THE POSITIVITY OF THE FIRST COEFFICIENTS OF NORMAL HILBERT POLYNOMIALS

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ABSTRACT. Let R be an analytically unramified local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. If R is unmixed, then $\bar{e}_I^1(R) \geq 0$ for every $\mathfrak{m}$ -primary ideal I in R , where $\bar{e}_I^1(R)$ denotes the first coefficient of the normal Hilbert polynomial of R with respect to I . Thus the positivity conjecture on $\bar{e}_I^1(R)$ posed by Wolmer V. Vasconcelos is settled affirmatively.

1. INTRODUCTION

Throughout this paper let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Assume that R is analytically unramified, whence the $\mathfrak{m}$ -adic completion $\hat{R}$ of R is reduced. We fix an $\mathfrak{m}$ -primary ideal I in R and denote by $\overline{I^{n+1}}$ (resp. $\lambda_R(R/\overline{I^{n+1}})$) the integral closure of I^{n+1} (resp. the length of $R/\overline{I^{n+1}}$) for each $n \geq 0$. Then the normal Hilbert function

$$\lambda_R(R/\overline{I^{n+1}})$$

of R with respect to I is of polynomial type with degree d , and we have integers $\{\bar{e}_I^i(R)\}_{0 \leq i \leq d}$ such that the equality

$$\lambda_R(R/\overline{I^{n+1}}) = \bar{e}_I^0(R) \binom{n+d}{d} - \bar{e}_I^1(R) \binom{n+d-1}{d-1} + \cdots + (-1)^d \bar{e}_I^d(R)$$

holds true for all $n \gg 0$. We call these integers $\bar{e}_I^i(R)$ the coefficients of the normal Hilbert polynomial of R with respect to I .

In this paper we are interested in the analysis of the first coefficient $\bar{e}_I^1(R)$ of the normal Hilbert polynomial. The main purpose is to study the positivity conjecture on $\bar{e}_I^1(R)$ posed by Wolmer V. Vasconcelos [V], and our result is stated as follows.

Theorem 1.1. *Let R be an analytically unramified local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. If R is unmixed, then*

$$\bar{e}_I^1(R) \geq 0$$

for every $\mathfrak{m}$ -primary ideal I in R .

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Here we should note that the conjecture holds true in the case where R is a Cohen-Macaulay local ring ([PUV, Theorem 2.2]). In fact, generally we have

$$\bar{e}_I^0(R) = e_I^0(R),$$

where $e_I^0(R)$ stands for the ordinary Hilbert-Samuel multiplicity of R with respect to I . Therefore $\bar{e}_I^1(R) \geq e_I^1(R)$, and so, if R is a Cohen-Macaulay local ring, we get

$$\bar{e}_I^1(R) \geq e_I^1(R) \geq 0,$$

because $e_I^1(R) \geq 0$ ([Nr, Corollary 1]). Mainly based on this fact, the third author, M. Mandal, along with B. Singh and J. Verma [MSV], gave several interesting answers in certain special cases, and our Theorem 1.1 now affirmatively settles the conjecture in full generality.

We shall prove Theorem 1.1 in Section 2. In Section 3 we will discuss a few results related to the positivity conjecture. We expect that the integral closure $\bar{R}$ of R is a regular ring and $I\bar{R}$ is normal; that is, $I^n\bar{R}$ is integrally closed for all $n \geq 1$, once $\bar{e}_I^1(R) = 0$ for some $\mathfrak{m}$ -primary ideal I in R . We shall give an affirmative answer in the case where $\bar{R}$ is a Cohen-Macaulay ring.

Throughout this paper, unless otherwise specified, we denote by R a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let $\bar{R}$ be the integral closure of R in its total quotient ring. For each finitely generated R -module M , let $\mu_R(M)$ (resp. $\lambda_R(M)$) stand for the number of elements in a minimal system of generators (resp. the length) of M .

2. PROOF OF THEOREM 1.1

The main purpose of this section is to prove Theorem 1.1.

Proof of Theorem 1.1. We have $\bar{e}_{I\bar{R}}^1(\hat{R}) = \bar{e}_I^1(R)$, since $\widehat{\mathfrak{a}\bar{R}} = \widehat{\mathfrak{a}}\hat{R}$ for every $\mathfrak{m}$ -primary ideal $\mathfrak{a}$ in R . Therefore, passing to the $\mathfrak{m}$ -adic completion $\hat{R}$ of R , without loss of generality we may assume that R is complete. If $d = 1$, we then have

$$\bar{e}_I^1(R) = \lambda_R(\bar{R}/R) \geq 0.$$

Suppose that $d \geq 2$ and let $S = \bar{R}$. For each $\mathfrak{p} \in \text{Ass} R$ we put $S(\mathfrak{p}) = \bar{R}/\mathfrak{p}$. Then $S(\mathfrak{p})$ is a module-finite extension of $R/\mathfrak{p}$, and we get

$$S = \prod_{\mathfrak{p} \in \text{Ass} R} S(\mathfrak{p}) \quad \text{and} \quad \overline{I^{n+1}} = \overline{I^{n+1}S} \cap R$$

for all $n \geq 0$. Hence

$$\begin{aligned} \lambda_R(R/\overline{I^{n+1}}) &\leq \lambda_R(S/\overline{I^{n+1}S}) = \sum_{\mathfrak{p} \in \text{Ass} R} \lambda_R(S(\mathfrak{p})/\overline{I^{n+1}S(\mathfrak{p})}) \\ &= \sum_{\mathfrak{p} \in \text{Ass} R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \lambda_{S(\mathfrak{p})}(S(\mathfrak{p})/\overline{I^{n+1}S(\mathfrak{p})}), \end{aligned}$$

where $\mathfrak{m}_{S(\mathfrak{p})}$ denotes the maximal ideal of $S(\mathfrak{p})$. Notice that, since $\dim S(\mathfrak{p}) = d$ for each $\mathfrak{p} \in \text{Ass} R$, we have

$$\begin{aligned} \bar{e}_I^0(R) = e_I^0(R) = e_I^0(S) &= \sum_{\mathfrak{p} \in \text{Ass} R} e_I^0(S(\mathfrak{p})) \\ &= \sum_{\mathfrak{p} \in \text{Ass} R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot e_{IS(\mathfrak{p})}^0(S(\mathfrak{p})) \\ &= \sum_{\mathfrak{p} \in \text{Ass} R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \bar{e}_{IS(\mathfrak{p})}^0(S(\mathfrak{p})), \end{aligned}$$

whence

$$\begin{aligned} 0 &\leq \lambda_R(S/\overline{I^{n+1}S}) - \lambda_R(R/\overline{I^{n+1}}) \\ &= \left[\bar{e}_I^1(R) - \sum_{\mathfrak{p} \in \text{Ass} R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \bar{e}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \right] \binom{n+d-1}{d-1} \\ &\quad + (\text{terms of lower degree}), \end{aligned}$$

so that

$$\bar{e}_I^1(R) \geq \sum_{\mathfrak{p} \in \text{Ass} R} \lambda_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \bar{e}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})).$$

Thus, in order to see $\bar{e}_I^1(R) \geq 0$, it suffices to show that $\bar{e}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq 0$ for each $\mathfrak{p} \in \text{Ass} R$. If $d = 2$, we get

$$\bar{e}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq e_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq 0,$$

because $S(\mathfrak{p})$ is a Cohen-Macaulay local ring. Hence $\bar{e}_I^1(R) \geq 0$.

Suppose that $d \geq 3$ and that our assertion holds true for $d - 1$. Then thanks to the above observation, passing to the ring $S(\mathfrak{p})$, we may assume that R is a normal complete local ring. Let $I = (a_1, a_2, \dots, a_\ell)$ with $a_i \in R$, where $\ell = \mu_R(I)$. Let

$$T = R[Z_1, Z_2, \dots, Z_\ell], \quad \mathfrak{q} = \mathfrak{m}T, \quad x = \sum_{i=1}^{\ell} a_i Z_i, \quad \text{and} \quad D = T/xT,$$

where $Z_1, Z_2, \dots, Z_\ell$ are indeterminates over R . Let

$$R' = T_{\mathfrak{q}}, \quad I' = IR', \quad \text{and} \quad D' = D_{\mathfrak{q}}.$$

We then have $\overline{I^{n+1}R'} = \overline{I^{n+1}}R'$ for all $n \geq 0$, so that $\lambda_{R'}(R'/\overline{I^{n+1}R'}) = \lambda_R(R/\overline{I^{n+1}})$, whence

$$\bar{e}_I^1(R) = \bar{e}_{I'}^1(R').$$

Here we notice that $\text{Ass} D' = \text{Assh} D'$, because R' is catenary and normal; hence D' is unmixed, as D' is a homomorphic image of a Cohen-Macaulay ring. The ring D' is analytically unramified. To see this, since D' is a Nagata local ring, by [M, Theorem 70] it suffices to show that D is reduced; that is, $D_P = T_P/xT_P$ is an integral domain for every $P \in \text{Ass}_T D$. Let $\mathfrak{p} = P \cap R$. Then since $\text{ht}_T P = 1$, we have $\text{ht}_R \mathfrak{p} \leq 1$, so that $I \not\subseteq \mathfrak{p}$, because $\text{ht}_R \mathfrak{p} \leq 1 < d = \dim R$. Without loss of generality we may assume that $a_\ell \notin \mathfrak{p}$. Then, because $x = \sum_{i=1}^{\ell} a_i Z_i$ and a_ℓ is a unit of $R_{\mathfrak{p}}$, we get

$$T_{\mathfrak{p}} = R_{\mathfrak{p}}[Z_1, Z_2, \dots, Z_\ell] = R_{\mathfrak{p}}[Z_1, Z_2, \dots, Z_{\ell-1}, x],$$

whence the ring

$$T_{\mathfrak{p}}/xT_{\mathfrak{p}} = R_{\mathfrak{p}}[Z_1, Z_2, \dots, Z_{\ell}]/xR_{\mathfrak{p}}[Z_1, Z_2, \dots, Z_{\ell}] = R_{\mathfrak{p}}[Z_1, Z_2, \dots, Z_{\ell-1}]$$

is an integral domain, as it is the polynomial ring with $\ell - 1$ indeterminates over $R_{\mathfrak{p}}$. Therefore for all $P \in \text{Ass}_T D$ the ring $D_P = T_P/xT_P$ is an integral domain, because it is a localization of $R_{\mathfrak{p}}[Z_1, Z_2, \dots, Z_{\ell-1}]$. Thus D is reduced, whence D' is analytically unramified and unmixed.

Let us denote by $\mathcal{A}$ the extended Rees ring of IT and by $\overline{\mathcal{A}}$ the integral closure of $\mathcal{A}$ in $T[t, t^{-1}]$, where t denotes an indeterminate. Similarly, let us denote by $\mathbb{T}$ the extended Rees ring of ID and by $\overline{\mathbb{T}}$ the integral closure of $\mathbb{T}$ in $D[t, t^{-1}]$. We put $N = (t^{-1}, It)$ in $\mathcal{A}$. We look at the homomorphism

$$\psi : T[t, t^{-1}] \rightarrow D[t, t^{-1}]$$

of graded T -algebras such that $\psi(t) = t$. Since $\psi(\mathcal{A}) = \mathbb{T}$ and $\overline{\mathbb{T}}$ is a module-finite extension of $\mathbb{T}$, the homomorphism ψ gives rise to the finite homomorphism

$$\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \longrightarrow \overline{\mathbb{T}}$$

of graded T -algebras. Let $\overline{\mathcal{B}}$ (resp. $\overline{\mathbb{U}}$) denote the integral closure of $\mathcal{B} = \mathcal{A}_{\mathfrak{q}}$ (resp. $\mathbb{U} = \mathbb{T}_{\mathfrak{q}}$). Then we get the homomorphism

$$\varphi_{\mathfrak{q}} : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \rightarrow \overline{\mathbb{U}}$$

of graded R' -algebras and, thanks to the proof of [HU, Theorem 2.1], we furthermore have the following. Let us include a brief proof for the sake of completeness.

Claim 1. The homomorphism

$$\varphi_P : [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_P \longrightarrow [\overline{\mathbb{T}}]_P$$

is an isomorphism for all $P \in \text{Spec } \mathcal{A} \setminus V(N)$. Hence the kernel and the cokernel of the homomorphism $\varphi_{\mathfrak{q}} : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \longrightarrow \overline{\mathbb{U}}$ of graded $\mathcal{B}$ -modules are of finite length, so they are finitely graded.

Proof. Because $\overline{\mathcal{A}}[t] = T[t, t^{-1}]$ and $xt\overline{\mathcal{A}}[t] = xT[t, t^{-1}]$, the homomorphism $\varphi_{t^{-1}}$ is an isomorphism, whence so is the homomorphism φ_P , if $t^{-1} \notin P$.

Suppose now that $It \not\subseteq P$. We may assume $a_{\ell}t \notin P$. Notice that

$$\begin{aligned} [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_{a_{\ell}t} &= \left[\overline{R[It, t^{-1}]}[Z_1, Z_2, \dots, Z_{\ell}] / xt \cdot \overline{R[It, t^{-1}]}[Z_1, Z_2, \dots, Z_{\ell}] \right]_{a_{\ell}t} \\ &= \left(\overline{R[It, t^{-1}]} \left[\frac{1}{a_{\ell}t} \right] \right) [Z_1, Z_2, \dots, Z_{\ell}] / \left(\sum_{i=1}^{\ell-1} \frac{a_i Z_i t}{a_{\ell}t} + Z_{\ell} \right) \\ &= \left(\overline{R[It, t^{-1}]} \left[\frac{1}{a_{\ell}t} \right] \right) [Z_1, Z_2, \dots, Z_{\ell-1}] \end{aligned}$$

and that

$$\begin{aligned}
 D[t, t^{-1}]_{a_\ell t} &= T[t, t^{-1}, \frac{1}{a_\ell t}] / x \cdot T[t, t^{-1}, \frac{1}{a_\ell t}] \\
 &= T[t, t^{-1}, \frac{1}{a_\ell}] / x \cdot T[t, t^{-1}, \frac{1}{a_\ell}] \\
 &= R[\frac{1}{a_\ell}, Z_1, Z_2, \dots, Z_\ell, t, t^{-1}] / x \cdot R[\frac{1}{a_\ell}, Z_1, Z_2, \dots, Z_\ell, t, t^{-1}] \\
 &= \left(R[\frac{1}{a_\ell}, t, t^{-1}] \right) [Z_1, Z_2, \dots, Z_\ell] / \left(\sum_{i=1}^{\ell-1} \frac{a_i Z_i}{a_\ell} + Z_\ell \right) \\
 &= \left([R[t, t^{-1}]] \left[\frac{1}{a_\ell t} \right] \right) [Z_1, Z_2, \dots, Z_{\ell-1}].
 \end{aligned}$$

Then we get the following commutative diagram:

$$\begin{array}{ccccc}
 [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_{a_\ell t} & \xrightarrow{\varphi_{a_\ell t}} & [\overline{\Gamma}]_{a_\ell t} & \xrightarrow{\quad} & D[t, t^{-1}]_{a_\ell t} \\
 \downarrow \simeq & & & & \downarrow \simeq \\
 ([\overline{R[It, t^{-1}]]}[\frac{1}{a_\ell t}])[Z_1, \dots, Z_{\ell-1}] & \xrightarrow{\quad} & ([R[t, t^{-1}]][\frac{1}{a_\ell t}])[Z_1, \dots, Z_{\ell-1}], & &
 \end{array}$$

where the vertical homomorphisms are isomorphisms, so the horizontal homomorphism $\varphi_{a_\ell t}$ is injective. Because $\left([\overline{R[It, t^{-1}]]}[\frac{1}{a_\ell t}] \right) [Z_1, Z_2, \dots, Z_{\ell-1}]$ is integrally closed in $\left([R[t, t^{-1}]][\frac{1}{a_\ell t}] \right) [Z_1, Z_2, \dots, Z_{\ell-1}]$ and $\varphi_{a_\ell t}$ is finite, $\varphi_{a_\ell t}$ is an isomorphism, whence

$$\varphi_P : [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_P \longrightarrow [\overline{\Gamma}]_P$$

is an isomorphism, too. This proves Claim 1. $\square$

The normal ring $\overline{\mathcal{B}}$ is catenary, since it is a finitely generated R' -algebra, while we get

$$\dim \overline{\mathcal{B}}/(xt, t^{-1})\overline{\mathcal{B}} = \dim \overline{\mathcal{U}}/t^{-1}\overline{\mathcal{U}} = d - 1$$

by Claim 1. Therefore t^{-1}, xt forms a regular sequence in the normal ring $\overline{\mathcal{B}}$. Hence xt is a non-zero-divisor in the associated graded ring

$$\overline{\mathcal{B}}/t^{-1}\overline{\mathcal{B}} = \bigoplus_{n \geq 0} \overline{I}^n R' / \overline{I}^{n+1} R'$$

of the filtration $\{\overline{I}^n R'\}_{n \in \mathbb{Z}}$ of integrally closed ideals in R' . Consequently, we have

$$\overline{e}_I^1(R) = \overline{e}_{I'}^1(R') = \overline{e}_{I D'}^1(D'),$$

since $\dim D' = \dim R' - 1 = d - 1 \geq 2$ and since the kernel and the cokernel of the homomorphism

$$\overline{\varphi}_q : \overline{\mathcal{B}}/(xt, t^{-1})\overline{\mathcal{B}} \longrightarrow \overline{\mathcal{U}}/t^{-1}\overline{\mathcal{U}}$$

induced from φ_q are finitely graded. Thus the hypothesis of induction on d yields the assertion that $\overline{e}_I^1(R) \geq 0$, which completes the proof of Theorem 1.1. $\square$

The condition in Theorem 1.1 that R is unmixed is not superfluous. Let us note the simplest example. See [MSV, Example 2.4] for more examples.

Example 2.1. We look at the local ring

$$R = k[[X, Y, Z]]/\mathfrak{a},$$

where $k[[X, Y, Z]]$ is the formal power series ring over a field k and $\mathfrak{a} = (X) \cap (Y, Z)$. Then $\dim R = 2$, R is mixed, and $\bar{e}_{\mathfrak{m}}^1(R) = \bar{e}_{\mathfrak{m}}^2(R) = -1$. Hence the famous bad example [N, p. 203, Example 2] of Nagata which is a non-regular local integral domain $(A, \mathfrak{n})$ of dimension 2 with $e_{\mathfrak{n}}^0(A) = 1$ possess $\bar{e}_{\mathfrak{n}}^1(A) = \bar{e}_{\mathfrak{n}}^2(A) = -1$, because

$$\hat{A} \cong k[[X, Y, Z]]/[(X) \cap (Y, Z)]$$

for some field k .

Proof. We put $T = k[[X, Y, Z]]$ and $\mathfrak{q} = (X, Y, Z)$ in T . Then $\bar{R} = T/(X) \oplus T/(Y, Z)$, and we have the exact sequence

$$(E) \quad 0 \rightarrow R \rightarrow T/(X) \oplus T/(Y, Z) \rightarrow T/\mathfrak{q} \rightarrow 0$$

of T -modules; hence $\mathfrak{m}\bar{R} \subseteq R$. Recall that $\mathfrak{m}$ is a normal ideal in R ; that is, $\overline{\mathfrak{m}^n} = \mathfrak{m}^n$ for all $n \geq 1$, since the associated graded ring

$$\mathrm{gr}_{\mathfrak{m}}(R) = k[X, Y, Z]/[(X) \cap (Y, Z)]$$

of $\mathfrak{m}$ is reduced. Therefore, as

$$\mathfrak{m}^{n+1} = \overline{\mathfrak{m}^{n+1}} = \overline{\mathfrak{m}^{n+1}\bar{R}} \cap R = \mathfrak{m}^{n+1}\bar{R} \cap R,$$

thanks to exact sequence (E) above, we get

$$0 \rightarrow R/\overline{\mathfrak{m}^{n+1}} \rightarrow T/[(X) + \mathfrak{q}^{n+1}] \oplus T/[(Y, Z) + \mathfrak{q}^{n+1}] \rightarrow T/\mathfrak{q} \rightarrow 0$$

for all $n \geq 0$. Hence

$$\lambda_R(R/\overline{\mathfrak{m}^{n+1}}) = \binom{n+2}{2} + \binom{n+1}{1} - 1,$$

so that $\bar{e}_{\mathfrak{m}}^1(R) = \bar{e}_{\mathfrak{m}}^2(R) = -1$. □

Let us note a consequence of Theorem 1.1.

Corollary 2.2 ([MTV, Theorem 1]). *Let R be an analytically unramified unmixed local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let I be a parameter ideal in R . If $\bar{e}_I^1(R) = e_I^1(R)$, then R is a regular local ring with $\mu_R(\mathfrak{m}/I) \leq 1$, whence I is normal.*

Proof. We get $e_I^1(R) \geq 0$ by Theorem 1.1, whence by [GhGHOPV, Theorem 2.1] R is a Cohen-Macaulay local ring with $e_I^1(R) = 0$. Because $\bar{e}_I^1(R) \geq e_I^1(R)$ and

$$e_I^1(R) \geq 0$$

([Nr, Corollary 1]), we furthermore have $e_I^1(R) = 0$, whence $\bar{I}$ is a parameter ideal in R ([Nr, Corollary 2]). Because parameter ideals contain no proper reductions ([NR]), we get $\bar{I} = I$, whence by [G, Theorem (3.1)] R is a regular local ring with $\mu_R(\mathfrak{m}/I) \leq 1$ and I is normal. □

Remark 2.3. In Corollary 2.2, unless I is a parameter ideal, R is not necessarily a regular local ring, even though $\bar{e}_I^1(R) = e_I^1(R)$. Let us note an example. We look at the local ring

$$R = k[[X, Y, Z]]/(Z^2 - XY),$$

where $k[[X, Y, Z]]$ is the formal power series ring over a field k of characteristic 0. Then R is a rational singularity, so $\bar{e}_I^1(R) = e_I^1(R)$ for every integrally closed $\mathfrak{m}$ -primary ideal I in R .

3. A FURTHER PROBLEM

Let R be an analytically unramified unmixed local ring and I an $\mathfrak{m}$ -primary ideal in R . We then expect that $\bar{R}$ is a regular ring and $I\bar{R}$ is normal; that is, all the powers $I^n\bar{R}$ are integrally closed, once $\bar{e}_I^1(R) = 0$. This is the case when $\bar{R}$ is a Cohen-Macaulay ring, as we will show in the following.

Theorem 3.1. *Let R be an analytically unramified local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let S be an overring of R and assume that S is a finitely generated R -module with $\dim_R S/R < d$. Let I be an $\mathfrak{m}$ -primary ideal in R such that $\bar{e}_I^1(R) = 0$. If $\text{depth}_R S = d$, then S is a regular ring, $S = \bar{R}$, and $I\bar{R}$ is normal.*

Proof. We may assume that R is complete. Let $Q(R)$ be the total quotient ring of R . We notice that S is a Cohen-Macaulay R -module with $\dim_R S = d$; hence R is unmixed. Therefore $S \subseteq Q(R)$, as $\dim_R S/R < d$, so that $S \subseteq \bar{R}$. Since R is complete, we get a decomposition $S = \prod_{i=1}^{\ell} S_i$ of S , where S_i is a Cohen-Macaulay local ring with $\dim S_i = d$. Consequently, for the same reason as in the proof of Theorem 1.1 we have

$$\bar{e}_I^1(R) \geq \sum_{i=1}^{\ell} \lambda_R(S_i/\mathfrak{m}_i) \cdot \bar{e}_{IS_i}^1(S_i) \geq 0,$$

where $\mathfrak{m}_i$ is the maximal ideal in S_i ; hence $\bar{e}_{IS_i}^1(S_i) = 0$ for each $1 \leq i \leq \ell$. As $\bar{e}_{IS_i}^1(S_i) \geq e_{IS_i}^1(S_i) \geq 0$, we have $e_{IS_i}^1(S_i) = 0$, so that $\overline{IS_i}$ is a parameter ideal in S_i . Hence $\overline{IS_i} = IS_i$. Therefore by [G, Theorem (3.1)] S_i is a regular local ring and IS_i is normal. Thus S is regular and IS is normal, whence $S = \bar{R}$. $\square$

Corollary 3.2. *Let R be a two-dimensional analytically unramified unmixed local ring with maximal ideal $\mathfrak{m}$ and let I be an $\mathfrak{m}$ -primary ideal in R . If $\bar{e}_I^1(R) = 0$, then $\bar{R}$ is a regular ring and $I\bar{R}$ is normal.*

Proof. Notice that $\bar{R}$ is a finitely generated R -module and $\text{depth}_R \bar{R} = 2$, because R is analytically unramified and unmixed with $\dim R = 2$, whence the assertion follows from Theorem 3.1, taking $S = \bar{R}$. $\square$

Remark 3.3. The ring R itself is, however, not necessarily a regular local ring even if $\dim R = 2$. Let us note an example. We look at the local ring

$$R = k[[X, Y, Z, W]]/[(X, Y) \cap (Z, W)],$$

where $k[[X, Y, Z, W]]$ is the formal power series ring over a field k . We then have $\bar{e}_{\mathfrak{m}}^1(R) = 0$ and $\bar{e}_{\mathfrak{m}}^2(R) = -1$. The ring R is Buchsbaum but not Cohen-Macaulay, while

$$\bar{R} = k[[X, Y]] \times k[[Z, W]]$$

is a regular ring.

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