions of generalized Ricci curvature on Alexandrov spaces.

Definition 2.2 ($\text{BG}(0)$; cf. [9]). An n -dimensional Alexandrov space X satisfies the infinitesimal Bishop-Gromov volume comparison $\text{BG}(0)$ at point $p \in X$ if

$$d(\Phi_{p,t*}H^n)(x) \geq t^n dH^n(x),$$

for any $x \in X$, $0 < t \leq 1$, where $\Phi_{p,t*}H^n$ is the push-forward measure by $\Phi_{p,t}$ of the n -dimensional Hausdorff measure H^n . We say that X satisfies the condition $\text{BG}(0)$ if it holds everywhere.

Remark 2.3. In addition, Sturm [22], [23] and Lott-Villani [14] defined another kind of generalized Ricci curvature, i.e. the curvature-dimension condition $CD(0, n)$ on a metric measure space (X, d, m) via optimal transport. For Riemannian manifolds (M, d, H^n) , the curvature-dimension condition $CD(0, n)$ is equivalent to $\text{Ric} \geq 0$ and $\dim M \leq n$ (cf. [23], [14] or [25]). For Alexandrov spaces (X, d, H^n) , Petrunin proved that $\text{Sec} \geq 0$ implies $CD(0, n)$ (cf. [18]), which indicates that the Ricci curvature definition via optimal transport is compatible with the definition in the sense of Alexandrov. On Alexandrov spaces, the condition $\text{BG}(0)$ is weaker than the curvature-dimension condition $CD(0, n)$ (cf. [23]). Hence, for global properties

of harmonic functions, it suffices to consider $BG(0)$. In the sequel, we always call $BG(0)$ the generalized nonnegative Ricci curvature.

Next we recall some basic definitions of harmonic functions on Alexandrov spaces (cf. [6], [8] or [19]). For some precompact domain $\Omega \Subset X$, the Sobolev space $W^{1,2}(\Omega)$ ($W_0^{1,2}(\Omega)$) is defined as the closure of Lipschitz functions (with compact support in Ω), $Lip(\Omega)$ ($Lip_0(\Omega)$), with respect to the norm

$$\|u\|_{W^{1,2}(\Omega)}^2 = \int_{\Omega} (u^2 + |\nabla u|^2).$$

Moreover, $u \in W_{loc}^{1,2}(X)$ if $u \in W^{1,2}(\Omega)$ for any precompact domain $\Omega \Subset X$.

Definition 2.4. A function $u : X \mapsto \mathbb{R}$ is harmonic (subharmonic, superharmonic) if $u \in W_{loc}^{1,2}(X)$ such that $\forall \varphi \in Lip_0(X)$ and $\varphi \geq 0$, we have

$$(2.1) \quad \int_X \nabla u \cdot \nabla \varphi = 0 \quad (\leq 0, \geq 0).$$

The infinitesimal Bishop-Gromov volume comparison $BG(0)$ implies the following Bishop-Gromov volume comparison theorem.

Theorem 2.5 (Bishop-Gromov; cf. [9]). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. Then $\forall p \in X, \forall 0 < r < r'$, we have*

$$(2.2) \quad \frac{H^n(B(p, r'))}{H^n(B(p, r))} \leq \left(\frac{r'}{r}\right)^n,$$

$$(2.3) \quad H^n(B(p, 2r)) \leq 2^n H^n(B(p, r)).$$

3. POINCARÉ INEQUALITY AND PROOFS OF THE MAIN RESULTS

Poincaré inequality. By the general theory of the Poincaré inequalities on metric spaces, following Saloff-Coste [20], we obtain the uniform Poincaré inequality on Alexandrov spaces with nonnegative Ricci curvature (cf. [6]).

Theorem 3.1 (Uniform Poincaré Inequality). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. Then there exists a constant $C = C(n)$, such that $\forall u \in W_{loc}^{1,2}(X)$, $\forall x \in X$, $\forall r > 0$, we have*

$$(3.1) \quad \int_{B(x,r)} |u - u_B|^2 \leq Cr^2 \int_{B(x,r)} |\nabla u|^2,$$

where $u_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} u$.

First we recall a lemma (Lemma 4.2 in Kuwae-Machigashira-Shioya [8]). Denote by $\gamma_{xy}(t), t \in [0, 1]$ a minimal geodesic joining x and y with parameter proportional to the arclength. Note that for almost all $(x, y) \in X \times X$, there is a unique minimal geodesic joining x and y (cf. [10] or [16]).

Lemma 3.2. *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. Then $\forall x \in X, r > 0$, nonnegative function $u : B(x, r) \rightarrow \mathbb{R}^+$ and $t \in (0, 1]$, we have*

$$(3.2) \quad \int_{B(x,r)} u(\gamma_{xy}(t)) dH^n(y) \leq \frac{1}{t^n} \int_{B(x,tr)} u(z) dH^n(z).$$

Proof. This is straightforward by the infinitesimal Bishop-Gromov volume comparison BG(0). \square

The uniform weak Poincaré inequality follows from the previous lemma. For simplicity, we always denote $dy = dH^n(y)$.

Lemma 3.3. *Let X be an n -dimensional Alexandrov space with generalized non-negative Ricci curvature. Then $\forall u \in W_{loc}^{1,2}(X)$, $\forall x \in X$, $\forall r > 0$, we have*

$$(3.3) \quad \int_{B(x,r)} |u - u_B|^2 \leq 2^{n+2} r^2 \int_{B(x,3r)} |\nabla u|^2.$$

Proof. It suffices to consider $u \in Lip_{loc}(X)$. We have

$$\begin{aligned} \int_{B(x,r)} |u - u_B|^2 &\leq \frac{1}{|B(x,r)|} \int_B \int_B |u(y) - u(z)|^2 dy dz \\ &\leq \frac{4r^2}{|B(x,r)|} \int_B \int_B \int_0^1 |\nabla u(\gamma_{yz}(t))|^2 dt dy dz \\ &= \frac{8r^2}{|B(x,r)|} \int_B \int_B \int_{1/2}^1 |\nabla u(\gamma_{yz}(t))|^2 dt dy dz. \end{aligned}$$

The last equality follows from a change of variable $t' = 1 - t$ on $[0, 1/2]$, by using the symmetry of y and z in the integrals and $\gamma_{yz}(t) = \gamma_{zy}(1 - t)$. That is the crucial trick, which is due to Korevaar and Schoen [11]. Then by Lemma 3.2 we get

$$\begin{aligned} \int_{B(x,r)} |u - u_B|^2 &\leq \frac{8r^2}{|B(x,r)|} \int_{B(x,r)} dy \int_{1/2}^1 dt \int_{B(y,2r)} |\nabla u(\gamma_{yz}(t))|^2 dz \\ &\leq \frac{8r^2}{|B(x,r)|} \int_{B(x,r)} dy \int_{1/2}^1 \frac{1}{t^n} dt \int_{B(y,2tr)} |\nabla u(w)|^2 dw \\ &\leq \frac{8r^2}{|B(x,r)|} \int_{B(x,r)} dy \int_{1/2}^1 \frac{1}{t^n} dt \int_{B(x,3r)} |\nabla u(w)|^2 dw \\ &\leq \frac{2^{n+2} r^2}{|B(x,r)|} \int_{B(x,r)} dy \int_{B(x,3r)} |\nabla u(w)|^2 dw \\ &= 2^{n+2} r^2 \int_{B(x,3r)} |\nabla u|^2, \end{aligned}$$

which proves the uniform weak Poincaré inequality. \square

Proof of Theorem 3.1. By the Whitney-type covering argument (cf. Corollary 5.3.5 in [20]), the uniform weak Poincaré inequality (3.3) is improved to the uniform Poincaré inequality as follows:

$$(3.4) \quad \int_{B(x,r)} |u - u_B|^2 \leq C(n) r^2 \int_{B(x,r)} |\nabla u|^2.$$

\square

Proofs of the main results. In what follows we shall provide the proofs of the main results of the present paper.

Proof of Theorem 1.1. By the uniform Poincaré inequality (3.1) and the volume doubling property (2.3), we use the Nash-Moser iteration to obtain the uniform

Harnack inequality and the Liouville theorem (cf. [6]). The details are included in the Appendix of this paper. \square

In order to prove the finite dimension theorem for polynomial growth harmonic functions, we need the following lemma.

Lemma 3.4. *For any finite dimensional subspace $K \subset H^d(X)$, there exists a constant $R_0(K)$ depending on K , such that for $\forall R \geq R_0$,*

$$\langle u, v \rangle = \int_{B_R} uv$$

is an inner product on K .

Proof. We shall prove the lemma by contradiction. Let $\{u_i\}_{i=1}^k$ be a basis of K . Suppose the lemma is not true. Then there exists a sequence $R_j \rightarrow \infty$ ($j \rightarrow \infty$) and $u_j = \sum_{i=1}^k a_j^i u_i \neq 0$ such that $\int_{B_{R_j}} u_j^2 = 0$. Without loss of generality, we may assume that $\sum_{i=1}^k (a_j^i)^2 = 1$. By the compactness of the unit sphere $S^{k-1} \subset \mathbb{R}^k$, there exists a subsequence a_j^i such that

$$a_j^i \rightarrow b^i \quad (j \rightarrow \infty).$$

Then

$$(3.5) \quad \sum_{i=1}^k (b^i)^2 = 1.$$

Then with $u = \sum_i b^i u_i$, it follows that for any $R > 0$,

$$\int_{B_R} u_j^2 \rightarrow \int_{B_R} u^2;$$

thus $\int_{B_R} u^2 = 0$; $u|_{B_R} \equiv 0$. Since this holds for arbitrary R , $u \equiv 0$ on X , which contradicts (3.5), and hence the lemma follows. \square

Remark 3.5. One key point of Colding-Minicozzi and Li's arguments to prove the finite dimension theorem for polynomial growth harmonic functions is that $\int_{B_R} uv$ is an inner product on K for any R . For fixed R , $\int_{B_R} u^2 = 0$ implies $u|_{B_R} \equiv 0$, but not $u \equiv 0$ on X , since the unique continuation property of harmonic functions is unknown on Alexandrov spaces. However this lemma states that for a fixed subspace K , the integration over B_R is an inner product on K for sufficiently large R . Then by Colding-Minicozzi and Li's arguments, we obtain the uniform dimension estimate, which does not depend on K , and hence the finite dimension theorem follows. In what follows, we shall indicate a proof of the finite dimension theorem by Colding-Minicozzi's argument and provide a detailed proof by Li's simplified argument respectively.

Proof 1 of Theorem 1.2. By Lemma 3.4, the Poincaré inequality (3.1) and the relative volume comparison (2.2), we can use Colding-Minicozzi's argument in [4] without essential modification on Alexandrov spaces with nonnegative Ricci curvature. The dimension estimate is asymptotically optimal, i.e. $\dim H^d(X) \leq C(n)d^{n-1}$. \square

By the Nash-Moser iteration, we obtain the mean value inequality on Alexandrov spaces with generalized nonnegative Ricci curvature.

Theorem 3.6 (Mean Value Inequality; cf. Theorem A.2 in the Appendix). *Let X be an n -dimensional Alexandrov space with nonnegative Ricci curvature. Then there exists a constant $C(n)$, for any harmonic function u on X and $p \in X$ such that*

$$(3.6) \quad |u(p)| \leq C \frac{1}{H^n(B_R(p))} \int_{B_R(p)} |u|.$$

Following Li's mean value inequality argument in [12] or [13], we shall prove the next two lemmas on Alexandrov spaces (cf. Lemma 13.3, Theorem 13.4 in [13]).

Lemma 3.7 (Li). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature and let K be a k -dimensional subspace of $H^d(X)$. For $p \in X, \beta > 1, \delta > 0, R_0 > 0$, there exists $R > R_0$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of K with respect to the inner product $A_{\beta R}(u, v) = \int_{B_{\beta R}(p)} uv$, then*

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 \geq k\beta^{-(2d+n+\delta)}.$$

Proof. For each $R > 0$, let A_R be the bilinear form defined on K given by $A_R(u, v) = \int_{B_R(p)} uv$. By Lemma 3.4, it suffices to consider A_R for $R \geq R_0(K)$ which are inner products on K . Let us denote by $\text{tr}_{R'} R$ ($\det_{R'} R$) the trace (determinant) of the inner product A_R with respect to $A_{R'}$, i.e. the trace (determinant) of the following matrix:

$$(A_R(v_i, v_j))_{i,j=1,\dots,k},$$

where $\{v_i\}_{i=1}^k$ is an orthonormal basis of K with respect to the inner product $A_{R'}$.

Suppose the lemma is not true. Then there exist p, β, δ, R_0 , such that $\forall R \geq R_0$,

$$\text{tr}_{\beta R} A_R = \sum_{i=1}^k \int_{B_R(p)} u_i^2 < k\beta^{-(2d+n+\delta)},$$

where $\{u_i\}_{i=1}^k$ is an orthonormal basis of K with respect to $A_{\beta R}$. On the other hand, the arithmetic-geometric means asserts that

$$(\det_{\beta R} A_R)^{\frac{1}{k}} \leq \frac{\text{tr}_{\beta R} A_R}{k} < \beta^{-(2d+n+\delta)}.$$

Since $\det_{\beta R} A_R = \frac{1}{\det_R A_{\beta R}}$, we have

$$\det_R A_{\beta R} > \beta^{k(2d+n+\delta)}.$$

Replacing R by $R, \beta R, \dots, \beta^j R, \dots$, and noting that $\det_{R_1} A_{R_2} \det_{R_2} A_{R_3} = \det_{R_1} A_{R_3}$, we obtain

$$(3.7) \quad \det_R A_{\beta^j R} > \beta^{jk(2d+n+\delta)}.$$

By the polynomial growth assumptions on K and the volume growth condition,

$$(3.8) \quad \det_R A_{\beta^j R} \leq k! C^k (\beta^j R)^{k(2d+n)}.$$

This contradicts (3.7) as $j \rightarrow \infty$, and hence the lemma is proved. \square

Lemma 3.8 (Li). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature and let K be a k -dimensional subspace of $H^d(X)$. Then*

there exists a constant $C(n)$ such that for any basis of K , $\{u_i\}_{i=1}^k$, $\forall p \in X, R > 0, 0 < \epsilon < \frac{1}{2}$, we have

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 \leq C(n) \epsilon^{-(n-1)} \sup_{u \in \langle A, U \rangle} \int_{B_{(1+\epsilon)R}(p)} u^2,$$

where $\langle A, U \rangle = \{v = \sum_i a_i u_i, \sum_i a_i^2 = 1\}$.

Proof. For fixed $x \in B_R(p)$, we set $K_x = \{u \in K \mid u(x) = 0\}$. The subspace $K_x \subset K$ is of at most codimension 1, since for any $v, w \notin K_x$, $v - \frac{v(x)}{w(x)}w \in K_x$. Then there exists an orthogonal transformation mapping $\{u_i\}_{i=1}^k$ to $\{v_i\}_{i=1}^k$, where $v_i \in K_x$, $i \geq 2$. By the mean value inequality (3.6), we have

$$\begin{aligned} \sum_{i=1}^k u_i^2(x) &= \sum_{i=1}^k v_i^2(x) = v_1^2(x) \leq C \int_{B_{(1+\epsilon)R-r(x)}(x)} v_1^2 \\ (3.9) \quad &\leq C |B_{(1+\epsilon)R-r(x)}(x)|^{-1} \sup_{u \in \langle A, U \rangle} \int_{B_{(1+\epsilon)R}(p)} u^2, \end{aligned}$$

where $r(x) = d(p, x)$. For simplicity, denote $V_p(t) = |B_t(p)|$ and $A_p(t) = |\partial B_t(p)|$.

By the Bishop-Gromov volume comparison (2.2), we have

$$V_x((1+\epsilon)R - r(x)) \geq \left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^n V_x(2R) \geq \left(\frac{(1+\epsilon)R - r(x)}{2R}\right)^n V_p(R).$$

Hence, substituting it into (3.9) and integrating over $B_R(p)$, we have

$$(3.10) \quad \sum_{i=1}^k \int_{B_R(p)} u_i^2 \leq \frac{C2^n}{V_p(R)} \sup_{u \in \langle A, U \rangle} \int_{B_{(1+\epsilon)R}(p)} u^2 \int_{B_R(p)} (1 + \epsilon - R^{-1}r(x))^{-n} dx.$$

Define $f(t) = (1 + \epsilon - R^{-1}t)^{-n}$. Then $f'(t) = \frac{n}{R}(1 + \epsilon - R^{-1}t)^{-(n+1)} \geq 0$ and

$$\int_{B_R(p)} f(r(x)) dx = \int_0^R f(t) A_p(t) dt.$$

Since $A_p(t) = V_p'(t)$ a.e., integrating by parts we obtain

$$\int_0^R f(t) A_p(t) dt = f(t) V_p(t) \Big|_0^R - \int_0^R V_p(t) f'(t) dt.$$

Noting that $f'(t) \geq 0$ and the Bishop-Gromov volume comparison (2.2), we have

$$\begin{aligned} \int_0^R V_p(t) f'(t) dt &\geq \frac{V_p(R)}{R^n} \int_0^R t^n f'(t) dt \\ &= \frac{V_p(R)}{R^n} \{t^n f(t) \Big|_0^R - n \int_0^R t^{(n-1)} f(t) dt\}. \end{aligned}$$

Therefore

$$\int_{B_R(p)} f(r(x)) dx \leq \frac{n V_p(R)}{R^n} \int_0^R t^{(n-1)} f(t) dt \leq \frac{n}{n-1} V_p(R) \epsilon^{-(n-1)}.$$

Combining this with (3.10), we prove the lemma. \square

By the previous two lemmas, we obtain the optimal dimension estimate and prove the finite dimension theorem for polynomial growth harmonic functions.

Proof 2 of Theorem 1.2. For any k -dimensional subspace $K \subset H^d(X)$, we set $\beta = 1 + \epsilon$. Let $\{u_i\}_{i=1}^k$ be an orthonormal basis of K with respect to $A_{\beta R}$. By Lemma 3.7, we have

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 \geq k(1 + \epsilon)^{-(2d+n+\delta)}.$$

Lemma 3.8 implies that

$$\sum_{i=1}^k \int_{B_R(p)} u_i^2 \leq C(n)\epsilon^{-(n-1)}.$$

Setting $\epsilon = \frac{1}{2d}$ and letting δ tend to 0, we have

$$(3.11) \quad k \leq C(n) \left(\frac{1}{2d}\right)^{-(n-1)} \left(1 + \frac{1}{2d}\right)^{(2d+n+\delta)} \leq Cd^{n-1}.$$

Noting that (3.11) holds for arbitrary subspace K , we prove the theorem. \square

APPENDIX

In this Appendix, we outline the proof of the Liouville theorem for harmonic functions by the Nash-Moser iteration on Alexandrov spaces with nonnegative Ricci curvature (see [6] for details). First we derive the Sobolev inequality.

Theorem A.1 (Sobolev Inequality). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. There exists a constant $C = C(n)$ such that $\forall p \in X, \forall r > 0, B := B(p, r), u \in W_0^{1,2}(B)$, we have*

$$(A.1) \quad \left(\int_B u^{2\chi} \right)^{\frac{1}{\chi}} \leq Cr^2 \int_B |\nabla u|^2,$$

where $\chi = \frac{n}{n-2}$ ($n \geq 3$), $\chi > 1$ ($n = 2$) and $\int_B u = \frac{1}{|B|} \int_B u$.

Proof. By means of the standard pseudo-Poincaré technique (see Theorems 5.2.3 and 3.2.9 in Saloff-Coste [20]) and the relative volume comparison (2.2), we obtain the Sobolev inequality; see [6] for details. \square

As long as the Poincaré and Sobolev inequalities are obtained, the standard Nash-Moser iteration can be carried out (cf. [15] or [7]). We shall carry out the estimates in all geodesic balls because the scaling technique is not suitable here.

Theorem A.2. *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. For any subharmonic function u on X and $\forall p > 0, 0 < \theta < \tau \leq 1$, there exists a constant $C = C(n, p, \theta, \tau)$ such that $\forall B_R := B(x, R)$,*

$$(A.2) \quad \sup_{B_{\theta R}} u \leq C \left(\int_{B_{\tau R}} |u|^p \right)^{1/p}.$$

Proof.

Step 1. We prove the theorem for $p \geq 2$. Set $u^+ = \max\{u, 0\}$, $\bar{u} = u^+ + k$ for some $k > 0$, and for some $m > 0$,

$$\bar{u}_m = \begin{cases} \bar{u}, & u < m, \\ k + m, & u \geq m. \end{cases}$$

Note that $\nabla \bar{u}_m = \nabla \bar{u} = \nabla u$ a.e. $0 < u < m$; otherwise $\nabla \bar{u}_m = 0$ a.e.

Set the test function $\varphi = \eta^2(\bar{u}_m^\beta \bar{u} - k^{\beta+1}) \in W_0^{1,2}(B_R)$ for any $\beta \geq 0$, $\eta \in Lip_0(B_R)$ and $\eta \geq 0$. A direct calculation shows that

$$0 \geq \int \nabla u \cdot \nabla \varphi \geq \frac{1}{2} \int \eta^2 \bar{u}_m^\beta |\nabla \bar{u}|^2 + \beta \int \eta^2 \bar{u}_m^\beta |\nabla \bar{u}_m|^2 - 2 \int \bar{u}_m^\beta |\bar{u}|^2 |\nabla \eta|^2, \quad (A.3)$$

$$\frac{1}{2} \int \eta^2 \bar{u}_m^\beta |\nabla \bar{u}|^2 + \beta \int \eta^2 \bar{u}_m^\beta |\nabla \bar{u}_m|^2 \leq 2 \int \bar{u}_m^\beta |\bar{u}|^2 |\nabla \eta|^2.$$

By setting $w = \bar{u}_m^{\frac{\beta}{2}} \bar{u}$, (A.3) implies that

$$\int |\nabla(\eta w)|^2 \leq 18(\beta + 1) \int w^2 |\nabla \eta|^2. \quad (A.4)$$

For given $\theta_0 \leq \theta$, any $\theta_0 \leq a < b \leq 1$, we choose $0 \leq \eta \in Lip_0(B_{bR})$, $\eta|_{B_{aR}} = 1$ and $|\nabla \eta| \leq \frac{1}{(b-a)R}$. The Sobolev inequality (A.1) and the Bishop-Gromov volume comparison $\frac{|B_{bR}|}{|B_{aR}|} \leq \frac{b^n}{a^n} \leq \frac{1}{\theta_0^n}$ imply that

$$\left(\int_{B_{aR}} |w|^{2\chi} \right)^{\frac{1}{\chi}} \leq \frac{C(n, \theta_0)(\beta + 1)}{(b-a)^2} \int_{B_{bR}} w^2.$$

By setting $\gamma = \beta + 2 \geq 2$ and letting $m \rightarrow \infty$, it follows that

$$\left(\int_{B_{aR}} \bar{u}^{\gamma\chi} \right)^{\frac{1}{\gamma\chi}} \leq \left(\frac{C(\gamma-1)}{(b-a)^2} \right)^{\frac{1}{\gamma}} \left(\int_{B_{bR}} \bar{u}^\gamma \right)^{\frac{1}{\gamma}}. \quad (A.5)$$

We start the Moser iteration as follows. Set $r_i = \theta R + \frac{(\tau-\theta)}{2^{i-1}}R$ and $\gamma_i = p\chi^{i-1}$ for $i = 1, 2, \dots$, and $I_i = \left(\int_{B_{r_i}} |\bar{u}|^{\gamma_i} \right)^{\frac{1}{\gamma_i}}$. Then

$$I_{i+1} \leq \left(\frac{C}{(\tau-\theta)^2} \right)^{\sum \frac{1}{\gamma_j}} 4^{\sum \frac{i-1}{\gamma_j}} \prod (\gamma_j - 1)^{\frac{1}{\gamma_j}} I_1. \quad (A.6)$$

Noting that $\sum \frac{1}{\gamma_j} \leq \frac{n}{2p}$, $\sum \frac{i-1}{\gamma_j} \leq C(n, p)$ and

$$\prod (\gamma_j - 1)^{\frac{1}{\gamma_j}} \leq \prod (\gamma_j)^{\frac{1}{\gamma_j}} \leq p^{\sum \frac{1}{\gamma_j}} \chi^{\sum \frac{i-1}{\gamma_j}} \leq C(n, p),$$

and letting $i \rightarrow \infty$, $k \rightarrow 0$ in (A.6), we obtain

$$\sup_{B_{\theta R}} u^+ \leq C(n, p, \theta_0) \left(\frac{1}{(\tau-\theta)^n} \int_{B_{\tau R}} (u^+)^p \right)^{1/p}.$$

Step 2. For the case $p < 2$, noting that by Young's inequality

$$\begin{aligned} \sup_{B_{\theta R}} u^+ &\leq C(n, \theta_0) \left(\frac{1}{(\tau-\theta)^n} \int_{B_{\tau R}} (u^+)^2 \right)^{1/2} \\ &\leq C \frac{1}{(\tau-\theta)^{\frac{n}{2}}} \left(\sup_{B_{\tau R}} u^+ \right)^{1-\frac{p}{2}} \left(\int_{B_{\tau R}} (u^+)^p \right)^{1/2} \\ &\leq \frac{1}{2} \sup_{B_{\tau R}} u^+ + \frac{C(n, p, \theta_0)}{(\tau-\theta)^{\frac{n}{p}}} \left(\int_{B_{\tau R}} (u^+)^p \right)^{1/p}, \end{aligned} \quad (A.7)$$

we prove the theorem by the following lemma (cf. Lemma 4.3 in [7]).

Lemma A.3. *Let f be a nonnegative and bounded function on $[\tau_0, \tau_1]$ with $\tau_0 \geq 0$. Suppose for $\tau_0 \leq t < s \leq \tau_1$ we have*

$$f(t) \leq \theta f(s) + \frac{A}{(s-t)^\alpha} + B$$

for some $\theta \in [0, 1)$. Then for any $\tau_0 \leq t < s \leq \tau_1$,

$$f(t) \leq c(\alpha, \theta) \left\{ \frac{A}{(s-t)^\alpha} + B \right\}. \quad \square$$

Theorem A.4. *Let X be an n -dimensional Alexandrov space with generalized non-negative Ricci curvature. For any nonnegative superharmonic function u on X and $\forall 0 < \theta < \tau < 1$, $0 < p < \frac{n}{n-2}$, there exists a constant $C = C(n, p, \theta, \tau)$ such that $\forall B_R := B(x, R)$,*

$$(A.8) \quad \inf_{B_{\theta R}} u \geq C \left(\int_{B_{\tau R}} |u|^p \right)^{1/p}.$$

Proof.

Step 1. The theorem holds for some $p_0 > 0$. Set $\bar{u} = u + k > 0$, for some $k > 0$. By letting $k \rightarrow 0$, it suffices to prove the theorem for \bar{u} . Set $v = \bar{u}^{-1}$, the test function $\varphi = \frac{\phi}{\bar{u}^2}$ for any $\phi \in W_0^{1,2}(X)$ and $\phi \geq 0$. A direct calculation shows that v is subharmonic. By Theorem A.2, $\forall p > 0$, $0 < \theta < \tau \leq 1$, there exists $C = C(n, p, \theta, \tau)$ such that

$$(A.9) \quad \begin{aligned} \sup_{B_{\theta R}} v &\leq C \left(\int_{B_{\tau R}} |v|^p \right)^{1/p}, \\ \inf_{B_{\theta R}} \bar{u} &\geq C \left(\int_{B_{\tau R}} |\bar{u}|^{-p} \right)^{-\frac{1}{p}} = C \left(\int_{B_{\tau R}} |\bar{u}|^{-p} \int_{B_{\tau R}} |\bar{u}|^p \right)^{-\frac{1}{p}} \left(\int_{B_{\tau R}} |\bar{u}|^p \right)^{1/p}. \end{aligned}$$

It suffices to prove that for some $p_0(n, \tau) > 0$,

$$(A.10) \quad \int_{B_{\tau R}} |\bar{u}|^{-p_0} \int_{B_{\tau R}} |\bar{u}|^{p_0} \leq C(n, \tau).$$

By setting $w := \log \bar{u} - \mu$, where $\mu = \int_{B_{\tau R}} \log \bar{u}$, (A.10) follows from

$$(A.11) \quad \int_{B_{\tau R}} e^{p_0|w|} \leq C(n, \tau).$$

Note that

$$e^{p_0|w|} = 1 + p_0|w| + \frac{(p_0|w|)^2}{2} + \cdots + \frac{(p_0|w|)^\alpha}{\alpha!} + \cdots,$$

where $\alpha \in \mathbb{N}$. Hence it suffices to estimate every term of the expression

$$\int_{B_{\tau R}} \frac{(p_0|w|)^\alpha}{\alpha!}.$$

First we derive the inequality for w . For any $\varphi \in W_0^{1,2}(X)$ and $\varphi \geq 0$, set the test function $\varphi^2 \bar{u}^{-1}$. A direct calculation shows that

$$(A.12) \quad \int \varphi^2 |\nabla w|^2 \leq 4 \int |\nabla \varphi|^2.$$

Choosing $\varphi|_{B_{\tau R}} = 1$, $\text{supp } \varphi \subset B_R$ and $|\nabla \varphi| \leq \frac{1}{(1-\tau)R}$, we obtain

$$\int_{B_{\tau R}} |\nabla w|^2 \leq \frac{4}{(1-\tau)^2 R^2} |B_R|.$$

Noting that $\int_{B_{\tau R}} w = 0$, we get by the Poincaré inequality that

$$(A.13) \quad \int_{B_{\tau R}} |w|^2 \leq C(n)\tau^2 R^2 \int_{B_{\tau R}} |\nabla w|^2 \leq C(n, \tau) \frac{|B_R|}{|B_{\tau R}|} \leq C \frac{1}{\tau^n} \leq C(n, \tau),$$

which is the required estimate for $\alpha = 2$.

Claim A.5. For any $\tau' \in (\tau, 1)$ we have

$$(A.14) \quad \int_{B_{\tau' R}} |w|^2 \leq C(n, \tau, \tau').$$

Proof of Claim A.5. Choosing $\varphi|_{B_{\tau' R}} = 1$ and $\text{supp} \varphi \subset B_R$, we have

$$\int_{B_{\tau' R}} |\nabla w|^2 \leq \frac{4}{(1 - \tau')^2 R^2} |B_R|.$$

The Poincaré inequality yields

$$\int_{B_{\tau' R}} |w - w_{B_{\tau' R}}|^2 \leq C(n)(\tau' R)^2 \int_{B_{\tau' R}} |\nabla w|^2 \leq C(n, \tau').$$

Hence noting that $\int_{B_{\tau R}} w = 0$, we obtain

$$(A.15) \quad \begin{aligned} \int_{B_{\tau' R}} |w|^2 &\leq C(\epsilon) \int_{B_{\tau' R}} |w - w_{B_{\tau' R}}|^2 + (1 + \epsilon) |w_{B_{\tau' R}}|^2 \\ &\leq C(n, \tau', \epsilon) + (1 + \epsilon) \frac{|B_{\tau' R} \setminus B_{\tau R}|}{|B_{\tau' R}|} \int_{B_{\tau' R}} w^2. \end{aligned}$$

By the Bishop-Gromov volume comparison (2.2), it follows that

$$(A.16) \quad \frac{|B_{\tau' R} \setminus B_{\tau R}|}{|B_{\tau' R}|} = 1 - \frac{|B_{\tau R}|}{|B_{\tau' R}|} \leq 1 - \left(\frac{\tau}{\tau'}\right)^n.$$

By choosing $\epsilon = \epsilon(\tau, \tau')$ such that $(1 + \epsilon)(1 - (\frac{\tau}{\tau'})^n) < 1$, the claim follows from (A.15) and (A.16).

Next we turn to estimate $\int_{B_{\tau R}} |w|^\alpha$ for any $\alpha \geq 2$. Set the test function $\varphi = \zeta^2 |w_m|^{2\beta} \bar{u}^{-1}$, where $\beta \geq 1$, $\zeta \in Lip_0(X)$ and $\zeta \geq 0$,

$$w_m = \begin{cases} m, & w > m, \\ w, & |w| \leq m, \\ -m, & w < -m. \end{cases}$$

By Young's and the Hölder inequalities, a direct calculation shows that

$$\int |\nabla w|^2 \zeta^2 |w_m|^{2\beta} \leq 16\beta^2 \int |w_m|^{2\beta} |\nabla \zeta|^2 + 2(2\beta)^{2\beta} \int \zeta^2 |\nabla w_m|^2.$$

By letting $m \rightarrow \infty$, Young's inequality and (A.12) imply that

$$(A.17) \quad \int |\nabla(\zeta |w|^\beta)|^2 \leq 128 \left\{ (2\beta)^{2\beta} \int |\nabla \zeta|^2 + \beta^2 \int |w|^{2\beta} |\nabla \zeta|^2 \right\}.$$

For any $\tau \leq a < b \leq 1$, choose $\zeta|_{B_{aR}} = 1$, $\text{supp} \zeta \subset B_{bR}$ and $|\nabla \zeta| \leq \frac{1}{(b-a)R}$. It follows from the Sobolev inequality (A.1) that

$$\left(\int_{B_{aR}} |w|^{2\beta X} \right)^{\frac{1}{X}} \leq \frac{C(n, \tau)(2\beta)^2}{(b-a)^2} \left\{ (2\beta)^{2\beta} + \int_{B_{bR}} |w|^{2\beta} \right\}.$$

Now we start the Moser iteration as follows. Setting $\beta_i = 2\chi^{i-1}$, $r_i = (\tau + \frac{1-\tau}{2^i})R$ and $I_i = (\int_{B_{r_i}} |w|^{\beta_i})^{\frac{1}{\beta_i}}$ for $i = 1, 2, \dots$, we have

$$I_{i+1} \leq C^{\sum \frac{i}{\beta_i}} \left(\prod \beta_i^{\frac{1}{\beta_i}} \right)^2 \left(\sum_{j=1}^i \beta_j + I_1 \right).$$

Noting that $\sum \frac{i}{\beta_i} < C$, $\prod \beta_i^{\frac{1}{\beta_i}} < C$ and $\sum_{j=1}^i \beta_j < C\beta_{i+1}$, we obtain

$$I_{i+1} \leq C(n, \tau)(\beta_{i+1} + I_1).$$

For any integer $\alpha \geq 2$ (the estimate of $\alpha = 0, 1$ is trivial), there exists $i \geq 1$, such that $\beta_i \leq \alpha < \beta_{i+1}$. Then by the Hölder inequality and the Bishop-Gromov volume comparison, this implies that

$$\begin{aligned} \left(\int_{B_{\tau R}} |w|^\alpha \right)^{\frac{1}{\alpha}} &\leq \left(\int_{B_{\tau R}} |w|^{\beta_{i+1}} \right)^{\frac{1}{\beta_{i+1}}} \leq C(\beta_{i+1} + I_1) \\ (A.18) \quad &\leq C(\alpha + I_1) \leq C_0(n, \tau)\alpha. \end{aligned}$$

The last step follows from $I_1 = (\int_{B_{\tau' R}} |w|^2)^{\frac{1}{2}} \leq C(n, \tau)$ for $\tau' = \frac{1+\tau}{2}$.

Hence, for any $\alpha \geq 2$, Sterling's formula implies that

$$\int_{B_{\tau R}} \frac{(p_0 |w|)^\alpha}{\alpha!} \leq \frac{(p_0 C_0 \alpha)^\alpha}{\alpha!} \leq (p_0 C_0 e)^\alpha.$$

Choosing $p_0 = (2C_0 e)^{-1}$, we draw the conclusion that

$$\int_{B_{\tau R}} e^{p_0 |w|} \leq C \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) \leq C(n, \tau).$$

Step 2. In order to prove the theorem for any $0 < p < \frac{n}{n-2}$, by iteration it suffices to prove the following claim.

Claim A.6. For any $\theta \leq l_1 < l_2 < 1$, and $0 < p_2 < p_1 < \frac{n}{n-2}$, there exists a constant $C = C(n, l_1, l_2, p_1, p_2, \theta)$ such that

$$(A.19) \quad \left(\int_{B_{l_1 R}} \bar{u}^{p_1} \right)^{\frac{1}{p_1}} \leq C \left(\int_{B_{l_2 R}} \bar{u}^{p_2} \right)^{\frac{1}{p_2}}.$$

Proof of Claim A.6. By setting the test function $\varphi = \bar{u}^{-\beta} \zeta^2$ for $\beta \in (0, 1)$, a direct calculation shows that

$$\int \bar{u}^{-\beta-1} \zeta^2 |\nabla \bar{u}|^2 \leq \frac{4}{\beta^2} \int \bar{u}^{1-\beta} |\nabla \zeta|^2.$$

Let $\gamma = 1 - \beta$, $w = \bar{u}^{\frac{\gamma}{2}}$. Then it follows that

$$\int |\nabla(\zeta w)|^2 \leq \frac{4}{(1-\gamma)^2} \int w^2 |\nabla \zeta|^2.$$

For any $\theta \leq a < b < 1$, by choosing $\zeta|_{B_{aR}} = 1$, $\text{supp} \zeta \subset B_{bR}$ and $|\nabla \zeta| \leq \frac{1}{(b-a)R}$, the Sobolev inequality (A.1) yields, for any $\gamma \in (0, 1)$,

$$(A.20) \quad \left(\int_{B_{aR}} \bar{u}^{\gamma \chi} \right)^{\frac{1}{\gamma \chi}} \leq \left(\frac{C(n, \theta)}{(1-\gamma)^2 (b-a)^2} \right)^{\frac{1}{\gamma}} \left(\int_{B_{bR}} \bar{u}^\gamma \right)^{\frac{1}{\gamma}}.$$

We start the Moser iteration as follows. Setting $\gamma_i = l_1 + \frac{l_2 - l_1}{2^{i-1}}$, $r_i = p_2 \chi^{i-1}$ and $I_i = \left(\int_{B_{r_i R}} \bar{u}^{\gamma_i} \right)^{\frac{1}{\gamma_i}}$, for $i = 1, 2, \dots$, we have

$$I_{i+1} \leq \frac{C^{\sum_{j=1}^i \frac{j}{\gamma_j}}}{(l_2 - l_1)^{2 \sum_{j=1}^i \frac{1}{\gamma_j}} \left(\prod_{j=1}^i (1 - \gamma_j)^{\frac{1}{\gamma_j}} \right)^2} I_1.$$

For $0 < p_2 < p_1 < \frac{n}{n-2}$, we may assume $p_2 < 1$. There exists $i \geq 1$ such that $\gamma_i \leq p_1 < \gamma_{i+1}$. Hence by the Hölder inequality and the Bishop-Gromov volume comparison, we have

$$\left(\int_{B_{l_1 R}} \bar{u}^{p_1} \right)^{\frac{1}{p_1}} \leq C I_{i+1} \leq \frac{C^{\sum_{j=1}^i \frac{j}{\gamma_j}}}{(l_2 - l_1)^{2 \sum_{j=1}^i \frac{1}{\gamma_j}} \left(\prod_{j=1}^i (1 - \gamma_j)^{\frac{1}{\gamma_j}} \right)^2} I_1.$$

Noting that $\sum_{j=1}^i \frac{j}{\gamma_j} < C(p_1, p_2)$, $\prod_{j=1}^i (1 - \gamma_j)^{\frac{1}{\gamma_j}} \geq \prod_{j=1}^i (1 - p_1)^{\frac{1}{\gamma_j}} \geq (1 - p_1)^{\frac{n}{2p_2}}$, we prove the claim

$$\left(\int_{B_{l_1 R}} \bar{u}^{p_1} \right)^{\frac{1}{p_1}} \leq C(n, l_1, l_2, p_1, p_2, \theta) \left(\int_{B_{l_2 R}} \bar{u}^{p_2} \right)^{\frac{1}{p_2}}.$$

Our conclusion follows directly from Step 1 and Step 2. \square

Now Theorems A.2 and A.4 imply the uniform Harnack inequality.

Theorem A.7 (Harnack Inequality). *Let X be an n -dimensional Alexandrov space with generalized nonnegative Ricci curvature. For any nonnegative harmonic function u on X and $r > 0$, there exists $C = C(n)$ such that*

$$(A.21) \quad \sup_{B_r} u \leq C \inf_{B_r} u.$$

Proof. The theorem follows by choosing $\theta = \frac{1}{4}$, $\tau = \frac{1}{2}$ and $p = 1$. \square

It follows from Saloff-Coste [20] that the uniform Harnack inequality (A.21) implies the Liouville Theorem.

Proof of Theorem 1.1. For $u \geq 0$, we know $\inf_X u \geq 0$. Applying the uniform Harnack inequality to $(u - \inf_X u)$, we have

$$\sup_{B_r} (u - \inf_X u) \leq C \inf_{B_r} (u - \inf_X u)$$

for any $r > 0$, and C does not depend on r . By letting $r \rightarrow \infty$, we observe that the right-hand side of the inequality tends to zero. Hence $u = \inf_X u = \text{constant}$. \square

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