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A ONE-PARAMETER FAMILY OF PICK FUNCTIONS DEFINED BY THE GAMMA FUNCTION AND RELATED TO THE VOLUME OF THE UNIT BALL IN n-SPACE

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Abstract. We show that

$$F_a(x) = \frac{\ln \Gamma(x+1)}{x \ln(ax)}$$

can be considered as a Pick function when $a \geq 1$, i.e. extends to a holomorphic function mapping the upper half-plane into itself. We also consider the function

$$f(x) = \left(\frac{\pi^{x/2}}{\Gamma(1+x/2)} \right)^{1/(x \ln x)}$$

and show that $\ln f(x+1)$ is a Stieltjes function and that $f(x+1)$ is completely monotonic on $]0, \infty[$. In particular, $f(n) = \Omega_n^{1/(n \ln n)}, n \geq 2$, is a Hausdorff moment sequence. Here Ω_n is the volume of the unit ball in Euclidean n-space.

1. Introduction and results

Since the appearance of the paper [3], monotonicity properties of the functions

$$F_a(x) = \frac{\ln \Gamma(x+1)}{x \ln(ax)}, \quad x > 0, a > 0, \tag{1}$$

have attracted the attention of several authors in connection with monotonicity properties of the sequence $\{\Omega_n\}$ of volumes of the unit ball in Euclidean n-space. Recent papers about inequalities involving Ω_n are [2], [15], [18]. See also the survey paper [4].

Let us first consider the case $a = 1$. In [10] the authors proved that F_1 is a *Bernstein function*, which means that it is positive and has a completely monotonic derivative, i.e.,

$$(-1)^n F_1^{(n+1)}(x) \geq 0, \quad x > 0, n \geq 0. \tag{2}$$

This extended monotonicity and concavity are proved in [5] and [13] respectively.

We actually proved a stronger statement than (2), namely that the reciprocal function $x \ln x / \ln \Gamma(x+1)$ is a Stieltjes transform, i.e. belongs to the Stieltjes cone

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$\mathcal{S}$ of functions of the form

$$g(x) = c + \int_0^\infty \frac{d\mu(t)}{x+t}, \quad x > 0, \tag{3}$$

where $c \geq 0$ and μ is a non-negative measure on $[0, \infty[$ satisfying

$$\int_0^\infty \frac{d\mu(t)}{1+t} < \infty.$$

The result was obtained using the holomorphic extension of the function F_1 to the cut plane $\mathcal{A} = \mathbb{C} \setminus]-\infty, 0]$, leading to an explicit formula for the measure μ in (3). Our derivation used the fact that the holomorphic function $\log \Gamma(z)$ vanishes in $\mathcal{A}$ only at the points $z = 1$ and $z = 2$, a result interesting in itself and included as an appendix in [10]. A simpler proof of the non-vanishing of $\log \Gamma(z)$ appeared in [11], where we also proved that F_1 is a Pick function and obtained the following representation formula:

$$F_1(z) = 1 - \int_0^\infty \frac{d_1(t)}{z+t} \, dt, \quad z \in \mathcal{A}, \tag{4}$$

with

$$d_1(t) = \frac{\ln |\Gamma(1-t)| + (k-1) \ln t}{t((\ln t)^2 + \pi^2)} \quad \text{for} \quad t \in]k-1, k[, \quad k = 1, 2, \ldots, \tag{5}$$

and $d_1(t)$ tends to infinity when t approaches $1, 2, \ldots$. Since $d_1(t) > 0$ for $t > 0$, (2) is an immediate consequence of (4).

We recall that a Pick function is a holomorphic function φ in the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ satisfying $\Im \varphi(z) \geq 0$ for $z \in \mathbb{H}$; cf. [12].

For $a = 2$ Anderson and Qiu proved in [5] that F_2 is strictly increasing on $[1, \infty[$, thereby proving a conjecture from [3]. Alzer proved in [2] that F_2 is concave on $[46, \infty[$. In [16] the concavity was extended to the optimal interval $]\frac{1}{2}, \infty[$.

We will now describe the main results of the present paper.

First a few words about notation. We use ln for the natural logarithm but only applied to positive numbers. The holomorphic extension of ln from the open half-line $]0, \infty[$ to the cut plane $\mathcal{A} = \mathbb{C} \setminus]-\infty, 0]$ is denoted $\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$, where $-\pi < \operatorname{Arg} z < \pi$ is the principal argument. The holomorphic branch of the logarithm of $\Gamma(z)$ for z in the simply connected domain $\mathcal{A}$, equal to $\ln \Gamma(x)$ for $x > 0$, is denoted $\log \Gamma(z)$. This branch can also be defined as the integral

$$\log \Gamma(z) = \int_1^z \frac{\Gamma'(w)}{\Gamma(w)} \, dw, \quad z \in \mathcal{A},$$

where the integration is along the segment from 1 to z. The imaginary part of $\log \Gamma(z)$ is a continuous branch of argument of $\Gamma(z)$ which we denote $\arg \Gamma(z)$, i.e.,

$$\log \Gamma(z) = \ln |\Gamma(z)| + i \arg \Gamma(z), \; z \in \mathcal{A}.$$

The expression

$$F_a(z) = \frac{\log \Gamma(z+1)}{z \operatorname{Log}(az)} \tag{6}$$

clearly defines a holomorphic extension of (1) to $\mathcal{A} \setminus \{1/a\}$, and $z = 1/a$ is a simple pole unless $a = 1$, where the residue $\ln \Gamma(1 + 1/a)$ vanishes. Thus $z = 1$ is a removable singularity for F_1.

Using the residue theorem we shall obtain:

Theorem 1.1. *For $a > 0$ the function F_a has the integral representation*

$$F_a(z) = 1 + \frac{\ln\Gamma(1+1/a)}{z-1/a} - \int_0^\infty \frac{d_a(t)}{z+t}\,dt, \quad z \in \mathcal{A}\setminus\{1/a\}, \tag{7}$$

where

$$d_a(t) = \frac{\ln|\Gamma(1-t)| + (k-1)\ln(at)}{t((\ln(at))^2+\pi^2)} \quad \text{for} \quad t \in \,]k-1,k[\,, \quad k=1,2,\ldots, \tag{8}$$

and $d_a(0) = 0, d_a(k) = \infty, k = 1, 2, \ldots$. We have $d_a(t) \geq 0$ for $t \geq 0, a \geq 1/2$. (This is slightly improved in Remark 2.6 below.) Furthermore, F_a is a Pick function for $a \geq 1$ but not for $0 < a < 1$.

From this follows the monotonicity property conjectured in [16]:

Corollary 1.2. *Assume $a \geq 1$. Then*

$$(-1)^n F_a^{(n+1)}(x) > 0, \quad x > 1/a, n = 0, 1, \ldots. \tag{9}$$

In particular, F_a is strictly increasing and strictly concave on the interval $]1/a, \infty[$.

The function

$$f(x) = \left(\frac{\pi^{x/2}}{\Gamma(1+x/2)}\right)^{1/(x\ln x)} \tag{10}$$

has been studied because the volume Ω_n of the unit ball in $\mathbb{R}^n$ is

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}, n = 1, 2, \ldots.$$

We prove the following integral representation of the extension of $\ln f(x+1)$ to the cut plane $\mathcal{A}$.

Theorem 1.3. *For $z \in \mathcal{A}$ we have*

$$\log f(z+1) = -\frac{1}{2} + \frac{\ln(2/\sqrt{\pi})}{z} + \frac{\ln(\sqrt{\pi})}{\mathrm{Log}(z+1)} + \frac{1}{2}\int_1^\infty \frac{d_2((t-1)/2)}{z+t}\,dt. \tag{11}$$

In particular, $1/2 + \ln f(x+1)$ is a Stieltjes function and $f(x+1)$ is completely monotonic.

We recall that completely monotonic functions $\varphi :]0, \infty[\to \mathbb{R}$ are characterized by Bernstein's theorem as

$$\varphi(x) = \int_0^\infty e^{-xt}\,d\mu(t), \tag{12}$$

where μ is a positive measure on $[0, \infty[$ such that the integrals above make sense for all $x > 0$.

We also recall that a sequence $\{a_n\}_{n\geq 0}$ of positive numbers is called a Hausdorff moment sequence if it has the form

$$a_n = \int_0^1 x^n\,d\sigma(x),\ n \geq 0, \tag{13}$$

where σ is a positive measure on the unit interval. Note that $\lim_{n\to\infty} a_n = \sigma(\{1\})$. Hausdorff proved that these sequences are exactly the same as completely monotonic sequences; see [20, p. 108] or [8, p. 134]. It is easy to see that if φ is completely monotonic with the integral representation (12), then $a_n = \varphi(n+1), n \geq 0$, is a Hausdorff moment sequence, because

$$a_n = \int_0^\infty e^{-(n+1)t}\, d\mu(t) = \int_0^1 x^n\, d\sigma(x),$$

where σ is the image measure of $e^{-t}\, d\mu(t)$ under e^{-t}. Since $\lim_{x\to\infty} f(x+1) = e^{-1/2}$ we get

Corollary 1.4. *The sequence*

$$f(n+2) = \Omega_{n+2}^{1/((n+2)\ln(n+2))}, n = 0, 1, \ldots, \tag{14}$$

is a Hausdorff moment sequence tending to $e^{-1/2}$.

A Hausdorff moment sequence $\{a_n\}_{n\geq 0}$ is easily seen to be decreasing and convex, and it is even logarithmically convex, meaning that $a_n^2 \leq a_{n-1}a_{n+1},\ n \geq 1$. In fact, by the Cauchy-Schwarz inequality

$$\begin{aligned} a_n^2 &= \left(\int_0^1 x^n\, d\sigma(x)\right)^2 \\ &= \left(\int_0^1 x^{(n-1)/2} x^{(n+1)/2}\, d\sigma(x)\right)^2 \leq \int_0^1 x^{n-1}\, d\sigma(x) \int_0^1 x^{n+1}\, d\sigma(x) = a_{n-1}a_{n+1}. \end{aligned}$$

The logarithmic convexity of $\{a_n\}_{n\geq 0}$ was obtained in [16] in a different way.

2. Properties of the function F_a

In this section we will study the holomorphic extension (6) of the function F_a defined in (1). We shall use the following property of $\log \Gamma(z)$; cf. [10, Lemma 2.1].

Lemma 2.1. *We have, for any* $k \geq 1$,

$$\lim_{z\to t, \Im z>0} \log\Gamma(z) = \ln|\Gamma(t)| - i\pi k$$

for $t \in]-k, -k+1[$ *and*

$$\lim_{z\to t, \Im z>0} |\log\Gamma(z)| = \infty$$

for $t = 0, -1, -2, \ldots$.

Lemma 2.2. *For* $a > 0$ *and* $t \leq 0$ *we have*

$$\lim_{y\to 0^+} \Im F_a(t+iy) = \pi d_a(-t), \tag{15}$$

where d_a *is given by* (8).

Proof. For $-1 < t < 0$ we get

$$\lim_{y\to 0^+} F_a(t+iy) = \frac{\ln\Gamma(1+t)}{t(\ln(a|t|)+i\pi)};$$

hence $\lim_{y\to 0^+} \Im F_a(t+iy) = \pi d_a(-t)$. For $-k < t < -k+1$, $k = 2, 3, \ldots$, we find, using Lemma 2.1, that

$$\lim_{y\to 0^+} F_a(t+iy) = \frac{\ln|\Gamma(1+t)| - i(k-1)\pi}{t(\ln(a|t|)+i\pi)}.$$

Hence $\lim_{y\to 0^+} \Im F_a(t+iy) = \pi d_a(-t)$ also in this case.

For $t = -k,\ k = 1, 2, \ldots$, we have

$$|F_a(-k+iy)| \geq \frac{|\ln|\Gamma(-k+1+iy)||}{|-k+iy||\operatorname{Log}(a(-k+iy))|} \to \infty$$

for $y \to 0^+$ because $\Gamma(z)$ has poles at $z = 0, -1, \ldots$. Finally, for $t = 0$ we get (15) from the next lemma. □

Lemma 2.3. *For $a > 0$ we have*

$$\lim_{z\to 0, z\in\mathcal{A}} |F_a(z)| = 0.$$

Proof. Since $\log\Gamma(z+1)/z$ has a removable singularity for $z = 0$, the result follows because $|\operatorname{Log}(az)| \geq |\ln(a|z|)| \to \infty$ for $|z| \to 0, z \in \mathcal{A}$. □

Lemma 2.4. *For $a > 0$ we have the radial behaviour*

$$\lim_{r\to\infty} F_a(re^{i\theta}) = 1 \textit{ for } -\pi < \theta < \pi, \tag{16}$$

and there exists a constant $C_a > 0$ such that for $k = 1, 2, \ldots$ and $-\pi < \theta < \pi$,

$$|F_a((k+\tfrac{1}{2})e^{i\theta})| \leq C_a. \tag{17}$$

Proof. We first note that

$$F_a(z) = F_1(z)\frac{\operatorname{Log}(z)}{\operatorname{Log}(az)}, \tag{18}$$

and since

$$\lim_{|z|\to\infty, z\in\mathcal{A}} \frac{\operatorname{Log}(z)}{\operatorname{Log}(az)} = 1,$$

it is enough to prove the results for $a = 1$. We do this by using a method introduced in [10, Prop. 2.4].

Define

$$R_k = \{z = x+iy \in \mathbb{C} \mid -k \leq x < -k+1,\ 0 < y \leq 1\} \text{ for } k \in \mathbb{Z}$$

and

$$R = \bigcup_{k=0}^{\infty} R_k, \quad S = \{z = x+iy \in \mathbb{C} \mid x \leq 1, |y| \leq 1\}.$$

The function F_1 is continuous on the punctured circle $|z| = (k+\frac{1}{2})e^{i\theta}, \theta \in]-\pi, \pi[$, and by Lemma 2.1 it has limits for $\theta \to \pm\pi$. These limits are complex conjugate of each other, and therefore $|F_1|$ has a continuous extension to the circle $|z| = k + \frac{1}{2}$. Hence

$$M_k = \sup_{|\theta|<\pi} |F_1((k+\tfrac{1}{2})e^{i\theta})| < \infty \tag{19}$$

for each $k = 1, 2, \ldots$. It is then enough to prove that M_k is bounded for $k \to \infty$.

Stieltjes ([19, formula 20]) found the following formula for $\log\Gamma(z)$ for z in the cut plane $\mathcal{A}$:

$$\log\Gamma(z+1) = \ln\sqrt{2\pi} + (z+1/2)\operatorname{Log} z - z + \mu(z). \tag{20}$$

Here

$$\mu(z) = \sum_{n=0}^{\infty} h(z+n) = \int_0^{\infty} \frac{P(t)}{z+t}dt,$$

where $h(z) = (z + 1/2)\,\mathrm{Log}(1 + 1/z) - 1$ and P is periodic with period 1 and $P(t) = 1/2 - t$ for $t \in [0, 1[$. A derivation of these formulas can also be found in [6]. The integral above is improper, and integration by parts yields

$$\mu(z) = \frac{1}{2}\int_0^\infty \frac{Q(t)}{(z+t)^2}\,dt, \tag{21}$$

where Q is periodic with period 1 and $Q(t) = t - t^2$ for $t \in [0, 1[$. Note that by (21) μ is a completely monotonic function. For further properties of Binet's function μ; see [14].

We claim that

$$|\mu(z)| \le \frac{\pi}{8} \text{ for } z \in \mathcal{A} \setminus S.$$

In fact, since $0 \le Q(t) \le 1/4$, we get for $z = x + iy \in \mathcal{A}$,

$$|\mu(z)| \le \frac{1}{8}\int_0^\infty \frac{dt}{(t+x)^2 + y^2}.$$

For $x > 1$ we have

$$\int_0^\infty \frac{dt}{(t+x)^2 + y^2} \le \int_0^\infty \frac{dt}{(t+1)^2} = 1,$$

and for $x \le 1, |y| \ge 1$ we have

$$\int_0^\infty \frac{dt}{(t+x)^2 + y^2} = \int_x^\infty \frac{dt}{t^2 + y^2} < \int_{-\infty}^\infty \frac{dt}{t^2+1} = \pi.$$

Since

$$F_1(z) = 1 + \frac{\ln\sqrt{2\pi} + 1/2\,\mathrm{Log}\,z - z + \mu(z)}{z\,\mathrm{Log}\,z},$$

for $z \in \mathcal{A}$, we immediately get (16) and

$$|F_1(z)| \le 2 \tag{22}$$

for all $z \in \mathcal{A} \setminus S$ for which $|z|$ is sufficiently large. In particular, there exists $N_0 \in \mathbb{N}$ such that

$$|F_1((k+\tfrac{1}{2})e^{i\theta})| \le 2 \text{ for } k \ge N_0,\ (k+\tfrac{1}{2})e^{i\theta} \in \mathcal{A} \setminus S. \tag{23}$$

By continuity the quantity

$$c = \sup\left\{|\log\Gamma(z)| \;\mid\; z = x + iy, \tfrac{1}{2} \le x \le 1, 0 \le y \le 1\right\} \tag{24}$$

is finite.

We will now estimate the quantity $|F_1((k+\frac{1}{2})e^{i\theta})|$ when $(k+\frac{1}{2})e^{i\theta} \in S$, and since $F_1(\overline{z}) = \overline{F_1(z)}$, it is enough to consider the case when $(k+\frac{1}{2})e^{i\theta} \in R_{k+1}$. To do this we use the relation

$$\log\Gamma(z+1) = \log\Gamma(z+k+1) - \sum_{l=1}^{k} \mathrm{Log}(z+l) \tag{25}$$

for $z \in \mathcal{A}$ and $k \in \mathbb{N}$. Equation (25) follows from the fact that the functions on both sides of the equality sign are holomorphic functions in $\mathcal{A}$, and they agree on the positive half-line by repeated applications of the functional equation for the Gamma function.

For $z = (k + \frac{1}{2})e^{i\theta} \in R_{k+1}$ we get $|\log\Gamma(z + k + 1)| \le c$ by (24), and hence by (25)

$$|\log\Gamma(z+1)| \le c + \sum_{l=1}^{k} |\operatorname{Log}(z+l)| \le c + k\pi + \sum_{l=1}^{k} |\ln|z+l||.$$

For $l = 1, \ldots, k-1$ we have $k-l < |z+l| < k+2-l$; hence $0 < \ln|z+l| < \ln(k+2-l)$. Furthermore, $1/2 \le |z+k| \le \sqrt{2}$; hence $-\ln 2 < \ln|z+k| \le (\ln 2)/2$. Inserting these inequalities, we get

$$|\log\Gamma(z+1)| \le c + k\pi + \sum_{j=2}^{k+1} \ln j < c + k\pi + k\ln(k+1).$$

From this we get for $z = (k + \frac{1}{2})e^{i\theta} \in R_{k+1}$

$$|F_1(z)| \le \frac{c + k\pi + k\ln(k+1)}{(k+\frac{1}{2})\ln(k+\frac{1}{2})}, \tag{26}$$

which tends to 1 for $k \to \infty$. Combined with (23) we see that there exists $N_1 \in \mathbb{N}$ such that

$$|F_1((k+\tfrac{1}{2})e^{i\theta})| \le 2 \text{ for } k \ge N_1,\ -\pi < \theta < \pi,$$

which shows that M_k from (19) is a bounded sequence. □

Lemma 2.5. *Let $a > 0$. For $k = 1, 2, \ldots$ there exists an integrable function $f_{k,a} :]-k, -k+1[\to [0, \infty]$ such that*

$$|F_a(x+iy)| \le f_{k,a}(x) \textit{ for } -k < x < -k+1, 0 < y \le 1. \tag{27}$$

Proof. For $z = x + iy$ as above we get using (25)

$$|\log\Gamma(z+1)| \le |\log\Gamma(z+k+1)| + \sum_{l=1}^{k} |\operatorname{Log}(z+l)| \le L + k\pi + \sum_{l=1}^{k} |\ln|z+l||,$$

where L is the maximum of $|\log\Gamma(z)|$ for $z \in \overline{R_{-1}}$. We only treat the case $k \ge 2$ because the case $k = 1$ is a simple modification combined with Lemma 2.3.

For $l = 1, \ldots, k-2$ we have $1 < |z+l| < 1 + k - l$, and for $l = k-1$, k we have $\ln|x+l| \le \ln|z+l| \le (1/2)\ln 2$, so we find

$$|\log\Gamma(z+1)| \le L + k\pi + \sum_{j=2}^{k} \ln j + |\ln|x+k-1|| + |\ln|x+k||, \tag{28}$$

so as $f_{k,1}$ we can use the right-hand side of (28) divided by $(k-1)\ln(k-1)$. Using (18) we next define

$$f_{k,a}(x) = f_{k,1}(x) \max_{z \in \overline{R_k}} \frac{|\operatorname{Log} z|}{|\operatorname{Log}(az)|}.$$ □

Proof of Theorem 1.1. For fixed $w \in \mathcal{A} \setminus \{1/a\}$ we choose $\varepsilon > 0, k \in \mathbb{N}$ such that $\varepsilon < |w|, 1/a < k + \frac{1}{2}$ and consider the positively oriented contour $\gamma(k, \varepsilon)$ in $\mathcal{A}$ consisting of the half-circle $z = \varepsilon e^{i\theta}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and the half-lines $z = x \pm i\varepsilon, x \le 0$ until they cut the circle $|z| = k + \frac{1}{2}$, which closes the contour. By the residue theorem we find that

$$\frac{1}{2\pi i}\int_{\gamma(k,\varepsilon)} \frac{F_a(z)}{z-w}\,dz = F_a(w) + \frac{\ln\Gamma(1+1/a)}{1/a - w}.$$

We now let $\varepsilon \to 0$ in the contour integration. By Lemma 2.3 the contribution from the half-circle with radius ε will tend to zero, and by Lemma 2.2 and Lemma 2.5 we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{F_a((k+\frac{1}{2})e^{i\theta})}{(k+\frac{1}{2})e^{i\theta}-w}(k+\tfrac{1}{2})e^{i\theta}\,d\theta+\int_{-k-\frac{1}{2}}^{0}\frac{d_a(-t)}{t-w}\,dt=F_a(w)+\frac{\ln\Gamma(1+1/a)}{1/a-w}.$$

For $k\to\infty$ the integrand in the first integral converges to 1 for each $\theta\in\,]-\pi,\pi[$ and by Lemma 2.4 Lebesgue's theorem on dominated convergence can be applied, so we finally get

$$F_a(w)=1+\frac{\ln\Gamma(1+1/a)}{w-1/a}-\int_0^\infty\frac{d_a(t)}{t+w}\,dt.$$

The last integral above appears as an improper integral, but we shall see that the integrand is Lebesgue integrable. We show below that $d_a(t)\geq 0$ when $a\geq 1/2$, and for these values of a the integrability is obvious. The function d_a tends to 0 for $t\to 0$ and has a logarithmic singularity at $t=1$, so d_a is integrable over $]0,1[$. For $k-1<t<k,\ k\geq 2$ we have

$$d_a(t)=\frac{(\ln(t))^2+\pi^2}{(\ln(at))^2+\pi^2}d_1(t)+\frac{(k-1)\ln a}{t\left((\ln(at))^2+\pi^2\right)},\tag{29}$$

and the factor in front of $d_1(t)$ is a bounded continuous function with limit 1 at 0 and at infinity. Therefore

$$\int_1^\infty\frac{|d_a(t)|}{t}\,dt<\infty$$

follows from the finiteness of the corresponding integral for $a=1$ provided that we establish

$$K:=\sum_{k=2}^{\infty}(k-1)\int_{k-1}^{k}\frac{dt}{t^2\left((\ln(at))^2+\pi^2\right)}<\infty.$$

Choosing $N\in\mathbb{N}$ such that $aN>1$, we can estimate

$$\begin{aligned}K&<\sum_{k=1}^{\infty}\int_{ka}^{(k+1)a}\frac{dt}{t(\ln^2(t)+\pi^2)}<\int_a^{Na}\frac{dt}{t(\ln^2(t)+\pi^2)}+\sum_{k=N}^{\infty}\int_{ka}^{(k+1)a}\frac{dt}{t\ln^2(t)}\\&=\int_a^{Na}\frac{dt}{t(\ln^2(t)+\pi^2)}+\frac{1}{\ln(aN)}<\infty.\end{aligned}$$

We next examine positivity of d_a.

For $0<t<1$ we have

$$d_a(t)=\frac{\ln|\Gamma(1-t)|}{t((\ln(at))^2+\pi^2)}>0$$

because $\Gamma(s)>1$ for $0<s<1$.

For $k\geq 2$ and $t\in\,]k-1,k[$ the numerator N_a in d_a can be written as

$$N_a(t)=\ln\Gamma(k-t)+\sum_{l=1}^{k-1}\ln\frac{ta}{t-l},$$

where we have used the functional equation for Γ. Hence

$$N_a(t)\geq\sum_{l=1}^{k-1}\ln\frac{k}{k-l}+(k-1)\ln a=(k-1)\ln k-\ln\Gamma(k)+(k-1)\ln a,$$

because $\Gamma(k-t) > 1$ and $t/(t-l)$ is decreasing for $k-1 < t < k$. From (20) we get

$$\ln\Gamma(k) = \ln\sqrt{2\pi} + (k-1/2)\ln k - k + \mu(k), \tag{30}$$

and in particular for $k = 2$

$$\mu(2) = 2 - \frac{3}{2}\ln 2 - \ln\sqrt{2\pi}.$$

Using (30) we find that

$$N_a(t) \geq k - \frac{1}{2}\ln k - \ln\sqrt{2\pi} - \mu(k) + (k-1)\ln a \geq k - \frac{1}{2}\ln k - 2 + \frac{3}{2}\ln 2 + (k-1)\ln a,$$

because μ is decreasing on $]0,\infty[$ as shown by (21).

For $a \geq 1/2$ and $k-1 < t < k$ with $k \geq 2$ we then get

$$N_a(t) \geq k(1-\ln 2) - \frac{1}{2}\ln k + \frac{5}{2}\ln 2 - 2 \geq 0,$$

because the sequence $c_k, k \geq 2$, on the right-hand side is increasing with $c_2 = 0$.

We also see that $d_a(t)$ tends to infinity for t approaching the end points of the interval $]k-1,k[$. For $z = 1/a + iy, y > 0$, we get from (7)

$$\Im F_a(1/a+iy) = -\frac{\ln\Gamma(1+1/a)}{y} + \int_0^\infty \frac{y d_a(t)}{(1/a+t)^2 + y^2}\,dt.$$

The last term tends to 0 for $y \to 0$, while the first term tends to $-\infty$ when $0 < a < 1$. This shows that F_a is not a Pick function for these values of a. □

Remark 2.6. We proved in Theorem 1.1 that $d_a(t)$ is non-negative on $[0,\infty[$ for $a \geq 1/2$. This is not best possible, and we shall explain that the smallest value of a for which $d_a(t)$ is non-negative is $a_0 = 0.3681154742\ldots$.

Replacing k by $k+1$ in the numerator N_a for d_a given by (8), we see that

$$N_a(t) = \ln|\Gamma(1-t)| + k\ln(at) \quad \text{for} \quad t \in]k,k+1[, \quad k = 1,2,\ldots,$$

is non-negative if and only if

$$\ln(1/a) \leq \ln(k+s) + \frac{1}{k}\ln|\Gamma(1-k-s)| \quad \text{for} \quad s \in]0,1[, \quad k = 1,2,\ldots,$$

and using the reflection formula for Γ this is equivalent to $\ln(1/a) \leq \rho(k,s)$ for all $0 < s < 1$ and all $k = 1,2,\ldots$, where

$$\rho(k,s) = \ln(k+s) - \frac{1}{k}\ln\left(\Gamma(k+s)\frac{\sin(\pi s)}{\pi}\right). \tag{31}$$

Using Stieltjes' formula (20), we find that

$$\begin{aligned} \rho(k,s) = {} & 1 + \frac{\ln(\pi/2)}{2k} \\ & -(1/k)\left[(s-1/2)\ln(s+k) + \ln\sin(\pi s) - s + \mu(s+k)\right] \end{aligned} \tag{32}$$

for all $s \in]0,1[$ and $k = 1,2,\ldots$. For fixed $s \in]0,1[$ we see that $\rho(k,s) \to 1$ as $k \to \infty$, so $\ln(1/a) \leq 1$ is a necessary condition for non-negativity of $d_a(t)$. This condition is not sufficient, because for $\ln(1/a) = 1$ the inequality $1 \leq \rho(k,s)$ is equivalent to

$$0 \geq (1/2)\ln(2/\pi) + (s-1/2)\ln(s+k) + \ln\sin(\pi s) - s + \mu(s+k),$$

which does not hold when k is sufficiently large and $1/2 < s < 1$.

For each $k = 1, 2, \ldots$ it is easy to verify that the function $\rho_k(s) = \rho(k, s)$ has a unique minimum m_k over $]0, 1[$, and clearly

$$\ln(1/a_0) = \inf\{m_k, k \geq 1\} \tag{33}$$

determines the smallest value of a for which $d_a(t)$ is non-negative. Using Maple one obtains that m_k is decreasing for $k = 1, \ldots, 510$ and increasing for $k \geq 510$ with limit 1. Therefore $m_{510} = \inf m_k = 0.9993586013...$ corresponding to $a_0 = 0.3681154742....$ We add that $m_1 = 1.6477352344...$, $m_{178} = 1.0000028637...$, $m_{179} = 0.9999936630....$

3. Properties of the function f

Proof of Theorem 1.3. The function

$$\ln f(x) = \frac{(x/2)\ln \pi - \ln \Gamma(1 + x/2)}{x \ln x}$$

clearly has a meromorphic extension to $\mathcal{A} \setminus 1$ with a simple pole at $z = 1$ with residue $\ln 2$. We denote this meromorphic extension as $\log f(z)$ and have

$$\log f(z+1) = \frac{\ln \sqrt{\pi}}{\operatorname{Log}(z+1)} - \frac{1}{2} F_2\left(\frac{z+1}{2}\right).$$

Using the representation (7), we immediately get (11). It is well-known that $1/\operatorname{Log}(z + 1)$ is a Stieltjes function (cf. [9, p. 130]), and the integral representation is

$$\frac{1}{\operatorname{Log}(z+1)} = \int_1^\infty \frac{dt}{(z+t)((\ln(t-1))^2 + \pi^2)}. \tag{34}$$

It follows that $\ln(\sqrt{e} f(x+1))$ is a Stieltjes function, in particular completely monotonic, showing that $\sqrt{e} f(x+1)$ belongs to the class $\mathcal{L}$ of logarithmically completely monotonic functions studied in [17] and in [7]. Therefore $f(x+1)$ is also completely monotonic. □

4. Representation of $1/F_a$

For $a > 0$ we consider the function

$$G_a(z) = 1/F_a(z) = \frac{z \operatorname{Log}(az)}{\log \Gamma(z+1)}, \tag{35}$$

which is holomorphic in $\mathcal{A}$ with an isolated singularity at $z = 1$, which is a simple pole with residue $\ln a/\Psi(2) = \ln a/(1-\gamma)$ if $a \neq 1$, while it is a removable singularity when $a = 1$. Here $\Psi(z) = \Gamma'(z)/\Gamma(z)$ and γ is Euler's constant.

Theorem 4.1. *For $a > 0$ the function G_a has the integral representation*

$$G_a(z) = 1 + \frac{\ln a}{(1-\gamma)(z-1)} + \int_0^\infty \frac{\rho_a(t)}{z+t}\, dt, \quad z \in \mathcal{A} \setminus \{1\}, \tag{36}$$

where

$$\rho_a(t) = t \frac{\ln|\Gamma(1-t)| + (k-1)\ln(at)}{(\ln|\Gamma(1-t)|)^2 + ((k-1)\pi)^2} \quad \text{for} \quad t \in]k-1, k[, \quad k = 1, 2, \ldots, \tag{37}$$

and $\rho_a(0) = 1/\gamma, \rho_a(k) = 0,\ k = 1, 2, \ldots,$ which makes ρ_a continuous on $[0, \infty[$. We have $\rho_a(t) \geq 0$ for $t \geq 0,\ a \geq a_0 = 0.3681154742...$ (cf. Remark 2.6), and $G_a(x+1)$ is a Stieltjes function for $a \geq 1$ but not for $0 < a < 1$.

Proof. We notice that for $-k < t < -k+1$, $k = 1, 2, \ldots$, we get using Lemma 2.1

$$\lim_{y\to 0^+} G_a(t+iy) = \frac{t(\ln(a|t|) + i\pi)}{\ln|\Gamma(1+t)| - i(k-1)\pi},$$

and for $t = -k, k = 1, 2, \ldots$, we get

$$\lim_{y\to 0^+} |G_a(-k+iy)| = 0$$

because of the poles of Γ; hence $\lim_{y\to 0^+} \Im G_a(t+iy) = -\pi\rho_a(-t)$ for $t < 0$.

For fixed $w \in \mathcal{A} \setminus \{1\}$ we choose $\varepsilon > 0, k \in \mathbb{N}$ such that $\varepsilon < |w|, 1 < k + \frac{1}{2}$, and we consider the positively oriented contour $\gamma(k,\varepsilon)$ in $\mathcal{A}$, which was used in the proof of Theorem 1.1.

By the residue theorem we find that

$$\frac{1}{2\pi i}\int_{\gamma(k,\varepsilon)} \frac{G_a(z)}{z-w}\,dz = G_a(w) + \frac{\ln a}{(1-\gamma)(1-w)}.$$

We now let $\varepsilon \to 0$ in the contour integration. The contribution from the ε-half circle tends to 0, and we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{G_a((k+\frac{1}{2})e^{i\theta})}{(k+\frac{1}{2})e^{i\theta} - w}(k+\tfrac{1}{2})e^{i\theta}\,d\theta - \int_{-k-\frac{1}{2}}^{0} \frac{\rho_a(-t)}{t-w}\,dt = G_a(w) + \frac{\ln a}{(1-\gamma)(1-w)}.$$

Finally, letting $k \to \infty$ we get (36), leaving the details to the reader. Clearly, $\rho_a \geq 0$ if and only if d_a defined in (8) is non-negative. It follows that $G_a(x+1)$ is a Stieltjes function for $a \geq 1$ but not for $0 < a < 1$, since in the latter case $\Im G_a(1+iy) > 0$ for $y > 0$ sufficiently small. □

Remark 4.2. The integral representation in Theorem 4.1 was established in [10, (6)] in the case of $a = 1$. Since

$$G_a(z) = G_1(z) + \ln(a)\frac{z}{\log\Gamma(z+1)},$$

the formula for G_a can be deduced from the formula for G_1 and the formula

$$\frac{z}{\log\Gamma(z+1)} = \frac{1}{(1-\gamma)(z-1)} + \int_0^\infty \frac{\tau(t)dt}{z+t}, \quad z \in \mathcal{A}\setminus\{1\}, \tag{38}$$

where

$$\tau(t) = \frac{(k-1)t}{(\ln|\Gamma(1-t)|)^2 + ((k-1)\pi)^2} \quad \text{for} \quad t \in\,]k-1, k[\,, \quad k = 1, 2, \ldots. \tag{39}$$

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