PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 139, Number 6, June 2011, Pages 2143–2149
S 0002-9939(2010)10643-3
Article electronically published on November 22, 2010

ON MATHER'S α-FUNCTION OF MECHANICAL SYSTEMS

WEI CHENG

(Communicated by Yingfei Yi)

Abstract. We study Mather's α-function for mechanical systems. We show that for mechanical systems, the α-function is differentiable at $c = 0$ in at least one direction. We also give a topological condition on the potential function to guarantee the existence of a flat part near $c = 0$ for general mechanical systems. Some examples are also given.

1. Introduction

Let $L : T\mathbb{T}^n \to \mathbb{R}$ be the Tonelli Lagrangian on the n-torus satisfying the following properties:

(L1) Smoothness: $L : T\mathbb{T}^n \to \mathbb{R}$ is of class at least C^2.

(L2) Convexity: The Hessian $\frac{\partial^2 L}{\partial \dot{x}^2}(x, \dot{x})$ is positively definite on each fibre $T_x\mathbb{T}^n$.

(L3) Superlinearity:

$$\lim_{|\dot{x}|\to\infty} \frac{L(x,\dot{x})}{|\dot{x}|} = \infty, \quad \text{uniformly on } x \in \mathbb{T}^n.$$

Let $\mathscr{M}(L)$ be the set of Φ_t-invariant Borel probability measures on $T\mathbb{T}^n$, where Φ_t is the Euler-Lagrange flow of L. For every $\mu \in \mathscr{M}(L)$, we can define its *average action*

$$A(\mu) = \int_{\mathbb{T}^n} L \; d\mu. \tag{1}$$

The integral is defined since L is bounded below. If $A(\mu) < +\infty$, we may associate to μ its *rotation vector* $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$. The rotation vector $\rho(\mu)$ is uniquely characterized by

$$\langle c, \rho(\mu)\rangle = \int \eta_c \; d\mu, \quad \text{for all} \quad c \in H^1(\mathbb{T}^n, \mathbb{R}),$$

where η_c is a representative of the de Rham cohomological class $c \in H^1(\mathbb{T}^n, \mathbb{R}) \cong \mathbb{R}^n$ and the bracket on the left side of the equality above is the canonical pairing of $H^1(\mathbb{T}^n, \mathbb{R})$ and $H_1(\mathbb{T}^n, \mathbb{R})$. The integral on the right is well defined, since an addition of an exact form to η_c does not change the integral (see [Mat1, Mat2]).

Received by the editors December 30, 2009 and, in revised form, June 11, 2010.
2010 *Mathematics Subject Classification.* Primary 37Jxx, 70Hxx.
Key words and phrases. Mather theory, α-function, mechanical systems.
This work was partially supported by the National Basic Research Program of China (Grant No. 2007CB814800) and Natural Scientific Foundation of China (Grant No. 10971093).

For every $h \in H_1(\mathbb{T}^n,\mathbb{R})$, we define Mather's β-function, $\beta : H_1(\mathbb{T}^n,\mathbb{R}) \to \mathbb{R}$, as

$$\beta(h) = \inf\{A(\mu) : \mu \in \mathscr{M}(L),\ \rho(\mu) = h\}. \tag{2}$$

It is easy to see that $\beta(h)$ is a convex function on $H_1(\mathbb{T}^n,\mathbb{R})$ with superlinear growth. We define Mather's α-function, $\alpha : H^1(\mathbb{T}^n,\mathbb{R}) \to \mathbb{R}$, as the Fenchel transformation of the β-function, i.e.,

$$\alpha(c) = \max\{\langle c, h\rangle - \beta(h) : h \in H_1(\mathbb{T}^n,\mathbb{R})\}, \quad \text{for all} \quad c \in H^1(\mathbb{T}^n,\mathbb{R}). \tag{3}$$

From the basic facts in convex analysis, $\alpha(c)$ is also a convex function on $H^1(\mathbb{T}^n,\mathbb{R})$ with superlinear growth. It is well known that

$$\alpha(c) = -\inf_{\mu\in\mathscr{M}(L)} \int_{T\mathbb{T}^n} L - c\ d\mu. \tag{4}$$

The following inf-max formula for the α-function is also useful (see [CIPP]):

$$\alpha(c) = \inf_{u\in C^1(\mathbb{T}^n)} \max_{x\in\mathbb{T}^n} H(x, du(x) + c). \tag{5}$$

Many authors contributed to the structure of the α-functions or β-functions; see e.g. [BIK, C, LPV, Mas1, Mas2, O]. This paper is motivated by the two following problems:

Problem 1. Is it true that for any autonomous Tonelli Lagragian the α-function is differentiable in at least one direction everywhere?

Problem 2. For mechanical systems with the form $L(x,v) = \frac{1}{2}|v|^2 - U(x)$, is there some relation between the topological structure of the level set $\{x \in \mathbb{T}^n : U(x) = \max_{x\in\mathbb{T}^n} U(x)\}$ and the regularity properties of the α-function?

In this paper, we give partial answers for the problems above in sections 2 and 3 respectively. We have:

(1) (Corollary 1) For the Tonelli Lagrangian in the form $\ell(v) - U(x)$, where ℓ is strictly convex, $\ell(0) = 0 = \min_{v\in\mathbb{R}^n} \ell(v)$ and $\max_{x\in\mathbb{T}^n} U(x) = 0$, the α-function is differentiable in some directions at $c = 0$.

(2) (Theorem 3) If there exist k vector fields X_i on $\mathbb{T}^n$ independently, $1 \leqslant k \leqslant n$ such that the X_i's satisfy (9) and (10) with $\int_{\mathbb{T}^n} X_i(x)dx \neq 0$, $i = 1, \ldots, k$, then the α-function has k-dimensional flat part near $c = 0$.

(3) (Theorem 4) If the critical set E of U of the system $L(x,v) = \frac{1}{2}|v|^2 - U(x)$ does not contain a simple closed homotopically nontrivial smooth curve, then the α-function has fully dimensional flat part near $c = 0$.

In section 3, we also give some examples of mechanical systems with arguments to expose the link between the topological structure of the projected Aubry set $\mathscr{A}_0$ and the regularity properties of the α-function.

2. Differentiability of the α-function

This section is motivated by the famous Hedlund example on the geodesic flow on $\mathbb{T}^3$; see [Ba] for details or [Y] for a similar example of mechanical systems.

Given any $c \in H^1(\mathbb{T}^n,\mathbb{R}) \cong \mathbb{R}^n$, c is specified to a real vector in $\mathbb{R}^n$ as its representative element in the cohomology class c. From now on, we identify the cohomology class c with $c \in \mathbb{R}^n$ for convenience, since the exact 1-forms do not contribute to the action.

Now let $L(x,v)=\frac{1}{2}|v|^2-U(x)$ be a Tonelli Lagrangian of the mechanical system on $\mathbb{T}^n$, where $U:\mathbb{T}^n\to\mathbb{R}$ is a C^2 function and $\max_{x\in\mathbb{T}^n}U(x)=0$. Define

$$E=\{x\in\mathbb{T}^n:\ U(x)=0=\max_{x\in\mathbb{T}^n}U(x)\}.$$

It is clear that $E\subset\mathbb{T}^n$ is a compact set.

For convenience, let us lift the torus to its universal covering space. Let $Q=[0,1)^n$ be the fundamental domain of the covering space $\mathbb{R}^n$, $\tilde{E}\subset Q$ be the natural quotient of the lift of E in Q.

Since the fundamental group $\pi_1(\mathbb{T}^n)\cong\mathbb{Z}^n$, for any $\mathbf{n}\in\mathbb{Z}^n$, let $\Gamma_{\mathbf{n}}$ be the C^1 closed curve in $\mathbb{T}^n$ whose lifts to the universal covering space $\mathbb{R}^n$ is a straight line segment having the endpoints $\tilde{x}_1$ and $\tilde{x}_2=\tilde{x}_1+\mathbf{n}$.

Theorem 1. *Let E admit a homotopically nontrivial C^1 simple closed curve $\Gamma_{\mathbf{n}}$ for some nonzero $\mathbf{n}\in\mathbb{Z}^n$; i.e., the natural quotient of $\tilde{\Gamma}_{\mathbf{n}}$ in Q is $\tilde{E}$. Then the corresponding α-function in the direction $\mathbf{n}$ can be represented by*

$$\alpha(r\mathbf{n})=\frac{1}{2}|r\mathbf{n}|^2,\quad r\in\mathbb{R}.$$

Proof. For convenience, we lift the Hamiltonian $H(x,p)=\frac{1}{2}|p|^2+U(x)$ to the universal covering space $\mathbb{R}^n\times\mathbb{R}^n$ of $\mathbb{T}^*\mathbb{T}^n$; that is, U can be regarded as a $\mathbb{Z}^n$-periodic function with respect to x. Given any smooth $\mathbb{Z}^n$-periodic function u on $\mathbb{R}^n$, there exists $t\in[0,1]$ such that the directional derivative with respect to $\mathbf{n}$ of u on $\tilde{\Gamma}_{\mathbf{n}}(t)$ is 0. Then for $c=c(r)=r\mathbf{n}$, $r>0$, the inf-max formulae (5) of the α-function implies that

$$\max_{x\in Q}\frac{1}{2}|du+c|^2+U(x)\geqslant\max_{x\in Q}\frac{1}{2}|c|^2+U(x)=\frac{1}{2}|c|^2.$$

Then we have $\alpha(c)\geqslant\frac{1}{2}|c|^2$. On the other hand, we have $\alpha(c)\leqslant\frac{1}{2}|c|^2$ by choosing u to be any constant function. □

Lemma 1. *Let $L_{U,\ell}$ be the Tonelli Lagrangian in the form $L_{U,\ell}(x,v)=\ell(v)-U(x)$ with strictly convex kinetic energy ℓ and potential U. Suppose $U(x)\leqslant\tilde{U}(x)$ for any $x\in\mathbb{T}^n$ and $\ell(v)\geqslant\tilde{\ell}(v)$ for any $v\in\mathbb{R}^n$. Then the relation between the α-function of the systems $L_{U,\ell}$ and $L_{\tilde{U},\tilde{\ell}}$ satisfies*

$$\alpha_{U,\ell}\leqslant\alpha_{\tilde{U},\tilde{\ell}}.$$

Proof. This is deduced directly from the definition and formula (4). □

Now suppose the potential U is not trivially constant. Then there exist both a minimum and a maximum of U since $\mathbb{T}^n$ is compact. Let y and $z\in\mathbb{T}^n$ be such that

$$U(y)=\min_{x\in\mathbb{T}^n}U(x)<U(z)=\max_{x\in\mathbb{T}^n}U(x).\tag{6}$$

Then we can find an alternative potential function U_δ on $\mathbb{T}^n$ for small $\delta>0$ such that

$$\max_{x\in\mathbb{T}^n}U_\delta(x)=U_\delta(z)=U(z),\quad U_\delta(x)\geqslant U(x)\text{ for any }x\in\mathbb{T}^n\tag{7}$$

and

$$U_\delta(x)\equiv U_\delta(z)\text{ for }x\notin B(y,\delta).\tag{8}$$

The potential U_δ can be obtained by the smooth Urysohn lemma since E is a compact subset in $\mathbb{T}^n$ and there exists a ball $B(y,\delta) \subset \mathbb{T}^n \setminus E$.

Theorem 2. *For any mechanical Tonelli Lagrangian $\frac{1}{2}|v|^2 - U(x)$, the corresponding α-function $\alpha(c)$ is differentiable in some directions at $c = 0$.*

Proof. If $U \equiv C$ is a contant, then the system is completely integrable and $\alpha(c) = \frac{1}{2}|c|^2$, which is trivial. Now suppose $U \not\equiv C$.

We apply Theorem 1 to the system $\frac{1}{2}|v|^2 - U_\delta(x)$ at first for some small $\delta > 0$. In that case, $\tilde{E} = Q \setminus B(y,\delta)$. Then there exists $\tilde{\Gamma}_{\mathbf{n}} \subset \tilde{E}$ as in Theorem 1 for some $\mathbf{n} \in \mathbb{Z}^n$. Each $\tilde{\Gamma}_{\mathbf{n}}$ determines a direction $\mathbf{n}$ such that $\alpha_\delta(r\mathbf{n}) = \frac{1}{2}|r\mathbf{n}|^2$, where α_δ is the α-function of the system with potential U_δ. The number of such $\mathbf{n}$'s is decided by δ. Here we omit the argument of that number. Then we have that α_δ is quadratic in the direction $\mathbf{n}$ and is differentiable at $c = 0$.

From the construction of the potential U_δ, together with Lemma 1, we easily have $\alpha_\delta(c) \geqslant \alpha(c)$ for all c. Then the differentiability of α at $c = 0$ along the direction $\mathbf{n}$ can be obtained since $\alpha_\delta(0) = \alpha(0)$ and both of them are convex functions. $\square$

Corollary 1. *For the Tonelli Lagrangian in the form $\ell(v) - U(x)$, where ℓ is strictly convex, $\ell(0) = 0 = \min_{v\in\mathbb{R}^n} \ell(v)$ and $\max_{x\in\mathbb{T}^n} U(x) = 0$, the α-function is differentiable in some directions at $c = 0$.*

Proof. We only discuss the nontrivial case. Let $L_1(x,v) = \ell(v) - U(x)$, $L_2(x,v) = \frac{\lambda^2}{2}|v|^2 - U_\delta(x)$ for some $\lambda > 0$ and small $\delta > 0$ as before. Then $L_1 \geqslant L_2$ by the strict convexity assumption of ℓ and the definition of U_δ. Denoting by α_i, $i = 1, 2$, the α-function of L_i respectively, we have $\alpha_1 \leqslant \alpha_2$, and the differentiability of α_1 in some directions follows from that of α_2, which has been proved in Theorem 2. $\square$

Remark 1. If Problem 1 in section 1 can be solved so as to be true, then the main result of [BC] can be improved. We will try it in other papers in the future.

3. Flat part of the α-function

Let $L^0(x,v) = \frac{1}{2}|v|^2 - U(x)$ with the potential $U(x) \leqslant 0$ and $\max_{x\in\mathbb{T}^n} U(x) = 0$, $\kappa(x) = \sqrt{2(-U(x))}$. We want to find a C^1 vector field $X(x)$ as a function $X : \mathbb{T}^n \to \mathbb{R}^n$ such that

$$|X(x)| = \kappa(x) \tag{9}$$

and

$$dX(x) = dX^*(x). \tag{10}$$

Condition (10) means that X is a *gradient-like vector field* corresponding to a closed 1-form on $\mathbb{T}^n$ in the following sense: the closed 1-form ω is defined by

$$\omega(x)(v) = \langle X(x), v\rangle, \quad v \in T_x\mathbb{T}^n \cong \mathbb{R}^n.$$

Now let us recall some basic facts on the construction of the closed 1-form on a closed smooth manifold M. Letting $f : M \to \mathbb{T}^1$ be a smooth map, the circle $\mathbb{T}^1$ is equipped with the canonical angular form $d\theta$, where $d\theta$ is a closed 1-form, which cannot be represented as a differential of a smooth function on $\mathbb{T}^1$. The pullback $f^*(d\theta)$ is a closed 1-form on M. It is not hard to show that a closed 1-form ω on M can be represented by this form if and only if the de Rham cohomology class $[\omega] \in H^1(M,\mathbb{Z})$ (see [Fab]).

In particular in the case of $M = \mathbb{T}^n$, we can construct the required closed 1-form as follows: Letting $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ be the local coordinates of $T\mathbb{T}^n$, any closed 1-form is locally exact by Poincaré's lemma. Then there exists an open cover $\{\Omega_i\}$ of $\mathbb{T}^n$ such that there exist smooth functions f_i defined on Ω_i such that $\omega|_{\Omega_i} = df_i|_{\Omega_i}$. More precisely, for the purpose of conditions (9) and (10), define the local vector field $X_i(x) = (0, \dots, \frac{\partial f_i}{\partial x_i}, \dots, 0)$ on Ω_i, where f_i is a smooth function on $\mathbb{T}^n$ such that $|\frac{\partial f_i}{\partial x_i}| = \kappa(x)$. The smoothness of the f_i's can be guaranteed if $\max_{x\in\mathbb{T}^n} U(x) < 0$. The global vector fields can be given by partition of unity. It is clear that the X_i's are independent.

Then define a new Lagrangian

$$L^1(x,v) = \frac{1}{2}|v - X(x)|^2 = L^0(x,v) - \langle X(x), v\rangle. \tag{11}$$

Denote by α_0 and α_1 the α-functions of L^0 and L^1 respectively.

Lemma 2. *If for the potential U of L^0 there exists a C^1 vector field X satisfying* (9) *and* (10) *and $c = \int_{\mathbb{T}^n} X(x)dx$, then*

$$\alpha_0(c) = \alpha_1(0) = 0 \tag{12}$$

with $|c| \leqslant \int_{\mathbb{T}^n} \kappa(x)dx$. Consequently, with the conditions above, the α-function has a flat part near 0 *in the direction c if $c \neq 0$.*

Proof. It is well known that for a positive definite Lagrangian L, $L - \lambda$ has the same Euler-Lagrange equation if the 1-form λ is closed. It is clear from (11) that if $c = \int_{\mathbb{T}^n} X(x)dx$, then $\alpha_1(0) = \alpha_0(c) = 0$. $|c| \leqslant \int_{\mathbb{T}^n} \kappa(x)dx$ follows easily from the definition of the vector field X. If $c \neq 0$, then (12) implies that $\alpha_0(rc) = 0$ for $0 \leqslant r \leqslant 1$ since $\alpha_0(0) = \min_{c\in\mathbb{R}^n} \alpha_0(c)$ and the α-function is convex. □

Theorem 3. *If there exist k vector fields X_i on $\mathbb{T}^n$ independently, $1 \leqslant k \leqslant n$, such that the X_i's satisfy* (9) *and* (10) *with $\int_{\mathbb{T}^n} X_i(x)dx \neq 0$, $i = 1, \dots, k$, then the α-function has k-dimensional flat part near $c = 0$.*

Proof. This is a direct consequence of Lemma 2. □

Now we apply Lemma 2 to some examples.

Example 1. When $n = 1$ and $\max_{x\in\mathbb{T}^n} U(x) = 0$, the flat part $|c| \leqslant \int_{\mathbb{T}^1} \kappa(x)dx$ of the α-function is well known; see e.g. [LPV]. Let $V_\varepsilon(x) = U(x) - \varepsilon$ for $\varepsilon > 0$. Then

$$L^0_\varepsilon(x,v) - \langle X_\varepsilon(x), v\rangle = \frac{1}{2}|v|^2 - V_\varepsilon(x) - \langle X_\varepsilon(x), v\rangle = \frac{1}{2}|v - X_\varepsilon(x)|^2 = L^1_\varepsilon(x,v),$$

where X_ε satisfies (9) and (10) for the potential V_ε. The existence of such an X_ε can easily be obtained by $X_\varepsilon = \sqrt{-V_\varepsilon}$. Denote by α^0_ε and α^1_ε the α-function of L^0_ε and L^1_ε respectively. Then we have $\alpha^1_\varepsilon(0) = \alpha^0_\varepsilon(c_\varepsilon) = 0$ by Lemma 2, where $c_\varepsilon = \int_{\mathbb{T}^1} X_\varepsilon(x)dx$. This implies that $\alpha^0(c_\varepsilon) - \varepsilon = 0$ for any $\varepsilon > 0$. So $\alpha(c_0) = 0$ by the continuity of c_ε with respect to ε, and $c_0 \neq 0$ if $U \not\equiv 0$. Thus the α-function has flat part on $[0, c_0]$, and the case of $[-c_0, 0]$ is similar by choosing $X_\varepsilon = -\sqrt{-V_\varepsilon}$.

Example 2. Let U be a smooth function on $\mathbb{T}^n$, and suppose that the critical set E defined in section 2 does not contain a simple closed homotopically nontrivial C^1 curve; e.g., U is a function of Morse type. Then the ε-trick in Example 1 and the construction of independent vector fields X_i as before imply the following:

Theorem 4. *If the critical set E of U of the system $L(x,v)=\frac{1}{2}|v|^2-U(x)$ does not contain a simple closed homotopically nontrivial smooth curve, then the α-function has fully dimensional flat part near $c=0$.*

Proof. If the critical set E of U does not contain a simple closed homotopically nontrivial smooth curve, then there exist n independent gradient-like vector fields $\{X_{i,\varepsilon}\}_{i=1}^n$ for $V_\varepsilon(x)=U(x)-\varepsilon$ as in Example 1. Applying the argument in Example 1 to each $X_{i,\varepsilon}$, we can see that there exists $c_{0,i}\neq 0$ such that the α-function has flat part in the direction of $c_{0,i}$. The independency of $X_{i,\varepsilon}$ means the independency of such $c_{0,i}$'s; thus we get the conclusion. □

Remark 1. For the mechanical systems with the form $L(x,v)=\frac{1}{2}|v|^2-U(x)$, the critical set $E=\{x:U(x)=\max_{x\in\mathbb{T}^n}U(x)\}$ is exactly the projected Aubry set $\mathscr{A}_0$. For the definition of *Aubry set* and *projected Aubry set*, see e.g. [Mat3, Be, Fat, FS]. So the existence of the flat part near $c=0$ is closely related to the topological structure of $\mathscr{A}_0$. Actually a very complicated structure of $\mathscr{A}_0$ exists, e.g., Mather's striking example ([Mat3]).

Example 3. Let U be a smooth function on $\mathbb{T}^n$, and let the critical set E defined in section 2 contain a simple closed homotopically nontrivial C^1 curve $\Gamma_{\mathbf{n}}$, i.e., the example described in Theorem 1. It is clear from the construction of gradient-like vector fields before that there exist n gradient-like vector fields $X_1,\dots,X_n$ on $\mathbb{T}^n$ for the ε-perturbed system as in Example 1 such that the closed 1-form related to X_n is exact and $X_1,\dots,X_{n-1}$ may be independent. Then a similar argument shows that there is no flat part along the direction of $\mathbf{n}\in\mathbb{Z}^n$ since X_n is a gradient field. Rewriting

$$\frac{1}{2}|v|^2-U(x)-\langle c,v\rangle=\frac{1}{2}|v-c|^2-\frac{1}{2}|c|^2-U(x),$$

we also have that the α-function is quadratic in the direction of $\mathbf{n}$, which is the same statement as in Theorem 1.

Remark 2. When we consider the case that the projected Aubry set $\mathscr{A}_0=E$ contains a general closed homotopically nontrivial C^1 curve, we need to find the obstacle to the existence of such a gradient-like vector field. The author hopes to try it in the future.

Acknowledgments

The author heartily thanks Zhou Min for carefully checking the draft of this paper. The author also thanks Wang Lin for some discussions on Problem 1.

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Department of Mathematics, Nanjing University, Nanjing, 210093, People's Republic of China

E-mail address: chengwei@nju.edu.cn