A CHARACTERIZATION OF THE SINGULAR TIME OF THE MEAN CURVATURE FLOW

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ABSTRACT. In this note we investigate the behaviour at finite-time singularities of the mean curvature flow of compact Riemannian submanifolds $M_t^m \hookrightarrow (N^{m+n}, h)$. We show that they are characterized by the blow-up of a trace $A = H \cdot \text{II}$ of the square of the second fundamental form.

1. INTRODUCTION

It is well known that the mean curvature flow $\partial_t F = H$ of submanifolds $F_t : M^m \hookrightarrow \mathbb{R}^{m+n}$ has finite-time singularities characterized by the blowup of the second fundamental form II :

Theorem 1.1 (Huisken [4]). *Suppose $T < \infty$ is the first singular time for a compact mean curvature flow. Then $\max_{M_t} |\text{II}| \rightarrow \infty$ as $t \rightarrow T$.*

We will prove that in fact it suffices to consider the tensor $A_{ij} = H^\alpha h_{ij\alpha}$, where $H = \text{tr } \text{II}$ is the mean curvature and h are the components of II :

Theorem 1.2. *Let (N, h) be a Riemannian manifold with bounded geometry. Suppose $T < \infty$ is the first singular time for a mean curvature flow of compact submanifolds of (N, h) . Then $\max_{M_t} |A| \rightarrow \infty$ as $t \rightarrow T$.*

By slightly modifying the proof of Theorem 1.2, we also obtain

Theorem 1.3. *Suppose that along the flow, $|\text{II}(x, t)|^p (T - t) \leq C$ for some $p > 1$. Then $\max_{M_t} |H| \rightarrow \infty$ as $t \rightarrow T$.*

2. PRELIMINARIES

First we recall some evolution equations for the flow. We use indices $1 \leq i, j, k, l \leq m$, $m + 1 \leq \alpha, \beta, \gamma \leq m + n$. Let $h_{j\alpha}$ denote the α th component of $\text{II}(\partial_i, \partial_j)$, let H^α denote the α th component of the mean curvature H , and let g_{ij} denote the induced metric on M . Also, let $\bar{R}$ with four indices denote the Riemannian curvature of (N, h) , and let $\bar{R}$ with two indices denote the Ricci curvature of (N, h) . Moreover, let ∇_i denote the tangential covariant derivative in the direction

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i , and let $\bar{\nabla}$ denote the covariant derivative of h . We use the summation convention on upper and lower indices.

Lemma 2.1 (Huisken [4], Wang [9]). *Along a mean curvature flow $M_t \hookrightarrow (N, h)$, we have*

$$\begin{aligned} (1) \quad & \partial_t g_{ij} = -2H^\alpha h_{ij\alpha}, \\ (2) \quad & \partial_t \text{dvol} = -|H|^2 \text{dvol}, \\ (3) \quad & \partial_t h_{ij\alpha} = \Delta h_{ij\alpha} + (\bar{\nabla}_k \bar{R})_{\alpha ij}{}^k + (\bar{\nabla}_j \bar{R})_{\alpha ki}{}^k - 2\bar{R}_{lij k} h_\alpha{}^{lk} \\ & + 2\bar{R}_{\alpha\beta j k} h_i{}^k{}^\beta + 2\bar{R}_{\alpha\beta i k} h_j{}^k{}^\beta - \bar{R}_{lki}{}^k h_j{}^l{}_\alpha - \bar{R}_{lkj}{}^k h_i{}^l{}_\alpha + \bar{R}_{\alpha k\beta}{}^k h_{ij}{}^\beta \\ & + h_{il\alpha}(h_{lk\gamma} h_j{}^{k\gamma} - h_{kj\gamma} H^\gamma) + h_{lk\alpha}(h^{lk}{}_\gamma h_{ij}{}^\gamma - h_{lj\gamma} h_i{}^{k\gamma}) \\ & + h_{ik\beta}(h_l{}^{k\beta} h_\alpha{}^l{}_j - h_{lj}{}^\beta h_\alpha{}^{lk}) - h_{\alpha j k} h_{\beta i}{}^k H^\beta + h_{ij\beta} \langle e_\beta, \bar{\nabla}_H e_\alpha \rangle. \end{aligned}$$

By integrating the evolution equation for $|\nabla^s \Pi|$ and using the Hölder and Morrey inequalities, one can obtain

Theorem 2.2 (Huisken [4]). *Along the mean curvature flow, $\sup_{M \times [0, T)} |\nabla^s \Pi|$ is bounded in terms of $\sup_{M \times [0, T)} |\Pi|$ and the ambient geometry bounds.*

We recall a few lemmas about one-parameter families of Riemannian metrics:

Lemma 2.3 (Glickenstein [2]). *Suppose a one-parameter family of complete Riemannian manifolds $(M, g(t))$ is uniformly continuous in t ; that is, for any $\epsilon > 0$ there exists $\delta > 0$ so that for any t_0 , $(1 - \epsilon)g(t_0) \leq g(t) \leq (1 + \epsilon)g(t_0)$ for $t \in [t_0, t_0 + \delta]$. Then for any $p \in M$, $r > 0$, the metric balls centred at p satisfy*

$$B_{g(t_0)}(p, \frac{r}{\sqrt{1+\epsilon}}) \subseteq B_{g(t)}(p, r) \subseteq B_{g(t_0)}(p, \frac{r}{\sqrt{1-\epsilon}}).$$

Proof. Let $p, q \in M$. Let $\gamma : [0, S] \rightarrow M$ be a minimising geodesic from p to q for the metric $g(t_0)$. Then the distance $d_{g(t_0)}(p, q)$ in the metric $g(t_0)$ satisfies

$$\begin{aligned} d_{g(t_0)}(p, q) &= \int_0^S |\dot{\gamma}|_{g(t_0)}(s) ds \\ (2.1) \quad &\geq \frac{1}{\sqrt{1+\epsilon}} \int_0^S |\dot{\gamma}|_{g(t)}(s) ds \\ &\geq \frac{1}{\sqrt{1+\epsilon}} d_{g(t)}(p, q) \end{aligned}$$

so that $\frac{1}{\sqrt{1+\epsilon}} d_{g(t)}(p, q) \leq d_{g(t_0)}(p, q)$. This immediately implies

$$(2.2) \quad B_{g(t_0)}(p, \frac{r}{\sqrt{1+\epsilon}}) \subset B_{g(t)}(p, r).$$

The other inclusion is analogous. $\square$

Lemma 2.4 (Hamilton [3]). *Let $(M, g(t))$ be a one-parameter family of compact Riemannian manifolds defined for $t \in [0, T]$. Suppose that*

$$\int_0^T \max_{M_t} \left| \frac{\partial g}{\partial t} \right|_{g(t)} dt < \infty.$$

Then the metrics $g(t)$ are uniformly equivalent and converge pointwise as $t \rightarrow T$ to a continuous positive-definite metric $g(T)$.

3. II AND THE INJECTIVITY RADIUS

We will prove Theorems 1.2 and 1.3 by a blow-up argument. In particular we will use the Cheeger-Gromov convergence theorem to extract a limit of some submanifolds $F_j : M \hookrightarrow \mathbb{R}^{m+n}$, thought of as Riemannian manifolds $(M, F_j^* dx^2)$. We therefore need the following relationship between injectivity radius and the second fundamental form.

Theorem 3.1. *Let $F : M^m \hookrightarrow \mathbb{R}^{m+n}$ be an immersion with $|\text{II}| \leq C$. Then $\text{inj}(M, F^* dx^2) \geq \frac{1}{2\sqrt{2}C}$.*

We begin by considering the case of the graph of a map $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^n$, as in [7]. We need to compare the standard square-norm of certain objects, e.g. $|D^2\psi|^2 = \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq i, j \leq m}} \left(\frac{\partial^2 \psi_\alpha}{\partial x_i \partial x_j} \right)^2$, with the norms of the tensors II and ∇II in the metric g induced by the immersion. To keep the norms straight, in this section we use $|\cdot|$ for the standard square-norm and $|\cdot|_g$ for the norm in g :

$$(3.1) \quad \begin{aligned} |\text{II}|_g^2 &= h_{ij\alpha} h_{kl\beta} g^{\alpha\beta} g^{ik} g^{jl}, \\ |\nabla \text{II}|_g^2 &= \nabla_i h_{jk\alpha} \nabla_p h_{qr\beta} g^{ip} g^{jq} g^{kr} g^{\alpha\beta}. \end{aligned}$$

Lemma 3.2. *Let $\psi : D_r^m \rightarrow \mathbb{R}^n$ be a C^2 function on the disc of radius r . Then*

$$|D^2\psi|^2 \leq (1 + |D\psi|^2)^3 |\text{II}|_g^2$$

where II is the second fundamental form of the graph of ψ .

Proof. The graph of ψ has immersion map $F(x_1, \dots, x_m) = (x_1, \dots, x_m, \psi_1, \dots, \psi_n)$. We use the following tangent and normal frames, where $1 \leq i \leq m$ and $1 \leq \alpha \leq n$:

$$(3.2) \quad \begin{aligned} e_i &= (0, \dots, 0, 1, 0, \dots, 0, \frac{\partial \psi_1}{\partial x_i}, \dots, \frac{\partial \psi_n}{\partial x_i}) = (0, \dots, 0, 1, 0, \dots, 0, D_i \psi), \\ \nu_\alpha &= (-\frac{\partial \psi_\alpha}{\partial x_1}, \dots, -\frac{\partial \psi_\alpha}{\partial x_m}, 0, \dots, 0, 1, 0, \dots, 0) = (-D\psi_\alpha, 0, \dots, 0, 1, 0, \dots, 0). \end{aligned}$$

These choices induce the metric on the tangent bundle of the graph, which we denote by g with Latin indices:

$$(3.3) \quad g_{ij} = e_i \cdot e_j = \delta_{ij} + D_i \psi \cdot D_j \psi.$$

We also get a metric on the normal bundle, which we denote by g with Greek indices:

$$(3.4) \quad g_{\alpha\beta} = \nu_\alpha \cdot \nu_\beta = \delta_{\alpha\beta} + D\psi_\alpha \cdot D\psi_\beta.$$

We will use g^{ij} to denote the inverse matrix to g_{ij} and $g^{\alpha\beta}$ to denote the inverse to $g_{\alpha\beta}$. We compute the second fundamental form. Note that $D^2 F = (0, D^2 \psi)$. So we have

$$(3.5) \quad \begin{aligned} \text{II}(e_i, e_j) &= \text{proj}^\perp(D^2 F(e_i, e_j)) \\ &= (D_{ij}^2 F \cdot \nu_\beta) g^{\alpha\beta} \nu_\alpha \\ &= \frac{\partial^2 \psi_\beta}{\partial x_i \partial x_j} g^{\alpha\beta} \nu_\beta. \end{aligned}$$

In components, $h_{ij\alpha} = \frac{\partial^2 \psi_\alpha}{\partial x_i \partial x_j}$.

Then the norm-squared of the second fundamental form is

$$(3.6) \quad |\Pi|_g^2 = \frac{\partial^2 \psi_\alpha}{\partial x_i \partial x_j} \frac{\partial^2 \psi_\beta}{\partial x_k \partial x_l} g^{\alpha\beta} g^{ik} g^{jl}.$$

We can think of $|\Pi|_g^2$ as the norm-squared of $D^2\psi$ in the metric g as opposed to the standard metric. We will compare $g^{\alpha\beta}$ and g^{ij} to the standard metric by giving estimates for the eigenvalues of $g^{\alpha\beta}$ and g^{ij} . To do this we estimate the eigenvalues of g_{ij} and $g_{\alpha\beta}$.

Each eigenvalue λ of $g_{\alpha\beta}$ has the form $\lambda = g(X, X) = g_{\alpha\beta} X^\alpha X^\beta$ for some eigenvector $X \in \mathbb{R}^n$ with $|X|^2 = \sum_\alpha (X^\alpha)^2 = 1$. Then

$$(3.7) \quad \begin{aligned} \lambda &= (\delta_{\alpha\beta} + D\psi_\alpha \cdot D\psi_\beta)(X^\alpha X^\beta) \\ &= |X|^2 + (X^\alpha D\psi_\alpha) \cdot (X^\beta D\psi_\beta) \\ &= 1 + |X^\alpha D\psi_\alpha|^2. \end{aligned}$$

Thus $1 \leq \lambda \leq 1 + |D\psi|^2$. Similarly for an eigenvalue μ of g_{ij} , we have

$$(3.8) \quad \begin{aligned} \mu &= (\delta_{ij} + D_i\psi \cdot D_j\psi)X^i X^j \\ &= |X|^2 + |D_X\psi|^2 \\ &= 1 + |D_X\psi|^2. \end{aligned}$$

So $1 \leq \mu \leq 1 + |D\psi|^2$.

Thus the eigenvalues of the inverse matrices $g^{\alpha\beta}$ and g^{ij} are bounded away from zero and infinity:

$$(3.9) \quad \begin{aligned} 1 &\geq |\lambda^{-1}| \geq \frac{1}{1 + |D\psi|^2}, \\ 1 &\geq |\mu^{-1}| \geq \frac{1}{1 + |D\psi|^2}. \end{aligned}$$

So we can estimate

$$(3.10) \quad \begin{aligned} |\Pi|_g^2 &= \frac{\partial^2 \psi_\alpha}{\partial x_i \partial x_j} \frac{\partial^2 \psi_\beta}{\partial x_k \partial x_l} g^{\alpha\beta} g^{ik} g^{jl} \\ &\geq \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq i, j \leq m}} \left(\frac{\partial^2 \psi_\alpha}{\partial x_i \partial x_j} \right)^2 \frac{1}{(1 + |D\psi|^2)(1 + |D\psi|^2)^2} \\ &= |D^2\psi|^2 \frac{1}{(1 + |D\psi|^2)^3}, \end{aligned}$$

which establishes our lemma. $\square$

Our next step is to show that any immersed submanifold $F : M \looparrowright \mathbb{R}^{m+n}$ can be written as a collection of graphs of functions ψ with small $|D\psi|$.

We introduce the following notation and notions, following [5]. Given $q \in M$, denote by A_q any Euclidean isometry which takes $F(q)$ to the origin and $T_{F(q)}F(M)$ to the plane $\{(x_1, \dots, x_m, 0)\}$. Let π be the projection of $\mathbb{R}^{n+m}$ to the plane $\{(x_1, \dots, x_m, 0)\}$. Define $U_{r,q} \subset M$ to be the component of $(\pi \circ A_q^{-1} \circ F)^{-1}(D_r)$ which contains q . We call $F : M \looparrowright \mathbb{R}^{n+m}$ a (r, α) -immersion if for each $q \in M$ there is some $\psi_q : D_r^m \rightarrow \mathbb{R}^n$ with $|D\psi_q| \leq \alpha$ so that $A_q^{-1} \circ F(U_{r,q}) = \text{graph}(\psi_q)$.

Lemma 3.3. *Let $0 < \alpha \leq 1$. Then for any C^2 -immersed submanifold $F : M^m \looparrowright \mathbb{R}^{n+m}$ and any r satisfying*

$$r \leq \frac{\alpha}{(1 + \alpha^2)^{3/2}} \frac{1}{\sup_M |\Pi|_g},$$

F is an (r, α) -immersion.

Proof. Let $q \in M$ be arbitrary. Every submanifold is locally a graph over its tangent plane; thus $A_q(F(U_{r,q}))$ can be written as a graph over D_r for small enough r . So we set $S_q = \sup\{r | F(U_{r,q}) = \text{graph}(\psi_{r,q})\}$. For any large K , if $F(U_{r,q}) = \text{graph}(\psi_{r,q})$ and $|D\psi_{r,q}| \leq \frac{K}{2}$, we can extend $\psi_{r,q}$ to have a larger domain and still have $|D\psi| \leq K$. Thus we have

$$(3.11) \quad \lim_{r \rightarrow S_q} \inf_{\psi} \sup_{D_r} |D\psi| = \infty$$

where the infimum is taken over all ψ of which $F(U_{r,q})$ is a graph. Thus for our given α there exist some $r_q, \psi_q : D_{r_q} \rightarrow \mathbb{R}^n$ with $\sup_{D_{r_q}} |D\psi_q| = \alpha$. Now we use the fundamental theorem of calculus and Lemma 3.2 to get

$$(3.12) \quad \alpha = \sup_{D_{r_q}} |D\psi_q| \leq r_q \sup_{D_{r_q}} |D^2\psi_q| \leq r_q (1 + \alpha^2)^{3/2} \sup_{D_{r_q}} |\Pi_{\psi_q}|_g,$$

which implies that

$$(3.13) \quad r_q \geq \frac{\alpha}{(1 + \alpha^2)^{3/2}} \frac{1}{\sup_{D_{r_q}} |\Pi_{\psi_q}|_g} \geq \frac{\alpha}{(1 + \alpha^2)^{3/2}} \frac{1}{\sup_M |\Pi|_g}.$$

So for r less than the right-hand side of (3.13), there is some $\psi : D_r \rightarrow \mathbb{R}^{m+n}$ which makes $F(U_{r,q})$ a graph and $|D\psi| \leq \alpha$. $\square$

Proof of Theorem 3.1. Now choose $\alpha = 1$, and let r be given by Lemma 3.3. It is clear that $B(q, r) \subset U_{r,q}$, since $A_q F(U_{r,q})$ is a graph over a disc of radius r . Thus $B(q, r)$ can be written as a graph over the tangent plane, and in particular $\text{inj}(q) \geq r$.

Since q was arbitrary, we have $\text{inj}(M) \geq r > 0$. $\square$

4. THE TENSOR A BLOWS UP

We will prove Theorem 1.2 by contradiction. To this end, assume $\max_M |A(t)| \leq C$ for all $t \in [0, T)$ and that the flow has a singularity at $T < \infty$.

In particular, we have $|H|^4 = (\text{tr } A)^2 \leq n|A|^2 \leq nC^2$. So $|H|$ is also bounded along the flow. We will use C to denote this bound as well.

By Theorem 1.1, we know that as $t \rightarrow T$, $\max_M |\Pi(t)| \rightarrow \infty$. Let (p_j, t_j) be a sequence in $M \times [0, T)$ with $t_j \rightarrow T$ and $\max_{t \leq t_j} |\Pi| = |\Pi(p_j, t_j)| =: Q_j \rightarrow \infty$.

Since $|\partial_t F| = |H| \leq C$, we know that $F_t(M)$ is contained in the CT tubular neighborhood of $F_0(M)$. Thus the $F(p_j, t_j)$ accumulate. Passing to a subsequence, we have $F(p_j, t_j) \rightarrow p_0$. For any $R > 0$, we may choose a j_0 so that $F(p_j, t_j)$ lies in the ambient ball of radius R about p_0 , $B^N(R)$ for all $j \geq j_0$. In particular, we will take R to be less than the injectivity radius of (N, h) .

Consider the flows given by scaling the ambient metric by Q_j and time by Q_j^{-2} :

$$(4.1) \quad F_j(p, t) = F(p, t_j + \frac{t}{Q_j^2}) : M \hookrightarrow (B^N, Q_j^2 h).$$

Lemma 4.1. *Each F_j is a mean curvature flow on $M \times [-Q_j^2 t_j, 0]$. The second fundamental form of F_j is bounded:*

$$(4.2) \quad \max_{t \leq 0} |\Pi_j| = |\Pi_j(p_j, 0)| = 1.$$

Proof. Clearly $\partial_t F_j = Q_j^{-2} \partial_t F$. We need to show that by scaling the ambient metric, we induce the same scaling in H . By definition

$$(4.3) \quad \begin{aligned} H_{Q_j^2 h}(F_j) &= \text{tr}_{Q_j^2 h} \Pi_{Q_j^2 h}(F_j) \\ &= (Q_j^2 h)^{pq} (\partial_p \partial_q F_j)^{\perp_{Q_j^2 h}} \end{aligned}$$

where $(Q_j^2 h)^{pq}$ is the inverse matrix of $(Q_j^2 h)(\partial_p F_j, \partial_q F_j)$ and $\perp_{Q_j^2 h}$ is the projection onto the normal bundle induced from $Q_j^2 h$.

$Q_j^2 h$ induces the same splitting into tangent and normal bundles as h , so we have

$$(4.4) \quad \begin{aligned} H_{Q_j^2 h}(F_j) &= Q_j^{-2} [h^{pq} (\partial_p \partial_q F_j)^{\perp_h}] \\ &= Q_j^{-2} \text{tr}_h \Pi_h(F_j) \\ &= Q_j^{-2} H_h(F_j). \end{aligned}$$

So H scales as required.

Similarly scaling the ambient metric by Q_j^2 scales $|\Pi|$ by Q_j^{-1} , so we have

$$(4.5) \quad \begin{aligned} \max_{t \leq 0} |\Pi_j| &= \max_{t \leq t_j} Q_j^{-1} |\Pi| \\ &= Q_j^{-1} |\Pi(p_j, t_j)| = 1. \end{aligned} \quad \square$$

It is clear that the $(B^N, Q_j^2 h, p_0)$ converge in the Cheeger-Gromov sense to $(\mathbb{R}^{m+n}, dx^2, 0)$, where dx^2 is the Euclidean metric. In particular, we have a monotone exhausting sequence of open sets $V_j \subset \mathbb{R}^{m+n}$ and embeddings $\psi_j: (V_j, 0) \rightarrow (B^N, p_0)$, such that $\psi_j^*(Q_j^2 h) \rightarrow dx^2$.

Let $s_0 = -Q_1^2 t_1$. After passing to a smaller spatial region $\tilde{M} \subset M$, we can assume $F_j(\tilde{M} \times [s_0, 0]) \subset \psi_j(V_j)$. We restrict our argument to this smaller region and write $\tilde{M}$ without confusion.

Define $\tilde{F}_j: \tilde{M} \times [s_0, 0] \rightarrow \mathbb{R}^{m+n}$ by $\tilde{F}_j = \psi_j^{-1} F_j$. Each $\tilde{F}_j$ is a mean curvature flow with respect to the metric $\psi_j^*(Q_j^2 h)$.

The second fundamental forms $\tilde{\Pi}_j$ of the $\tilde{F}_j$ are uniformly bounded, so Theorem 2.2 gives uniform bounds on the covariant derivatives of the $\tilde{\Pi}_j$.

Since $\partial_t \tilde{F}_j = \tilde{H}_j$, we get bounds on the time derivative of $\tilde{F}_j$. In fact the evolution of H gives a bound

$$(4.6) \quad \begin{aligned} |\partial_t^2 \tilde{F}_j| &= |\partial_t \tilde{H}_j| \leq |\Delta \tilde{H}_j| + C_1 |\nabla \tilde{H}_j| + C_2 |\tilde{H}_j| |\tilde{\Pi}_j|^2 \\ &\leq |\nabla^2 \tilde{\Pi}_j| + C_1 |\nabla \tilde{\Pi}_j| + C_2 |\tilde{\Pi}_j|^3. \end{aligned}$$

Similarly, any iterated time derivative $\partial_t^s \tilde{F}_j = \partial_t^{s-1}(\tilde{H}_j)$ is controlled in terms of $|\nabla^r \tilde{\Pi}_j|$ for $r \leq 2(s-1)$. The mixed derivatives $\partial_t^r \nabla^s \tilde{\Pi}_j$ are similarly controlled by $|\nabla^l \tilde{\Pi}_j|$ for $l \leq 2r + s$.

Since $Q_j^2 h \rightarrow dx^2$ in C^k for any k , our bounds on the $|\nabla^s \tilde{\Pi}_j|$ give bounds on $|\bar{\nabla}^s \bar{\Pi}_j|$, where $\bar{\nabla}$ and $\bar{\Pi}_j$ are the connection and second fundamental form of $\tilde{F}_j$ with respect to the metric dx^2 .

Let $t \in [s_0, 0]$. Theorem 3.1 gives $\text{inj}(M, \tilde{F}_j(t)^* dx^2) \geq \frac{1}{2\sqrt{2}}$. The Gauss equation guarantees that the Riemannian curvature and its covariant derivatives of $(M, \tilde{F}_j(t)^* dx^2, p_j)$ are bounded uniformly in j . Thus by Cheeger-Gromov there is a limit Riemannian manifold $(M_\infty, g_\infty(t), p_\infty)$.

Adapting the ideas of [8], we consider the growth of balls in $(M_\infty, g_\infty(0))$. We will write g_∞ for $g_\infty(0)$.

Proposition 4.2. (M, g_∞) has Euclidean volume growth about p_∞ .

Proof. Let us use the following conventions for balls and volume forms. $B_\infty(\rho)$ will denote the metric ball in g_∞ centered at p_∞ ; $B_j(\rho)$ will denote the metric ball in $F_{t_j}^*(Q_j^2 h)$ centred at p_j ; $B_{t_j}(\rho)$ will denote the metric ball in $F_{t_j}^* h$ centred at p_j . Also, vol_∞ will denote the volume form of g_∞ ; vol_j will denote the volume form of $F_{t_j}^*(Q_j^2 h)$; vol_{t_j} will denote the volume form of $F_{t_j}^* h$. Note that

$$(4.7) \quad \begin{aligned} B_j(\rho) &= B_{t_j}\left(\frac{\rho}{Q_j}\right), \\ \text{vol}_j &= Q_j^m \text{vol}_{t_j}. \end{aligned}$$

We have, for any $r > 0$,

$$(4.8) \quad \begin{aligned} \frac{\text{vol}_\infty(B_\infty(r))}{r^m} &= \lim_j \frac{\text{vol}_j(B_j(r))}{r^m} \\ &= \lim_j \frac{\text{vol}_{t_j}(B_{t_j}(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m}. \end{aligned}$$

The evolution of g is

$$(4.9) \quad \partial_t g_{ij} = -2H^\alpha h_{ij\alpha} = -2A_{ij},$$

so we have $|\partial_t g| \leq C$, and in particular g is uniformly continuous in time in the sense of Lemma 2.3.

Thus we may apply Lemma 2.3 to estimate the metric balls at any time t_j by the metric ball at time t_{j_0} , so long as $t_j - t_{j_0} \leq \delta$. Since $t_j \rightarrow T$, we can pick a j_0 so that this condition holds for all $j \geq j_0$. So we can estimate (4.8) by

$$(4.10) \quad \lim_j \frac{\text{vol}_{t_j}(B_{t_j}(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m} \leq \lim_j \frac{\text{vol}_{t_{j_0}}(B_{t_{j_0}}(\frac{r}{\sqrt{1-\epsilon}Q_j}))}{(\frac{r}{Q_j})^m}.$$

The evolution of the volume form shows that the flow is pointwise volume-reducing. So $\text{vol}_{t_j} \leq \text{vol}_{t_{j_0}}$ for $j \geq j_0$. Thus we can estimate (4.10) by

$$(4.11) \quad \begin{aligned} \lim_j \frac{\text{vol}_{t_{j_0}}(B_{t_{j_0}}(\frac{r}{\sqrt{1-\epsilon}Q_j}))}{(\frac{r}{Q_j})^m} &\leq \lim_j \frac{\text{vol}_{t_{j_0}}(B_{t_{j_0}}(\frac{r}{\sqrt{1-\epsilon}Q_j}))}{(\frac{r}{Q_j})^m} \\ &= (1-\epsilon)^{-\frac{m}{2}} \lim_j \frac{\text{vol}_{t_{j_0}}(B_{t_{j_0}}(\frac{r}{\sqrt{1-\epsilon}Q_j}))}{(\frac{r}{\sqrt{1-\epsilon}Q_j})^m}. \end{aligned}$$

The only dependence of the right-hand side on j is in the Q_j .

The limit on the right-hand side of (4.11) is the local volume comparison at p_{j_0} for the Riemannian manifold $(M, F_{t_{j_0}}^* h)$. It is well-known that this limit is ω_m , the volume of the Euclidean unit m -ball. Therefore we have

$$(4.12) \quad \frac{\text{vol}_\infty(B_\infty(r))}{r^m} \leq (1-\epsilon)^{-\frac{m}{2}} \omega_m.$$

Since ϵ was arbitrary, we have shown $\text{vol}_\infty(B_\infty(r)) \leq \omega_m r^m$.

To show the reverse inequality, we make a similar argument starting from (4.8), this time using the first inclusion of Lemma 2.3. We now seek to estimate vol_{t_j} below by $\text{vol}_{t_{j_0}}$. Since we have assumed $|H| \leq C$, the evolution of vol implies that

$$(4.13) \quad \text{vol}_{t_j} \geq e^{-C^2(t_j - t_{j_0})} \text{vol}_{t_{j_0}},$$

and taking j_0 large enough we may ensure that $e^{-C^2(t_j - t_{j_0})} \geq 1 - \epsilon$. Then we can estimate (4.8) by

$$(4.14) \quad \lim_j \frac{\text{vol}_{t_j}(B_{t_j}(\frac{r}{Q_j}))}{(\frac{r}{Q_j})^m} \geq \lim_j (1 - \epsilon)(1 + \epsilon)^{-\frac{m}{2}} \frac{\text{vol}_{t_{j_0}}(B_{t_{j_0}}(\frac{r}{\sqrt{1+\epsilon}Q_j}))}{(\frac{r}{\sqrt{1+\epsilon}Q_j})^m}.$$

Again we can take the limit in j to get

$$(4.15) \quad \frac{\text{vol}_\infty(B_\infty(r))}{r^m} \geq (1 - \epsilon)(1 + \epsilon)^{-\frac{m}{2}} \omega_m.$$

Since ϵ was arbitrary, we have shown $\text{vol}_\infty(B_\infty(r)) \geq \omega_m r^m$. $\square$

Proof of Theorem 1.2. To finish the proof of the theorem, we want to use the volume growth of (M_∞, g_∞) to obtain a contradiction. The expansion for the volume of balls about p in r is

$$(4.16) \quad \text{vol}_\infty(B_\infty(r)) = \omega_m r^m (1 - \frac{R_\infty(p)}{6(m+2)} r^2 + O(r^3))$$

where $R_\infty(p_\infty)$ is the scalar curvature [1]. So Proposition 4.2 immediately implies that $R_\infty(p_\infty) = 0$.

On the other hand, tracing the Gauss equation twice gives that

$$(4.17) \quad \begin{aligned} R_\infty(p_\infty) &= \lim_j R_j(p_j) \\ &= \lim_j |H_j(p_j)|^2 - |\Pi_j(p_j)|^2 \\ &\leq \lim_j \frac{C}{Q_j^2} - 1 \\ &= -1. \end{aligned}$$

This is the desired contradiction. $\square$

5. A CONDITION FOR THE BLOW UP OF H

Definition 5.1. A singularity at time $T < \infty$ is of *type I* if $|\Pi(x, t)|^2(T - t) \leq C < \infty$ for all $x \in M_t$ and all $t \in [0, T)$.

This is the slowest possible rate of singularity formation and is attained in the case of a shrinking sphere or cylinder.

We can prove that the mean curvature blows up under a slightly more general condition, namely

Theorem 5.1. Suppose that along the flow, $|\Pi(x, t)|^p(T - t) \leq C$ for some $p > 1$. Then $\max_{M_t} |H| \rightarrow \infty$ as $t \rightarrow T$.

Proof. For the purpose of contradiction, suppose $|H| \leq C$ all along the flow.

As in the proof of Theorem 1.2, we consider the parabolic rescales

$$(5.1) \quad F_j(p, t) = F(p, t_j + \frac{t}{Q_j^2}) : M \hookrightarrow (N, Q_j^2 h)$$

where $Q_j = \max_{t \leq t_j} |\text{II}|$. Each F_j is a mean curvature flow, and we have

$$(5.2) \quad |H_j| \leq \frac{C}{Q_j}.$$

As in the proof of Theorem 1.2, we want to obtain a limit manifold (M_∞, g_∞) whose volume growth yields a contradiction.

To proceed to a contradiction as in the proof of Theorem 1.2, we need to establish that the balls of the metrics $g(t)$ can be compared as in the conclusion of Lemma 2.3.

Consider the evolution of the metric:

$$(5.3) \quad \begin{aligned} \left| \frac{\partial g}{\partial t} \right| &= 2|A| \leq 2|H||\text{II}| \\ &\leq 2C(T-t)^{-\frac{1}{p}}. \end{aligned}$$

The function $(T-t)^{-\frac{1}{p}}$ is integrable, so Lemma 2.4 guarantees that the metrics $g(t)$ converge to a positive definite $g(T)$ as $t \nearrow T$. We thus have for any $\varepsilon_1 > 0$ the existence of a $\delta > 0$ such that $t \geq T - \delta$ guarantees $1 - \varepsilon_1 \leq \frac{g(t)}{g(T)} \leq 1 + \varepsilon_1$. If $t, s \geq T - \delta$, then we have $\frac{1-\varepsilon_1}{1+\varepsilon_1} \leq \frac{g(t)}{g(s)} \leq \frac{1+\varepsilon_1}{1-\varepsilon_1}$. Choosing the δ which corresponds to $\varepsilon_1 = \frac{\epsilon}{2+\epsilon}$ then guarantees that $(1-\epsilon)g(s) \leq g(t) \leq (1+\epsilon)g(s)$.

We may then apply the proof of Lemma 2.3 to estimate balls in the metrics $g(t_j)$ for j sufficiently large and proceed as in the proof of Theorem 1.2, starting from equation (4.10). $\square$

Remark 1. The tensors H and A are either both zero or both nonzero. A bound on A implies that H is bounded. Theorem 1.2 says that H cannot decay “too fast” relative to II . The following question presents itself:

Question 1. Suppose $T < \infty$ is the first singular time for a compact mean curvature flow. Is it generally true that $\max_{M_t} |H| \rightarrow \infty$ as $t \rightarrow T$?

Remark 2. The blowup of the mean curvature in the type I case has been independently discovered by Le and Šešum [6].

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REFERENCES

- [1] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian Geometry*. Springer-Verlag, 1987. MR909697 (88k:53001)
- [2] David Glickenstein. Precompactness of solutions to the Ricci flow. *Geometry and Topology*, 7:487–510, 2003. MR2026540 (2004k:53099)
- [3] Richard Hamilton. Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, 17:255–306, 1982. MR664497 (84a:53050)
- [4] Gerhard Huisken. Flow by mean curvature of convex hypersurfaces into spheres. *Journal of Differential Geometry*, 20:237–266, 1984. MR772132 (86j:53097)

- [5] Joel Langer. A compactness theorem for surfaces with l_p -bounded second fundamental form. *Mathematische Annalen*, 270:223–234, 1985. MR771980 (86i:53035)
- [6] N. Q. Le, N. Šešum. The mean curvature at the first singular time of the mean curvature flow, *Ann. I. H. Poincaré-AN*, 2010, doi:10.1016/j.anihpc.2010.09.002.
- [7] Robert C. Reilly. On the Hessian of a function and the curvatures of its graph. *Michigan Math. J.*, 20:373–383, 1973. MR0334045 (48:12364)
- [8] Nataša Šešum. Curvature tensor under the Ricci flow. *American Journal of Mathematics*, 127:1315–1324, 2005. MR2183526 (2006f:53097)
- [9] Mu Tao Wang. Mean curvature flow of surfaces in Einstein four-manifolds, *J. Differential Geom.*, 57:301–338, 2001. MR1879229 (2003j:53108)

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