ON THE ASYMPTOTIC BEHAVIOR OF WEAKLY LACUNARY SERIES

C. AISTLEITNER, I. BERKES, AND R. TICHY

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ABSTRACT. Let f be a measurable function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x) dx = 0, \quad \text{Var}_{[0,1]} f < +\infty,$$

and let $(n_k)_{k \geq 1}$ be a sequence of integers satisfying $n_{k+1}/n_k \geq q > 1$ ($k = 1, 2, \dots$). By the classical theory of lacunary series, under suitable Diophantine conditions on n_k , $(f(n_k x))_{k \geq 1}$ satisfies the central limit theorem and the law of the iterated logarithm. These results extend for a class of subexponentially growing sequences $(n_k)_{k \geq 1}$ as well, but as Fukuyama showed, the behavior of $f(n_k x)$ is generally not permutation-invariant; e.g. a rearrangement of the sequence can ruin the CLT and LIL. In this paper we construct an infinite order Diophantine condition implying the permutation-invariant CLT and LIL without any growth conditions on $(n_k)_{k \geq 1}$ and show that the known finite order Diophantine conditions in the theory do not imply permutation-invariance even if $f(x) = \sin 2\pi x$ and $(n_k)_{k \geq 1}$ grows almost exponentially. Finally, we prove that in a suitable statistical sense, for almost all sequences $(n_k)_{k \geq 1}$ growing faster than polynomially, $(f(n_k x))_{k \geq 1}$ has permutation-invariant behavior.

1. INTRODUCTION

Let f be a measurable function satisfying

$$(1) \quad f(x+1) = f(x), \quad \int_0^1 f(x) dx = 0, \quad \text{Var}_{[0,1]} f < +\infty$$

and let $(n_k)_{k \geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition

$$(2) \quad n_{k+1}/n_k \geq q > 1 \quad (k = 1, 2, \dots).$$

In the case $n_k = 2^k$, Kac [14] proved that $f(n_k x)$ satisfies the central limit theorem

$$(3) \quad N^{-1/2} \sum_{k=1}^N f(n_k x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

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with respect to the probability space $[0, 1]$ equipped with the Lebesgue measure, where

$$\sigma^2 = \int_0^1 f^2(x) dx + 2 \sum_{k=1}^{\infty} \int_0^1 f(x) f(2^k x) dx.$$

Gaposhkin [11] extended (3) to the case when the fractions n_{k+1}/n_k are all integers or if $n_{k+1}/n_k \rightarrow \alpha$, where α^r is irrational for $r = 1, 2, \dots$. On the other hand, an example of Erdős and Fortet (see [15], p. 646) shows that the CLT (3) fails if $n_k = 2^k - 1$. Gaposhkin also showed (see [12]) that the asymptotic behavior of $\sum_{k=1}^N f(n_k x)$ is intimately connected with the number of solutions of the Diophantine equation

$$an_k + bn_l = c, \quad 1 \leq k, l \leq N.$$

Improving these results, Aistleitner and Berkes [1] gave a necessary and sufficient condition for the CLT (3). For related laws of the iterated logarithm, see [5], [11], [13], and [17].

The previous results show that for arithmetically “nice” sequences $(n_k)_{k \geq 1}$, the system $f(n_k x)$ behaves like a sequence of independent random variables. However, as an example of Fukuyama [9] shows, this result is not permutation-invariant: a rearrangement of $(n_k)_{k \geq 1}$ can change the variance of the limiting Gaussian law or ruin the CLT altogether. A complete characterization of the permutation-invariant CLT and LIL for $f(n_k x)$ under the Hadamard gap condition (2) is given in our forthcoming paper [3]. In particular, it is shown there that in the harmonic case $f(x) = \cos 2\pi x$, $f(x) = \sin 2\pi x$ the CLT and LIL for $f(n_k x)$ hold after any permutation of $(n_k)_{k \geq 1}$.

For subexponentially growing $(n_k)_{k \geq 1}$ the situation changes radically. Note that in the case $f(x) = \cos 2\pi x$, $f(x) = \sin 2\pi x$, the unpermuted CLT and LIL remain valid under the weaker gap condition

$$n_{k+1}/n_k \geq 1 + ck^{-\alpha}, \quad 0 < \alpha < 1/2;$$

see Erdős [8], Takahashi [18], [19]. However, as the following theorem shows, the slightest weakening of the Hadamard gap condition (2) can ruin the permutation-invariant CLT and LIL.

Theorem 1. *For any positive sequence $(\varepsilon_k)_{k \geq 1}$ tending to 0, there exists a sequence $(n_k)_{k \geq 1}$ of positive integers satisfying*

$$(4) \quad n_{k+1}/n_k \geq 1 + \varepsilon_k, \quad k \geq k_0$$

and a permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ of the positive integers such that

$$(5) \quad N^{-1/2} \sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x - b_N \xrightarrow{\mathcal{D}} G,$$

where G is a non-Gaussian distribution with characteristic function given by (14)-(16) and $(b_N)_{N \geq 1}$ is a numerical sequence with $b_N = O(1)$. Moreover, there exists a permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ of the positive integers such that

$$(6) \quad \limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x}{\sqrt{2N \log \log N}} = +\infty \quad \text{a.e.}$$

As we will see, with a slight change of the norming sequence $N^{-1/2}$ in (5) the limit distribution G can also be chosen as the Cauchy distribution with density $\pi^{-1}(1+x^2)^{-1}$. The proof of Theorem 1 will also show that (6) can be improved to

$$(7) \quad \limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x}{\sqrt{N \log N}} > 0 \quad \text{a.e.}$$

It should be noted that the subexponential gap condition (4) does not imply the permutation-invariant behavior of $f(n_k x)$ even for arithmetically “nice” sequences $(n_k)_{k \geq 1}$. Indeed, the sequence $(n_k)_{k \geq 1}$ in Theorem 1 can be chosen so that it satisfies conditions **B**, **C**, **G** in our paper [7] implying very strong independence properties of $\cos 2\pi n_k x$, $\sin 2\pi n_k x$, including the CLT and LIL. In fact, it is not easy to construct subexponential sequences $(n_k)_{k \geq 1}$ satisfying the permutation-invariant CLT and LIL: the only known example (see [2]) is the Hardy-Littlewood-Pólya sequence, i.e. the sequence generated by finitely many primes and arranged in increasing order; the proof uses deep number-theoretic tools. The purpose of this paper is to introduce a new, infinite order Diophantine condition $\mathbf{A}_\omega$ which implies the permutation-invariant CLT and LIL for $f(n_k x)$ and then to show that, in a suitable statistical sense, almost all sequences $(n_k)_{k \geq 1}$ growing faster than polynomially satisfy $\mathbf{A}_\omega$. Thus, despite the difficulties to construct explicit examples, the permutation-invariant CLT and LIL are rather the rule than the exception.

Given a nondecreasing sequence $\omega = (\omega_1, \omega_2, \dots)$ of positive numbers tending to $+\infty$, let us say that a sequence $(n_k)_{k \geq 1}$ of different positive integers satisfies

Condition $\mathbf{A}_\omega$ if for any $N \geq N_0$ the Diophantine equation

$$(8) \quad a_1 n_{k_1} + \dots + a_r n_{k_r} = 0, \quad 2 \leq r \leq \omega_N, \quad 0 < |a_1|, \dots, |a_r| \leq N^{\omega_N}$$

with different indices k_j and nonzero integer coefficients a_j has only such solutions, where all n_{k_j} belong to the smallest N elements of the sequence $(n_k)_{k \geq 1}$.

Clearly, this property is permutation-invariant and it implies that for any fixed nonzero integer coefficients a_j the number of solutions of (8) with different indices k_j is at most N^r .

Theorem 2. *Let $\omega = (\omega_1, \omega_2, \dots)$ be a nondecreasing sequence tending to $+\infty$ and let $(n_k)_{k \geq 1}$ be a sequence of different positive integers satisfying condition $\mathbf{A}_\omega$. Then for any f satisfying (1) we have*

$$(9) \quad N^{-1/2} \sum_{k=1}^N f(n_k x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \|f\|^2),$$

where $\|f\|$ denotes the $L_2(0, 1)$ norm of f . If $\omega_k \geq (\log k)^\alpha$ for some $\alpha > 0$ and $k \geq k_0$, then we also have

$$(10) \quad \limsup_{N \rightarrow \infty} \frac{\sum_{k=1}^N f(n_k x)}{(2N \log \log N)^{1/2}} = \|f\| \quad \text{a.e.}$$

Condition $\mathbf{A}_\omega$ is different from the usual Diophantine conditions in lacunarity theory, which typically involve 4 or fewer terms. In contrast, $\mathbf{A}_\omega$ is an “infinite order” condition; namely, it involves equations with arbitrary large order. As noted, the usual Diophantine conditions do not suffice in Theorem 2. Given any $\omega_k \uparrow \infty$, it is not hard to see that any sufficiently rapidly growing sequence $(n_k)_{k \geq 1}$ satisfies $\mathbf{A}_\omega$; on the other hand, we do not have any “concrete” subexponential examples for

$\mathbf{A}_\omega$. However, we will show that, in a suitable statistical sense, almost all sequences growing faster than polynomially satisfy condition $\mathbf{A}_\omega$ for some appropriate ω . To make this precise requires defining a probability measure over the set of such sequences, or, equivalently, a natural random procedure to generate such sequences. A simple procedure is to choose n_k independently and uniformly from the integers in the interval

$$(11) \quad I_k = [a(k-1)^{\omega_{k-1}}, ak^{\omega_k}), \quad k = 1, 2, \dots$$

Note that the length of I_k is at least $a\omega_k(k-1)^{\omega_{k-1}} \geq a\omega_1$ for $k = 2, 3, \dots$ and equals a for $k = 1$, and thus choosing a large enough, each I_k contains at least one integer. Let μ_ω be the distribution of the random sequence $(n_k)_{k \geq 1}$ in the product space $I_1 \times I_2 \times \dots$.

Theorem 3. *Let $\omega_k \uparrow \infty$ and let f be a function satisfying (1). Then with probability one with respect to μ_ω the sequence $(f(n_k x))_{k \geq 1}$ satisfies the CLT (9) after any permutation of its terms, and if $\omega_k \geq (\log k)^\alpha$ for some $\alpha > 0$ and $k \geq k_0$, $(f(n_k x))_{k \geq 1}$ also satisfies the LIL (10) after any permutation of its terms.*

The sequences $(n_k)_{k \geq 1}$ provided by μ_ω satisfy $n_k = O(k^{\omega_k})$; for slowly increasing ω_k , the so-obtained sequences grow much slower than exponentially, in fact they grow barely faster than polynomial speed. If ω_k grows so slowly that $\omega_k - \omega_{k-1} = o((\log k)^{-1})$, then the so-obtained sequence $(n_k)_{k \geq 1}$ has the precise speed $n_k \sim k^{\omega_k}$. We do not know if there exist polynomially growing sequences $(n_k)_{k \geq 1}$ satisfying the permutation-invariant CLT or LIL. The proof of Theorem 3 will also show that with probability 1, the sequences provided by μ_ω satisfy $\mathbf{A}_{\omega^*}$ with $\omega^* = (c\omega_1^{1/2}, c\omega_2^{1/2}, \dots)$.

2. PROOFS

2.1. Proof of Theorem 1. We begin with the CLT part. Let $(\varepsilon_k)_{k \geq 1}$ be a positive sequence tending to 0. Let $m_1 < m_2 < \dots$ be positive integers such that $m_{k+1}/m_k \geq 2^{k^2}$, $k = 1, 2, \dots$ and all the m_k are powers of 2; let $r_1 \leq r_2 \leq \dots$ be positive integers satisfying $1 \leq r_k \leq k^2$. Put $I_k = \{m_k, 2m_k, \dots, r_k m_k\}$; clearly the sets I_k , $k = 1, 2, \dots$ are disjoint. Define the sequence $(n_k)_{k \geq 1}$ by

$$(12) \quad (n_k)_{k \geq 1} = \bigcup_{j=1}^{\infty} I_j.$$

Clearly, if $n_k, n_{k+1} \in I_j$, then $n_{k+1}/n_k \geq 1 + 1/r_j$ and thus if r_j grows sufficiently slowly, the sequence $(n_k)_{k \geq 1}$ satisfies the gap condition (4). Also, if r_j grows sufficiently slowly, there exists a subsequence $(n_{k_\ell})_{\ell \geq 1}$ of $(n_k)_{k \geq 1}$ which has exactly the same structure as the sequence in (12), just with $r_k \sim k$. By the proof of Theorem 1 in [4], $(\cos 2\pi n_{k_\ell} x)_{\ell \geq 1}$ satisfies

$$(13) \quad \frac{1}{\sqrt{N}} \sum_{\ell=1}^N \cos 2\pi n_{k_\ell} x - b_N \xrightarrow{\mathcal{D}} G,$$

where $(b_N)_{N \geq 1}$ is a numerical sequence with $b_N = O(1)$ and G is a non-Gaussian infinitely divisible distribution with characteristic function

$$(14) \quad \exp \left\{ \int_{\mathbb{R} \setminus \{0\}} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dL(x) \right\},$$

where

$$(15) \quad L(x) = \begin{cases} -\frac{1}{\pi} \int_x^1 \frac{F(t)}{t} dt & \text{if } 0 < x \leq 1, \\ \frac{1}{\pi} \int_{-x}^1 \frac{G(t)}{t} dt & \text{if } -1 \leq x < 0, \\ 0 & \text{if } |x| > 1 \end{cases}$$

and

$$(16) \quad F(t) = \lambda\{x > 0 : \sin x/x \geq t\}, \quad G(t) = \lambda\{x > 0 : \sin x/x \leq -t\} \quad (t > 0),$$

where λ is the Lebesgue measure. Define a permutation σ in the following way:

- for $k \notin \{1, 2, 4, \dots, 2^m, \dots\}$, $\sigma(k)$ takes the values of the set $\{k_1, k_2, \dots\}$ in consecutive order;
- for $k \in \{1, 2, 4, \dots, 2^m, \dots\}$, $\sigma(k)$ takes the values of the set $\mathbb{N} \setminus \{k_1, k_2, \dots\}$ in consecutive order.

Then σ is a permutation of $\mathbb{N}$ and the sums

$$\sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x \quad \text{and} \quad \sum_{l=1}^N \cos 2\pi n_{k_l} x$$

differ at most in $2 \log_2 N$ terms. Therefore, (13) implies (5), proving the first part of Theorem 1.

The proof of the LIL part of Theorem 1 is modeled after the proof of Theorem 1 in Berkes and Philipp [6]. Similarly as above, we construct a sequence $(n_k)_{k \geq 1}$ satisfying (4) that contains a subsequence $(\mu_k)_{k \geq 1}$ of the form (12) with $I_k = \{m_k, 2m_k, \dots, r_k m_k\}$, where $r_k \sim k \log k$ and $(m_k)_{k \geq 1}$ is growing fast; specifically we choose m_k in such a way that it is a power of 2 and $m_{k+1} \geq r_k 2^{2k} m_k$. Let $\mathcal{F}_i$ denote the σ -field generated of the dyadic intervals

$$[\nu 2^{-(\log_2 m_i)-i}, (\nu+1) 2^{-(\log_2 m_i)-i}), \quad 0 \leq \nu < 2^{(\log_2 m_i)+i}.$$

Write

$$X_i = \cos 2\pi m_i x + \dots + \cos 2\pi r_i m_i x$$

and

$$Z_i = \mathbb{E}(X_i | \mathcal{F}_i).$$

Then for all $x \in (0, 1)$,

$$|X_i(x) - Z_i(x)| \ll r_i^2 m_i 2^{-(\log_2 m_i)-i}$$

and thus

$$\sum_{i \geq 1} |X_i(x) - Z_i(x)| < \infty \quad \text{for all } x \in (0, 1).$$

As in [6, Lemma 2.1], the random variables $Z_1, Z_2, \dots$ are independent, and as in [6, Lemma 2.2], for almost every $x \in (0, 1)$ we have

$$\limsup_{i \rightarrow \infty} X_i/r_i \geq 2/\pi.$$

Assume that for a fixed x and some $i \geq 1$ we have $X_i/r_i \geq 1/\pi$. Then either

$$|X_1 + \cdots + X_{i-1}| \geq r_i/2\pi$$

or

$$|X_1 + \cdots + X_i| \geq r_i/2\pi.$$

Since the total number of summands in $X_1, \dots, X_i$ is $\ll i^2 \log i$, and since

$$r_i \gg (i^2 \log i)^{1/2} (\log(i^2 \log i))^{1/2},$$

we have

$$\limsup_{N \rightarrow \infty} \frac{\left| \sum_{k=1}^N \cos 2\pi \mu_k x \right|}{\sqrt{N \log N}} > 0 \quad \text{a.e.},$$

and, in particular,

$$\limsup_{N \rightarrow \infty} \frac{\left| \sum_{k=1}^N \cos 2\pi \mu_k x \right|}{\sqrt{2N \log \log N}} = +\infty \quad \text{a.e.}$$

Thus we constructed a subsequence of $(n_k)_{k \geq 1}$ failing the LIL and similarly as above, we can construct a permutation $(n_{\sigma(k)})_{k \geq 1}$ of (n_k) failing the LIL as well.

Note that the just completed proof of Theorem 1 provides the stronger relation (7) instead of (6). Also, if relation $r_k \sim k$ just preceding relation (13) is replaced by $r_k \sim k \log \log k$, then the proof of Theorem 2 in [4] shows that (5) will be replaced by

$$a_N^{-1} \sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x - b_N \xrightarrow{\mathcal{D}} G,$$

where G is the Cauchy distribution with density $\pi^{-1}(1+x^2)^{-1}$ and $a_N \sim c\sqrt{N/\log \log N}$ with some $c > 0$ and $b_N = O(1)$.

2.2. Proof of Theorem 2.

Lemma 1. *Let $\omega_k \uparrow \infty$ and let $(n_k)_{k \geq 1}$ be a sequence of different positive integers satisfying condition $\mathbf{A}_\omega$. Let f satisfy (1) and put $S_N = \sum_{k=1}^N f(n_k x)$, $\sigma_N = (\mathbb{E} S_N^2)^{1/2}$. Then for any $p \geq 3$ we have*

$$\mathbb{E} S_N^p = \begin{cases} \frac{p!}{(p/2)!} 2^{-p/2} \sigma_N^p + O(T_N) & \text{if } p \text{ is even,} \\ O(T_N) & \text{if } p \text{ is odd,} \end{cases}$$

where

$$T_N = \exp(p^2) N^{(p-1)/2} (\log N)^p$$

and the constants implied by the O are absolute.

Proof. Fix $p \geq 2$ and choose the integer N so large that $\omega_{[N^{1/4}]} \geq 8p$. Without loss of generality we may assume that f is an even function and that $\|f\|_\infty \leq 1$, $\text{Var}_{[0,1]} f \leq 1$; the proof in the general case is similar. Let

$$(17) \quad f \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x$$

be the Fourier series of f . $\text{Var}_{[0,1]} f \leq 1$ implies that

$$(18) \quad |a_j| \leq j^{-1}$$

(see Zygmund [20, p. 48]), and writing

$$g(x) = \sum_{j=1}^{N^p} a_j \cos 2\pi jx, \quad r(x) = f(x) - g(x),$$

we have

$$\|g\|_\infty \leq \text{Var}_{[0,1]} f + \|f\|_\infty \leq 2, \quad \|r\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq 3$$

by (4.12) of Chapter II and (1.25) and (3.5) of Chapter III of Zygmund [20]. Letting $\|\cdot\|$ and $\|\cdot\|_p$ denote the $L_2(0,1)$, resp. $L_p(0,1)$, norms, (18) yields for any positive integer n ,

$$(19) \quad \|r(nx)\|^2 = \|r(x)\|^2 = \frac{1}{2} \sum_{j=N^p+1}^{\infty} a_j^2 \leq N^{-p}.$$

By Minkowski's inequality,

$$\|S_N\|_p \leq \left\| \sum_{k=1}^N g(n_k x) \right\|_p + \left\| \sum_{k=1}^N r(n_k x) \right\|_p$$

and

$$(20) \quad \left\| \sum_{k=1}^N r(n_k x) \right\|_p \leq 3 \sum_{k=1}^N \|r(n_k x)/3\|_p \leq 3 \sum_{k=1}^N \|r(n_k x)/3\|^{2/p} \leq 3 \sum_{k=1}^N N^{-1} \leq 3.$$

Similarly,

$$\left| \|S_N\| - \left\| \sum_{k=1}^N g(n_k x) \right\| \right| \leq \left\| \sum_{k=1}^N r(n_k x) \right\| \leq N^{-\frac{p}{2}+1},$$

and therefore

$$\begin{aligned} & \left| \|S_N\|^p - \left\| \sum_{k=1}^N g(n_k x) \right\|^p \right| \\ & \leq p \max \left(\|S_N\|^{p-1}, \left\| \sum_{k=1}^N g(n_k x) \right\|^{p-1} \right) \cdot \left| \|S_N\| - \left\| \sum_{k=1}^N g(n_k x) \right\| \right| \\ & \ll p (N(\log \log N)^2)^{\frac{p-1}{2}} N^{-\frac{p}{2}+1} \\ (21) \quad & \ll p(\log \log N)^{p-1} N^{1/2} \end{aligned}$$

since by a result of Gál [10] and Koksma [16],

$$(22) \quad \|S_N\|^2 \ll N(\log \log N)^2 \quad \text{and} \quad \left\| \sum_{k=1}^N g(n_k x) \right\|^2 \ll N(\log \log N)^2,$$

where the implied constants are absolute.

By expanding and using elementary properties of the trigonometric functions we get

$$\begin{aligned} & \mathbb{E} \left(\sum_{k=1}^N g(n_k x) \right)^p \\ (23) \quad & = 2^{-p} \sum_{1 \leq j_1, \dots, j_p \leq N^p} a_{j_1} \cdots a_{j_p} \sum_{1 \leq k_1, \dots, k_p \leq N} \mathbb{I}\{\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0\}, \end{aligned}$$

with all possibilities of the signs $\pm$ within the indicator function. Assume that $j_1, \dots, j_p$ and the signs $\pm$ are fixed, and consider a solution of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$. Then the set $\{1, 2, \dots, p\}$ can be split into disjoint sets $A_1, \dots, A_l$ such that for each such set A we have $\sum_{i \in A} \pm j_i n_{k_i} = 0$ and no further subsums of these sums are equal to 0. Group the terms of $\sum_{i \in A} \pm j_i n_{k_i}$ with equal k_i . If after grouping there are at least two terms, then by the restriction on subsums, the sum of the coefficients j_i in each group will be different from 0 and will not exceed

$$pN^p \leq \omega_{[N^{1/4}]} N^{\frac{1}{8}\omega_{[N^{1/4}]}} \leq 2^{\frac{1}{8}\omega_{[N^{1/4}]}} N^{\frac{1}{8}\omega_{[N^{1/4}]}} \leq N^{\frac{1}{4}\omega_{[N^{1/4}]}} \quad (N \geq N_0).$$

Also the number of terms after grouping will be at most $p \leq \omega_{[N^{1/4}]}$ and thus applying condition $\mathbf{A}_\omega$ with the index $[N^{1/4}]$ shows that within a block A the n_{k_i} belong to the smallest $[N^{1/4}]$ terms of the sequence. Thus letting $|A| = m$, the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ is at most $N^{m/4}$. If after grouping there is only one term, then all the k_i are equal and thus the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ is at most N . Thus if $m \geq 3$, then the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ in a block is at most $N^{m/3}$. If $m = 2$, then the number of solutions is clearly at most N . Thus if $s_i = |A_i|$ ($1 \leq i \leq l$) denotes the cardinality of A_i , the number of solutions of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$ admitting such a decomposition with fixed $A_1, \dots, A_l$ is at most

$$\begin{aligned} & \prod_{\{i:s_i \geq 3\}} N^{s_i/3} \prod_{\{i:s_i=2\}} N = N^{\frac{1}{3} \sum_{\{i:s_i \geq 3\}} s_i + \sum_{\{i:s_i=2\}} 1} \\ & = N^{\frac{1}{3} \sum_{\{i:s_i \geq 3\}} s_i + \frac{1}{2} \sum_{\{i:s_i=2\}} s_i} = N^{\frac{1}{3} \sum_{\{i:s_i \geq 3\}} s_i + \frac{1}{2}(p - \sum_{\{i:s_i \geq 3\}} s_i)} \\ & = N^{\frac{p}{2} - \frac{1}{6} \sum_{\{i:s_i \geq 3\}} s_i}. \end{aligned}$$

If there is at least one i with $s_i \geq 3$, then the last exponent is at most $(p-1)/2$ and since the number of partitions of the set $\{1, \dots, p\}$ into disjoint subsets is at most $p! 2^p$, we see that the number of solutions of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$, where at least one of the sets A_i has cardinality ≥ 3 , is at most $p! 2^p N^{(p-1)/2}$. If p is odd, there are no other solutions and thus using (18) the inner sum in (23) is at most $p! 2^p N^{(p-1)/2}$ and consequently, taking into account the 2^p choices for the signs ± 1 ,

$$\begin{aligned} & \left| \mathbb{E} \left(\sum_{k \leq N} g(n_k x) \right)^p \right| \\ & \leq p! 2^p N^{(p-1)/2} \sum_{1 \leq j_1, \dots, j_p \leq N^p} |a_{j_1} \cdots a_{j_p}| \ll \exp(p^2) N^{(p-1)/2} (\log N)^p. \end{aligned}$$

If p is even, there are also solutions where each A has cardinality 2. Clearly, the contribution of the terms in (23), where $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}, \dots$, is

$$\begin{aligned} & \left(\frac{1}{4} \sum_{1 \leq i, j \leq N^{2p}} \sum_{1 \leq k, \ell \leq N} a_i a_j \mathbb{I}_{\{\pm i n_k \pm j n_\ell = 0\}} \right)^{p/2} = \left(\mathbb{E} \left(\sum_{k \leq N} g(n_k x) \right)^2 \right)^{p/2} \\ &= \left\| \sum_{k \leq N} g(n_k x) \right\|^p \\ &= \|S_N\|^p + \mathcal{O} \left(p (\log \log N)^{p-1} N^{1/2} \right) \end{aligned}$$

by (21).

Since the splitting of $\{1, 2, \dots, p\}$ into pairs can be done in $\frac{p!}{(p/2)!} 2^{-p/2}$ different ways, we proved that

$$(24) \quad \mathbb{E} \left(\sum_{k \leq N} g(n_k x) \right)^p = \begin{cases} \frac{p!}{(p/2)!} 2^{-\frac{p}{2}} \sigma_N^p + O(T_N), \\ O(T_N) \end{cases}$$

according as p is even or odd; here

$$T_N = \exp(p^2) N^{(p-1)/2} (\log N)^p.$$

Now, letting $G_N = \sum_{k \leq N} g(n_k x)$ we get, using (20), (22) and (24),

$$\begin{aligned} & |\mathbb{E} S_N^p - \mathbb{E} G_N^p| \\ & \leq p \max(\|S_N\|_p^{p-1}, \|G_N\|_p^{p-1}) \cdot \|\|S_N\|_p - \|G_N\|_p\| \\ & \ll p \left(\frac{p!}{(p/2)!} 2^{-\frac{p}{2}} \sigma_N^p \right)^{\frac{p-1}{p}} \\ & \ll T_N, \end{aligned}$$

where in the last step we used the fact that $\sigma_N \ll \sqrt{N} \log \log N$ by (22). This completes the proof of Lemma 1. $\square$

Lemma 2. *Let $\omega_k \uparrow \infty$ and let $(n_k)_{k \geq 1}$ be a sequence of different positive integers satisfying condition $\mathbf{A}_\omega$. Then for any f satisfying (1) we have*

$$(25) \quad \int_0^1 \left(\sum_{k=1}^N f(n_k x) \right)^2 dx \sim \|f\|^2 N \quad \text{as } N \rightarrow \infty.$$

Proof. Clearly, $\omega_{[N^{1/4}]} \geq 4$ for sufficiently large N and thus applying Condition $\mathbf{A}_\omega$ for the index $[N^{1/4}]$ it follows that for $N \geq N_0$ the Diophantine equation

$$(26) \quad j_1 n_{i_1} + j_2 n_{i_2} = 0, \quad i_1 \neq i_2, \quad 0 < |j_1|, |j_2| \leq N$$

has only such solutions, where n_{i_1}, n_{i_2} belong to the set J_N of $[N^{1/4}]$ smallest elements of the sequence $(n_k)_{k \geq 1}$. Write $p_N(x)$ for the N -th partial sum of the Fourier series of f , and r_N for the N -th remainder term. Then we have for any f

satisfying (1),

(27)

$$\left\| \sum_{k=1}^N f(n_k x) \right\| \geq \left\| \sum_{k \in [1, N] \setminus J_N} p_N(n_k x) \right\| - \left\| \sum_{k \in J_N} f(n_k x) \right\| - \left\| \sum_{k \in [1, N] \setminus J_N} r_N(n_k x) \right\|.$$

Using the previous remark on the number of solutions of (26) we get, as in (23),

$$\left\| \sum_{k \in [1, N] \setminus J_N} p_N(n_k x) \right\| = (N - [N^{1/4}])^{1/2} \|p_N\| \sim N^{1/2} \|f\|,$$

since $\|p_N\| \rightarrow \|f\|$. Further, $\|r_N\| \ll N^{-1/2}$ by (18) and thus using Minkowski's inequality and the results of Gál and Koksma mentioned in (22), we get

$$\left\| \sum_{k \in J_N} f(n_k x) \right\| \ll N^{1/4}, \quad \left\| \sum_{k \in [1, N] \setminus J_N} r_N(n_k x) \right\| \ll \sqrt{N} \log \log N \|r_N\| \ll \log \log N.$$

These estimates, together with (27), prove Lemma 2.

Lemma 1 and Lemma 2 imply that for any fixed $p \geq 2$, the p -th moment of S_N/σ_N converges to $\frac{p!}{(p/2)!} 2^{-p/2}$ if p is even and to 0 if p is odd; in other words, the moments of S_N/σ_N converge to the moments of the standard normal distribution. By $\sigma_N \sim \|f\| \sqrt{N}$ and a well-known result in probability theory, this proves the CLT part of Theorem 2. The proof of the LIL part of Theorem 2 is more involved, and we will give just a sketch of the proof. The details can be modeled after the proof of [2, Theorem 1]. The crucial ingredient is Lemma 3 below, which yields the LIL part of Theorem 2, just as [2, Theorem 1] follows from [2, Lemma 3].

Let $\theta > 1$ and define $\Delta'_M = \{k \in \mathbb{N} : \theta^M < k \leq \theta^{M+1}\}$ and $T'_M = \sum_{k \in \Delta'_M} f(n_k x)$. By the standard method of proof of the LIL, we need precise bounds for the tails of T'_M and also, a near independence relation for the T'_M for the application of the Borel-Cantelli lemma in the lower half of the LIL. From the set Δ'_M we remove its $[\theta^{M/4}]$ elements with the smallest value of n_k (recall that the sequence $(n_k)_{k \geq 1}$ is not assumed to be increasing) and denote the remaining set by Δ_M . Since the number of removed elements is $\ll |\Delta'_M|^{1/4}$, this operation does not influence the partial sum asymptotics of T'_M . As in the proof of Lemma 1, we assume that we have a representation of f in the form (17) and that (18) holds. Define

$$g_M(x) = \sum_{j=1}^{[\theta^M]^2} a_j \cos 2\pi jx, \quad \sigma_M^2 = \int_0^1 \left(\sum_{k \in \Delta_M} g_M(n_k x) \right)^2 dx$$

and

$$T_M = \sum_{k \in \Delta_M} g_M(n_k x), \quad Z_M = T_M/\sigma_M.$$

From Lemma 2 it follows easily that

$$(28) \quad \sigma_M \gg |\Delta_M|^{1/2}.$$

Assume that $(n_k)_{k \geq 1}$ satisfies Condition $\mathbf{A}_\omega$ for a sequence $(\omega_k)_{k \geq 1}$ with $\omega_k \geq (\log k)^\alpha$ for some $\alpha > 0$, $k \geq k_0$. Without loss of generality we may assume

$0 < \alpha < 1/2$. Choose $\delta > 0$ so small that for sufficiently large r ,

$$(29) \quad \left(\log \theta^{\sqrt{r}/4} \right)^\alpha > 4 (\log \theta^r)^\delta. \quad \square$$

Lemma 3. *For sufficiently large M, N satisfying $N^{1-\alpha/2} \leq M \leq N$, and for positive integers p, q satisfying $p + q \leq (\log \theta^N)^\delta$ we have*

$$\mathbb{E} Z_M^p Z_N^q = \begin{cases} \frac{p!}{(p/2)! 2^{p/2}} \frac{q!}{(q/2)! 2^{q/2}} + \mathcal{O}(R_{M,N}) & \text{if } p, q \text{ are even,} \\ \mathcal{O}(R_{M,N}) & \text{otherwise,} \end{cases}$$

where

$$R_{M,N} = 2^{p+q} (p+q)! (\log M)^{p+q} |\Delta_M|^{-1/2}.$$

Proof. Note that $N \leq M^{1+3\alpha/4}$ and thus, setting $L = [\theta^{M/4}]$, relation $p + q \leq (\log \theta^N)^\delta$ and (29) imply

$$p + q \leq \frac{1}{4} (\log \theta^{\sqrt{N}/4})^\alpha \leq \frac{1}{4} (\log \theta^{M/4})^\alpha \leq \omega_L$$

and by a simple calculation,

$$(p+q) (\theta^N)^2 \leq [\theta^{M/4}]^{(\log[\theta^{M/4}])^\alpha} \leq L^{\omega_L}$$

provided N is large enough. Applying condition $\mathbf{A}_\omega$ we get that for all solutions of the equation

$$(30) \quad \pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} \pm j_{p+1} n_{k_{p+1}} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0$$

with different indices $k_1, \dots, k_{p+q}$, where

$$(31) \quad 1 \leq j_i \leq (p+q) (\theta^N)^2,$$

the n_{k_j} belong to the $[\theta^{M/4}]$ smallest elements of $(n_k)_{k \geq 1}$. By construction not a single one of these elements is contained in Δ_M or Δ_N . Thus the equation (30) subject to (31) has no solution $(k_1, \dots, k_{p+q})$, where $k_1, \dots, k_{p+q}$ are different and satisfy

$$k_1, \dots, k_p \in \Delta_M, \quad k_{p+1}, \dots, k_{p+q} \in \Delta_N.$$

Now

$$(32) \quad \mathbb{E} Z_M^p Z_N^q = \frac{2^{-p-q}}{\sigma_M^p \sigma_N^q} \sum_{\substack{1 \leq j_1, \dots, j_p \leq [\theta^{M/4}]^2, \\ 1 \leq j_{p+1}, \dots, j_{p+q} \leq [\theta^{N/4}]^2}} \times \sum_{\substack{k_1, \dots, k_p \in \Delta_M, \\ k_{p+1}, \dots, k_{p+q} \in \Delta_N}} a_{j_1} \dots a_{j_{p+q}} \mathbf{1}\{\pm j_1 n_{k_1} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0\}.$$

If for some $k_1, \dots, k_{p+q}$ we have (note that in (32) these indices need not be different)

$$(33) \quad \pm j_1 n_{k_1} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0,$$

then grouping the terms of the equation according to identical indices, we get a new equation of the form

$$j'_1 n_{l_1} + \dots + j'_s n_{l_s} = 0, \quad l_1 < \dots < l_s, \quad s \leq p+q, \quad j'_i \leq (p+q) (\theta^N)^2$$

and using the above observation, all the coefficients $j'_1, \dots, j'_s$ must be equal to 0. In other words, in any solution of (33) the terms can be divided into groups such that in each group the n_{k_j} are equal and the sum of the coefficients is 0. Consider

first the solutions where all groups have cardinality 2. This can happen only if both p and q are even, and similarly to the proof of Lemma 1, the contribution of such solutions in (32) is

$$\frac{p!}{(p/2)!2^{p/2}} \frac{q!}{(q/2)!2^{q/2}}.$$

Consider now the solutions of (33), where at least one group has cardinality ≥ 3 . Clearly the sets $\{k_1, \dots, k_p\}$ and $\{k_{p+1}, \dots, k_{p+q}\}$ are disjoint; let us denote the number of groups within these two sets by R and S , respectively. Evidently $R \leq p/2$, $S \leq q/2$, and at least one of the inequalities is strict. Fixing $j_1, \dots, j_{p+q}$ and the number of such solutions cannot exceed

$$|\Delta_M|^R |\Delta_N|^S \leq |\Delta_M|^{p/2} |\Delta_N|^{q/2} |\Delta_M|^{-1/2} \ll \sigma_M^p \sigma_N^q |\Delta_M|^{-1/2},$$

where we used (28) and the fact that $|\Delta_M| \leq |\Delta_N|$. Since the number of partitions of the set $\{1, 2, \dots, p+q\}$ into disjoint subsets is at most $(p+q)!2^{p+q}$ and since the number of choices for the signs $\pm$ in (33) is at most 2^{p+q} , we see, after summing over all possible values of $j_1, \dots, j_{p+q}$, that the contribution of the solutions containing at least one group with cardinality ≥ 3 in (32) is at most $2^{p+q}(p+q)!|\Delta_M|^{-1/2}(\log[\theta^N])^{p+q}$. This completes the proof of Lemma 3. $\square$

The rest of the proof of the LIL part of Theorem 2 can be modeled following the lines of Lemma 4, Lemma 5, Lemma 6 and the proof of Theorem 1 in [2].

2.3. Proof of Theorem 3. Let $\omega_k \uparrow \infty$ and set $\eta_k = \frac{1}{2}\omega_k^{1/2}$, $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots)$. Clearly

$$(34) \quad (2k)^{\eta_k^2 + 2\eta_k} \leq (2k)^{\omega_k/2} \leq k^{-2}|I_k| \quad \text{for } k \geq k_0$$

since, as we noted, $|I_k| \geq a\omega_k(k-1)^{\omega_k-1} \geq (k/2)^{\omega_k-1}$ for large k . We choose n_k , $k = 1, 2, \dots$, independently and uniformly from the integers of the intervals I_k in (11). We claim that, with probability 1, the sequence $(n_k)_{k \geq 1}$ is increasing and satisfies condition $\mathbf{A}_{\boldsymbol{\eta}}$. To see this, let $k \geq 1$ and consider the numbers of the form

$$(35) \quad (a_1 n_{i_1} + \dots + a_s n_{i_s})/d,$$

where $1 \leq s \leq \eta_k$, $1 \leq i_1, \dots, i_s \leq k-1$, $a_1, \dots, a_s, d$ are nonzero integers with $|a_1|, \dots, |a_s|, |d| \leq k^{\eta_k}$. Since the number of values in (35) is at most $(2k)^{\eta_k^2 + 2\eta_k}$, and by (34), the probability that n_k equals any of these numbers is at most k^{-2} . Thus by the Borel-Cantelli lemma, with probability 1 for $k \geq k_1$, n_k will be different from all the numbers in (35) and thus the equation

$$a_1 n_{i_1} + \dots + a_s n_{i_s} + a_{s+1} n_k = 0$$

has no solution with $1 \leq s \leq \eta_k$, $1 \leq i_1 < \dots < i_s \leq k-1$, $0 < |a_1|, \dots, |a_{s+1}| \leq k^{\eta_k}$. By monotonicity, the equation

$$a_1 n_{i_1} + \dots + a_s n_{i_s} = 0$$

has no solutions provided the indices i_ν are all different, the maximal index is at least k , the number of terms is at most η_k , and $0 < |a_1|, \dots, |a_s| \leq k^{\eta_k}$. In other words, (n_k) satisfies condition $\mathbf{A}_{\boldsymbol{\eta}}$. Now using Theorem 2, we get Theorem 3.

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INSTITUTE OF MATHEMATICS A, GRAZ UNIVERSITY OF TECHNOLOGY, STEYRERGASSE 30, 8010 GRAZ, AUSTRIA

E-mail address: aistleitner@math.tugraz.at

INSTITUTE OF STATISTICS, GRAZ UNIVERSITY OF TECHNOLOGY, MÜNZGRABENSTRASSE 11, 8010 GRAZ, AUSTRIA

E-mail address: berkes@tugraz.at

INSTITUTE OF MATHEMATICS A, GRAZ UNIVERSITY OF TECHNOLOGY, STEYRERGASSE 30, 8010 GRAZ, AUSTRIA

E-mail address: tichy@tugraz.at