THE GL-L.U.ST. CONSTANT AND ASYMMETRY OF THE KALTON-PECK TWISTED SUM IN FINITE DIMENSIONS

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Dedicated to the memory of Nigel J. Kalton

ABSTRACT. We prove that the Kalton-Peck twisted sum Z_2^n of n -dimensional Hilbert spaces has a GL-l.u.st. constant of order $\log n$ and bounded GL constant. This is the first concrete example which shows different explicit orders of growth in the GL and GL-l.u.st. constants. We also discuss the asymmetry constants of Z_2^n .

1. INTRODUCTION

Local unconditional structure, or l.u.st., is an important notion in the study of the geometry of Banach spaces (see for instance [GL1], [KT], [PW]). The variant of l.u.st. that we investigate here was introduced by Gordon and Lewis [GL1] and is frequently referred to as *GL-l.u.st.*; another, formally more restrictive, notion of l.u.st. was introduced in [FJT].

This notion was not, however, fully studied in the finite-dimensional case, where the asymptotic values of the various constants that it involves are not yet fully understood. We show in this paper that in the finite-dimensional setting of n -dimensional normed spaces, the GL constant and the l.u.st. constant can be of significantly different orders of magnitude, by considering the $2n$ -dimensional Kalton-Peck twisted sum of n -dimensional Hilbert spaces.

Let us now recall some definitions ([GL1]). A basis $B = (b_i)_{i \in I}$ for a Banach space E is called *unconditional* if there is a constant $C > 0$ such that for every $x \in E$, $x = \sum_{i \in I} \xi_i b_i$ and every choice of signs $\varepsilon_i = \pm 1$, with $\varepsilon_i = 1$ for all but a finite number of $i \in I$, one has

$$\|\sum_{i \in I} \varepsilon_i \xi_i b_i\| \leq C \|x\|.$$

The smallest C satisfying this is called *the unconditional constant of B* and denoted by $\chi(B)$. The *unconditional constant of E* is

$$\chi(E) := \inf\{\chi(B); B \text{ is a basis of } E\}.$$

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More generally, define the local unconditional structure constant of E (see [GL1]), $\chi_u(E)$, by

$$\chi_u(E) := \sup_{F \subset E} \left(\inf_{A, B, U} \|B\| \chi(U) \|A\| \right),$$

where the sup is taken over all finite-dimensional subspaces F of E and the infimum over all Banach spaces U and all continuous operators $A : F \rightarrow U$ and $B : U \rightarrow E$ such that $BA = I_F$, the identity on F . Clearly $\chi_u(E) \leq \chi(E)$. By [FJT], $\chi_u(E) = \chi_u(E')$ and $\chi_u(E)$ is finite iff E'' is isomorphic to a complemented subspace of a Banach lattice.

Given a subset $\{x_i; i \in I\}$ of E , we denote

$$\varepsilon_1(\{x_i; i \in I\}) := \sup \left\{ \sum_{i \in I} |x'(x_i)|; x' \in E', \|x'\| \leq 1 \right\}.$$

Let E and F be Banach spaces. The *weakly nuclear norm* of a linear operator $u : E \rightarrow F$, which has a representation $u = \sum_{i \in I} x'_i \otimes y_i$, which converges unconditionally in $L(E, F)$, is defined by

$$\eta(u) = \inf \varepsilon_1(\{x'_i \otimes y_i, i \in I\}),$$

where the infimum is taken over all such representations of u .

By Proposition 1.2. of [GL2], one has for all $u \in L(E, F)$

$$\eta(u) = \inf \|A\| \chi(U) \|B\|,$$

where the infimum is taken over all factorizations $u = BA$, with $A : E \rightarrow U$ compact, $B : U \rightarrow F$, and U is a Banach space with an unconditional basis. Hence $\eta(I_E) = \chi_u(E)$. It is easy to show, using e.g. techniques from [R], that when $E = F$ is finite dimensional and $u = I_E$, one may restrict this infimum to spaces U satisfying $\chi(U) = 1$ and $A : E \rightarrow U$ to be an isometric embedding.

The *Gordon-Lewis constant* of E , denoted by $\text{gl}(E)$, is defined to be

$$\text{gl}(E) = \inf \{c > 0; \gamma_1(A) \leq c \Pi_1(A); F \text{ a Banach space and } A : E \rightarrow F\}.$$

Here

$$\gamma_1(A) = \inf \{ \|\alpha\| \|\beta\|; \beta\alpha = i_F A \},$$

where the infimum ranges over all Banach spaces F and all $\alpha : E \rightarrow L^1(\mu)$, $\beta : L^1(\mu) \rightarrow F''$, $i_F : F \rightarrow F''$ is the canonical inclusion, and

$$\Pi_1(A) = \inf \left\{ C > 0; \sum_{i=1}^n \|Ax_i\| \leq C \sup_{\pm} \left\| \sum_{i=1}^n \pm x_i \right\| \right. \\ \left. \text{for every } n \geq 1 \text{ and } x_1, \dots, x_n \in E \right\}$$

is the classical *1-absolutely summing norm* of A . It was proved in [GL1] that $\text{gl}(E) \leq \chi_u(E)$, and it is well known that there exist infinite-dimensional separable Banach spaces E such that $\text{gl}(E)$ is finite and $\chi_u(E)$ is infinite (the first such example is presented in [JLS]). In particular, it follows that there exists an increasing sequence of $2n$ -dimensional Banach spaces E_n for which $\chi_u(E_n) \rightarrow \infty$ and $(\text{gl}(E_n))_n$ is bounded.

We prove in the sequel that, for some constant $c > 0$, one has

$$c_n = \sup \left\{ \frac{\chi_u(E_n)}{\text{gl}(E_n)}; E_n \text{ is an } n\text{-dimensional normed space} \right\} \geq c \log n.$$

To do this, we refine the proof, given by [JLS], that the Kalton-Peck space Z_2 (see [KP]) has bounded gl-constant but that $\chi_u(Z_2)$ is infinite. The spaces E_n mentioned above are the subspaces Z_2^n of Z_2 spanned by first $2n$ coordinates. Note that always $c_n \leq \sqrt{n}$. An interesting problem is, how big can c_n be? For example, is $c_n \geq cn^\alpha$ for some $\alpha > 0$ and c an absolute positive constant? Perhaps this is true even with $\alpha = 1/2$.

Z_2 (and Z_p) has unconditional finite-dimensional decomposition into 2-dimensional subspaces (2-UFDD). Many of the concrete examples of spaces with GL and without l.u.st. possess such structure (see [KT] for references). The paper [CK] presents a general treatment of spaces having uniform UFDD (that is, UFDD for which the dimensions of the blocks are uniformly bounded). It is shown there that such a (infinite-dimensional) space either has an unconditional basis or fails to have l.u.st. Observing the computations made in the works mentioned above, it is plausible that if $c_n \geq cn^\alpha$ for some $\alpha > 0$, then the examples showing this would not be with uniform UFDD. In this respect we may quote a conjecture that Nigel Kalton sent us, together with other helpful suggestions, in response to a preprint version of the present paper:

Conjecture. *If we restrict the definition of c_n to kn -dimensional initial blocks of spaces with a k -UFDD, then c_n is equivalent to $(\log n)^{k-1}$.*

The concept of asymmetry of an n -dimensional Banach space was introduced in [GG] and generalized the notion of the asymmetry of a basis. This was followed up in [GL1] as well. Here we study related notions of asymmetry and apply them to Z_2^n .

For general terminology concerning the geometry of Banach spaces we refer the reader to [LT1, LT2]. Terminology concerning normed ideals of operators may be found in [P] and [T].

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2. THE L.U.ST. CONSTANT OF Z_2^n

Let Z_2^n be the $2n$ -dimensional (real) Banach space which is the subspace of the Kalton-Peck [KP] space Z_2 , spanned by the first $2n$ coordinates. More precisely, for $a, b \in \mathbb{R}^n$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, we define

$$\begin{aligned} \|(a, b)\|_{Z_2^n} &= \|(a_1, \dots, a_n, b_1, \dots, b_n)\|_{Z_2^n} = \left\| \sum_{j=1}^n (a_j e_j + b_j f_j) \right\|_{Z_2^n} \\ &= \left(\sum_{j=1}^n b_j^2 \right)^{1/2} + \left(\sum_{j=1}^n \left(a_j - b_j \log \left(|b_j| \left(\sum_{i=1}^n b_i^2 \right)^{-1/2} \right) \right)^2 \right)^{1/2}. \end{aligned}$$

As it is proved in [KP], this quasi-norm on $\mathbb{R}^{2n}$ is (uniformly in n) equivalent to a norm.

As in [JLS] we consider Z_2^n as an unconditional sum

$$(1) \quad Z_2^n = \sum_{j=1}^n \bigoplus E_j,$$

where $E_j = [e_j, f_j]$.

It was proved in [JLS] that Z_2 fails to have l.u.st. They observed that if a Banach space E is of the form $E = \bigoplus_{j=1}^{\infty} E_j$, a 1-unconditional sum of finite-dimensional subspaces, then

$$gl(E) \leq \sup_j \dim(E_j),$$

and thus $gl(Z_2) \leq 2c$. Hence it follows that $gl(Z_2^n) \leq 2c$, but that the l.u.st. constants of Z_2^n tend to infinity with n . Here we establish the order of growth of these constants.

Theorem 1. *One has $gl(Z_2^n) \leq 2c$, and the l.u.st. constant $\chi_u(Z_2^n)$ of Z_2^n satisfies*

$$(2) \quad \chi_u(Z_2^n) \sim \log n.$$

Throughout the proof the letters $C, c, c_1, c_2, \dots$, etc. will denote absolute constants which do not depend on n . The same letter c (etc.) may denote different constants in different lines.

It is shown in [KP] that the Banach-Mazur distance $d(Z_2^n, \ell_2^{2n})$ from Z_2^n to ℓ_2^{2n} is of the order of $\log n$. Using Proposition 2 of [JLS], there exists a Banach space Y_n with a 1-unconditional basis $\{y_{j,i}, j = 1, \dots, n, i = 1, \dots, k_j\}$, such that

- Z_2^n is a subspace of Y_n and, for each $1 \leq j \leq n$, $E_j \subset [\{y_{j,i}, i = 1, \dots, k_j\}]$.
- There exists a projection $P_n : Y_n \rightarrow Z_2^n$ such that $P_n([\{y_{j,i}, i = 1, \dots, k_j\}]) = E_j$ for all j , $1 \leq j \leq n$. ($[A]$ denotes the span of A .)
- $\|P_n\| = K_n \leq c\chi_u(Z_2^n)$. To justify this, one first shows (see [FJT]) that for any Banach space E ,

$$\chi_u(E) = \inf \|P\|_{L \rightarrow E},$$

where the infimum is taken over all Banach lattices L such that E is isometrically embedded in L and all projections $P : L \rightarrow E$. Then observing that $E = Z_2^n$ have bounded cotype- q constants for a fixed $q < \infty$, we use [R] to reduce to the case when moreover L is supposed to have the same bound on its cotype- q constant. Then, we follow the lines of Proposition 2 of [JLS], using Rademacher embedding.

We shall show that $K_n \geq c \log n$, thus proving (2) because it is clear that $K_n \leq d(Z_2^n, \ell_2^{2n}) \sim \log n$.

Notation. Let $T : Z_2^n \rightarrow Z_2^n$ be a linear operator that satisfies $T(E_j) \subset E_j$ for $1 \leq j \leq n$. We say that T *splits through* $\{E_j\}$ or, for short, that T *splits*. If T splits, the matrices representing $T|_{E_j}$ in the basis $\{e_j, f_j\}$, $1 \leq j \leq n$, will be denoted by

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}.$$

If $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, the *support* of a is $\text{supp}(a) := \{j : 1 \leq j \leq n, a_j \neq 0\}$. If $a, b \in \mathbb{R}^n$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$, we denote $(a, b) = \sum_{j=1}^n (a_j e_j + b_j f_j) \in Z_2^n$. We also define a vector $ab \in \mathbb{R}^n$ by $ab = (a_1 b_1, \dots, a_n b_n)$.

Lemma 2. *For any T that splits we have*

$$\max_j \max\{|\alpha_j|, |\beta_j|, |\gamma_j|, |\delta_j|\} \leq \|T\|.$$

Proof. If $x = e_j$, $1 \leq j \leq n$, then $\|Tx\| = |\gamma_j| + |\alpha_j| \leq \|T\|\|x\| = \|T\|$. If $x = f_j$, one similarly obtains $|\delta_j| + |\beta_j| \leq \|T\|$. $\square$

Lemma 3. *With the preceding notation, there exists $C > 0$ such that for every $T : Z_2^n \rightarrow Z_2^n$ that splits and for all $(a, b) \in \mathbb{R}^{2n}$, we have*

$$\left(\sum_{j=1}^n \left(\delta_j a_j - \alpha_j a_j + \gamma_j a_j \log \frac{|a_j|}{\|a\|_2} \right)^2 \right)^{\frac{1}{2}} \leq C \left(\|T\| \|(a, b)\|_{Z_2^n} + \left(\max_{j \in \text{supp}(a)} |\gamma_j| \right) \|a\|_2 \right).$$

Proof. It is established in [KP] that the function $F : \ell_2^n \rightarrow \ell_2^n$ given by

$$F(b) = \left(b_1 \log \frac{|b_1|}{\|b\|_2}, \dots, b_n \log \frac{|b_n|}{\|b\|_2} \right) \text{ for } b = (b_1, \dots, b_n) \in \mathbb{R}^n$$

is *quasi-linear* in the sense that for every $a, b \in \mathbb{R}^n$,

$$(3) \quad \|F(a+b) - F(a) - F(b)\|_2 \leq C (\|a\|_2 + \|b\|_2).$$

We have

$$\|(\gamma F(a), \gamma a)\|_{Z_2^n} = \|\gamma a\|_2 + \|\gamma F(a) - F(\gamma a)\|_2,$$

so that

$$\|\gamma F(a) - F(\gamma a)\|_2 \leq \|(\gamma F(a), \gamma a)\|_{Z_2^n}.$$

Hence

$$(4) \quad \|(\delta - \alpha)a + \gamma F(a)\|_2 \leq \|(\delta - \alpha)a + F(\gamma a)\|_2 + \|(\gamma F(a), \gamma a)\|_{Z_2^n}.$$

Also, unconditionality of the sum (1) implies for every $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$,

$$(5) \quad \|(\theta a, \theta b)\|_{Z_2^n} \leq C \max_j |\theta_j| \|(a, b)\|_{Z_2^n}.$$

Defining $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^b$, β , γ and δ related to T as above, one has

$$\begin{aligned} \|(a, b)\|_{Z_2^n} &= \|b\|_2 + \|a - F(b)\|_2, \\ \|T(a, b)\|_{Z_2^n} &= \|\gamma a + \delta b\|_2 + \|\alpha a + \beta b - F(\gamma a + \delta b)\|_2 \end{aligned}$$

and

$$(6) \quad \|F(\gamma a + \delta b) - \alpha a\|_2 \leq 2\|T\| \|(a, b)\|_{Z_2^n},$$

because

$$\begin{aligned} \|F(\gamma a + \delta b) - \alpha a\|_2 &\leq \|\beta b\|_2 + \|\alpha a + \beta b - F(\gamma a + \delta b)\|_2 + \|\gamma a + \delta b\|_2 \\ &\leq \|T(a, b)\|_{Z_2^n} + \max_j |\beta_j| \|b\|_2 \leq 2\|T\| \|(a, b)\|_{Z_2^n}. \end{aligned}$$

Applying Lemma 2, (3), (5), (6) and the fact that $\|(F(a), a)\|_{Z_2^n} = \|a\|_2$, we get from (4)

$$\begin{aligned} &\|(\delta - \alpha)a + \gamma F(a)\|_2 \leq \|(\delta - \alpha)a + F(\gamma a)\|_2 + \|(\gamma F(a), \gamma a)\|_{Z_2^n} \\ &\leq \|F(\gamma a) + F(\delta b) - F(\gamma a + \delta b)\|_2 + \|F(\gamma a + \delta b) - \alpha a\|_2 + \|\delta a - F(\delta b)\|_2 \\ &\quad + \left(\max_{j \in \text{supp}(a)} |\gamma_j| \right) \|(F(a), a)\|_{Z_2^n} \\ &\leq C (\|\gamma a\|_2 + \|\delta b\|_2) + 2\|T\| \|(a, b)\|_{Z_2^n} + 2\|T\| \|(a, b)\|_{Z_2^n} + \max_{j \in \text{supp}(a)} |\gamma_j| \|a\|_2 \\ &\leq C \left(\|T\| \|(a, b)\|_{Z_2^n} + \max_{j \in \text{supp}(a)} |\gamma_j| \|a\|_2 \right). \quad \square \end{aligned}$$

Lemma 4. *In the context of the preceding lemmas, for every subset $A \subset \{1, \dots, n\}$, $|A| = k > 1$, one has*

$$\left(\frac{1}{k} \sum_{j \in A} \gamma_j^2 \right)^{\frac{1}{2}} \leq \frac{4\|T\|}{\log k}.$$

Proof. Let $x = (a, b) \in Z_2^n$, where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are given by

$$a_j = \frac{\log \frac{1}{\sqrt{k}}}{\sqrt{k}}, \quad b_j = \frac{1}{\sqrt{k}} \text{ if } j \in A, \quad a_j = b_j = 0 \text{ otherwise.}$$

Then $\|x\|_{Z_2^n} = 1$ and therefore

$$\|T\| \geq \|Tx\|_{Z_2^n} \geq \left(\frac{1}{k} \sum_{j \in A} (-\gamma_j \log \sqrt{k} + \delta_j)^2 \right)^{\frac{1}{2}} \geq \log \sqrt{k} \left(\frac{1}{k} \sum_{j \in A} \gamma_j^2 \right)^{\frac{1}{2}} - \max_j |\delta_j|.$$

We now use Lemma 2 to complete the proof. $\square$

Corollary 5. *In the context of the preceding lemmas, let $A \subset \{1, \dots, n\}$, $|A| = k > 1$. Then*

a) *There exists a subset $A' \subset A$, with $|A'| \geq \frac{k}{2}$, such that*

$$(7) \quad \max_{j \in A'} |\gamma_j| \leq \frac{4\sqrt{2}\|T\|}{\log k}.$$

b) *If $\text{supp}(a) \subset A'$, where $A' \subset \{1, \dots, n\}$, $|A'| \leq k$ and (7) is satisfied, then*

$$\left(\sum_{j=1}^n \left(\delta_j a_j - \alpha_j a_j + \gamma_j a_j \log \frac{|a_j|}{\|a\|_2} \right)^2 \right)^{\frac{1}{2}} \leq C\|T\| \|(a, b)\|_{Z_2^n}.$$

Proof. Let $C = \{j \in A; |\gamma_j| > \frac{4\sqrt{2}\|T\|}{\log k}\}$. To prove a), it is enough to show that $|C| \leq k/2$. If it was not true, one would have

$$\left(\frac{1}{k} \sum_{j \in A} \gamma_j^2 \right)^{\frac{1}{2}} \geq \left(\frac{1}{k} \sum_{j \in C} \gamma_j^2 \right)^{\frac{1}{2}} > \frac{4\|T\|}{\log k},$$

which contradicts Lemma 4. Part b) follows from Lemma 3 and the fact (see [KP]) that the norm of the operator $p_k : Z_2^k \rightarrow \ell_2^k$, $p_k(a, b) = a$ is equivalent to $\log k$. $\square$

Lemma 6. *Let $A \subset \{1, \dots, n\}$, $|A| = k > k_0 > 1$. Then there exists a subset $A'' \subset A$, with $|A''| \geq \frac{\sqrt{k}}{3}$, such that for all $j \in A''$ we have*

$$|\gamma_j| \leq \frac{C\|T\|}{(\log k)^2} \quad \text{and} \quad |\delta_j - \alpha_j| \leq \frac{C\|T\|}{\log k}.$$

Proof. Let $A' \subset A$ with $k' = |A'| \geq \frac{k}{2}$ satisfy (7). Let $x = (a, b) \in Z_2^n$ be defined as in the proof of Lemma 4, with A' replacing A . Then, by Corollary 5b), we have (since $\|x\| = 1$)

$$\left(\frac{1}{k'} \sum_{j \in A'} (\delta_j - \alpha_j - \gamma_j \log \sqrt{k'})^2 \right)^{\frac{1}{2}} \leq \frac{C\|T\|}{\log \sqrt{k'}}.$$

As in the proof of Corollary 5, there exists a subset $B \subset A'$ with $k'' = |B| \geq \frac{k'}{2} \geq \frac{k}{4}$ such that

$$(8) \quad |\delta_j - \alpha_j - \gamma_j \log \sqrt{k'}| \leq \frac{\sqrt{2}C\|T\|}{\log \sqrt{k'}} = \frac{2C_1\|T\|}{\log k'} \text{ for all } j \in B.$$

Now take any subset $B' \subset B$ with $|B'| = k''' = \sqrt{k'} \leq \frac{k'}{2} \leq k''$ and a new vector $x \in Z_2^n$ like the one above, with B' replacing A' (the implicit assumption that $\sqrt{k'}$ is an integer can easily be dealt with). The same argument as above provides a subset $A'' \subset B'$ with

$$|A''| \geq \frac{k'''}{2} = \frac{\sqrt{k'}}{2} \geq \frac{\sqrt{k}}{2\sqrt{2}}$$

such that

$$(9) \quad |\delta_j - \alpha_j - \gamma_j \log \sqrt{k'''}| \leq \frac{\sqrt{2}C\|T\|}{\log \sqrt{k'''}} = \frac{4C_1\|T\|}{\log k'} \text{ for all } j \in A''.$$

Adding (8) and (9) together we get for $j \in A''$:

$$\begin{aligned} \frac{|\gamma_j| \log k'}{4} &= \left| \frac{1}{2} \gamma_j \log k' - \frac{1}{4} \gamma_j \log k' \right| \\ &= |(-\delta_j + \alpha_j + \gamma_j \log \sqrt{k'}) + (\delta_j - \alpha_j - \gamma_j \log \sqrt{k'''})| \leq \frac{6C_1\|T\|}{\log k'}, \end{aligned}$$

and thus

$$(10) \quad |\gamma_j| \leq \frac{24C_1\|T\|}{(\log k')^2} \leq \frac{C_2\|T\|}{(\log k)^2}.$$

From (9) and (10) together we get

$$|\delta_j - \alpha_j| \leq \frac{4C_1\|T\|}{\log k'} + |\gamma_j| \log \sqrt{k'''} \leq \frac{4C_1\|T\|}{\log k'} + \frac{24C_1\|T\|}{(\log k')^2} \cdot \frac{1}{4} \log k' \leq \frac{C_3\|T\|}{\log k} \text{ for } j \in A''.$$

□

Proof of Theorem 1. We now recall the construction from [JLS]: Under the hypotheses that Z_2 has l.u.st., they constructed an operator $T : Z_2 \rightarrow Z_2$ that provided a counterexample to that hypotheses. In the context of Z_2^n , this construction, instead of providing a counterexample, will provide a lower bound estimate of $\chi_u(Z_2^n)$.

Let the Banach space Y_n with the unconditional basis $\{y_{j,i}; j = 1, \dots, n, i = 1, \dots, k_j\}$ and the projection $P_n : Y_n \rightarrow Z_2^n$ be those presented at the opening. For a fixed $1 \leq j \leq n$ a set $A_j \subset \{1, \dots, k_j\}$ is selected and an operator $T_{A_j} : [e_j, f_j] \rightarrow [e_j, f_j]$ is defined by

$$T_{A_j}(x) = P_n \left(\sum_{i \in A_j} y_{j,i}^*(x) y_{j,i} \right),$$

where $\{y_{j,i}^*\} \subset Y_n^*$ are the bi-orthogonal functionals of $\{y_{j,i}\}$. Thus $T_{A_j} = \sum_{i \in A_j} T_{j,i}$, where $T_{j,i}$ is the rank-1 operator $T_{j,i}(x) = P_n(y_{j,i}^*(x) y_{j,i})$. We may assume that $T_{j,i} \neq 0$, as otherwise $y_{j,i}$ can be dropped from the basis. Being a rank-1 operator, the matrix representing $T_{j,i}$ is of the form

$$\begin{pmatrix} a_i & b_i \\ \theta_i a_i & \theta_i b_i \end{pmatrix} \text{ (case I) or } \begin{pmatrix} 0 & 0 \\ a_i & b_i \end{pmatrix} \text{ (case II)}$$

(note that we dropped the index j for simplicity). In case II we define θ_i to be 1. We have $\sum_{i=1}^{k_j} \theta_i b_i = 1$ (because $\sum_i T_{j,i} f_j = (\sum_i \theta_i b_i) f_j = f_j$) and $\sum_{i=1}^{k_j} |b_i| \leq K_n$.

Define

$$B_j := \left\{ i \in \{1, \dots, k_j\}; |\theta_i| \geq \frac{1}{2K_n} \right\}.$$

We have

$$\sum_{i \in B_j} \theta_i b_i > \frac{1}{2}.$$

For each $i \in B_j$ at least one of the following four possibilities holds:

- (ι) $\theta_i a_i > \frac{1}{4K_n} |\theta_i b_i|$,
- (υ) $-\theta_i a_i > \frac{1}{4K_n} |\theta_i b_i|$,
- ($\iota\iota$) $a_i - \theta_i b_i \geq \frac{1}{2} |\theta_i b_i|$ ($-\theta_i b_i \geq \frac{1}{2} |\theta_i b_i|$ in case II),
- ($\iota\nu$) $\theta_i b_i - a_i \geq \frac{1}{2} |\theta_i b_i|$ ($\theta_i b_i \geq \frac{1}{2} |\theta_i b_i|$ in case II).

In fact, if $|\theta_i a_i| > \frac{1}{4K_n} |\theta_i b_i|$, then (ι) or (υ) holds. Otherwise $|\theta_i a_i| \leq \frac{1}{4K_n} |\theta_i b_i|$. Since $i \in B_j$ this implies in case I that

$$\frac{1}{2K_n} |a_i| \leq \frac{1}{4K_n} |\theta_i b_i|,$$

which is

$$\left| \frac{a_i}{\theta_i} \right| \leq \frac{1}{2} |b_i|.$$

We thus have

$$\left| \frac{a_i}{\theta_i} - b_i \right| \geq |b_i| - \left| \frac{a_i}{\theta_i} \right| \geq |b_i| - \frac{1}{2} |b_i| = \frac{1}{2} |b_i|,$$

which is either ($\iota\iota$) or ($\iota\nu$). The remarks about case II are trivial.

It follows that there exists a subset $A_j \subset B_j$ such that one and the same possibility out of (ι)–($\iota\nu$) holds for all $i \in A_j$ and $\sum_{i \in A_j} \theta_i b_i > \frac{1}{8}$. The operator $T : Z_2^n \rightarrow Z_2^n$ that splits and is defined by $T|_{E_j} = T_{A_j}$ satisfies $\|T\| \leq K_n$. Assume that the matrix representing T_{A_j} in the basis $\{e_j, f_j\}$ is, as before,

$$\begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}.$$

From the above it follows that $\delta_j > \frac{1}{8}$ for all j .

If (ι) or (υ) is satisfied for all $i \in A_j$, then we have

$$(11) \quad |\gamma_j| > \frac{1}{4K_n} \delta_j > \frac{1}{32K_n}.$$

If ($\iota\iota$) or ($\iota\nu$) is satisfied for all $i \in A_j$, then we have

$$(12) \quad |\alpha_j - \delta_j| > \frac{1}{2} \delta_j > \frac{1}{16}.$$

We can now select a subset $D \subset \{1, \dots, n\}$, with $|D| \geq \frac{n}{2}$, such that either (11) holds for all $j \in D$ or (12) holds for all $j \in D$. We take D to be the set A of Lemma 6. Let $A'' \subset D$ be as in Lemma 6 and $j \in A''$.

If the members of D satisfy (11), then we have

$$\frac{1}{32K_n} \leq |\gamma_j| \leq \frac{C\|T\|}{(\log \frac{n}{2})^2} \leq \frac{C_1 K_n}{(\log n)^2},$$

and we conclude

$$(13) \quad K_n \geq c \log n$$

for some positive constant c .

If the members of D satisfy (12), then we have

$$\frac{1}{16} \leq |\delta_j - \alpha_j| \leq \frac{C\|T\|}{\log \frac{n}{2}} \leq \frac{C_1 K_n}{\log n};$$

thus

$$(14) \quad K_n \geq c \log n. \quad \square$$

3. THE ASYMMETRY OF Z_2^n

Definition. We say that a subgroup G of $GL_k(\mathbb{R})$ is *rich* if it is a compact group and that every operator in $L(\mathbb{R}^k)$ which commutes with all elements of G is a scalar multiple of the identity on $\mathbb{R}^k$. It is well known that for any compact subgroup G of $GL_k(\mathbb{R})$, there exists a compact subgroup H of $\mathcal{O}_k$ (the orthogonal group) such that $G = \{V^{-1}hV; h \in H\}$, where $V \in L(\mathbb{R}^k)$ is some linear invertible operator. If G is a rich subgroup of $GL_k(\mathbb{R})$, we define the *measure of symmetry* $s_G(E)$ of a k -dimensional normed space E with respect to G by

$$s_G(E) := \int_G \|g\|_{E \rightarrow E} d\mu(g),$$

where μ is the normalized Haar measure on G . We define

$$s(E) = \inf\{s_G(E); G \text{ a rich subgroup}\}.$$

Note that

$$1 \leq s(E) \leq S(E) := \inf\{\sup_{g \in G} \|g\|_{E \rightarrow E}; G \text{ a rich subgroup}\}.$$

The quantity $S(E)$, called the *asymmetry constant of E* , was defined originally in [GG] and further discussed in [GL1]. If $S(E) = 1$ we say that E has *enough symmetries*. This is the case for spaces with 1-symmetric basis (take the rich group of permutations and changes of signs on the basis). In connection with this quantity we have the following partial result:

Theorem 7. *For every rich subgroup G of $\mathcal{O}_{2n}$ one has*

$$(15) \quad s_G(Z_2^n) = \int_G \|g\|_{Z_2^n \rightarrow Z_2^n} d\mu(g) \sim \log n \sim \max_{g \in G} \|g\|_{Z_2^n \rightarrow Z_2^n},$$

where μ denotes the normalized Haar measure on G .

We use the following lemma.

Lemma 8. *Let a couple E and F of k -dimensional normed spaces, an ideal norm α and a rich subgroup G of $GL_k(\mathbb{R})$ be given. Then for every invertible operator $S \in L(\mathbb{R}^k)$, one has*

$$(16) \quad k \leq \alpha(S : E \rightarrow F) \alpha^*(S^{-1} : F \rightarrow E) \leq k \int_G \|S^{-1}gS\|_{E \rightarrow E} \|g^{-1}\|_{F \rightarrow F} d\mu(g),$$

where α^* denotes the conjugate ideal norm of α . In particular, if all elements of G are isometries of F , then

$$(17) \quad k \leq \alpha(S : E \rightarrow F) \alpha^*(S^{-1} : F \rightarrow E) \leq k \int_G \|S^{-1}gS\|_{E \rightarrow E} d\mu(g).$$

Proof. We first prove for any k -dimensional normed spaces D and F on $\mathbb{R}^n$ that if $I : \mathbb{R}^k \rightarrow \mathbb{R}^k$ denotes the identity operator, then

$$(18) \quad k \leq \alpha(I : D \rightarrow F) \alpha^*(I^{-1} : F \rightarrow D) \leq k \int_G \|g\|_{D \rightarrow D} \|g^{-1}\|_{F \rightarrow F} d\mu(g).$$

By the definition of α^* , there exists $U \in L(\mathbb{R}^k)$ such that

$$\alpha(U : D \rightarrow F) \alpha^*(I^{-1} : F \rightarrow D) = \text{trace}(UI^{-1}) = \text{trace}(U).$$

Let $V = \int_G g^{-1}Ug d\mu(g)$ be considered as an operator from D to F . Since G is a rich subgroup, one has $V = cI$ for some $c \in \mathbb{R}$, so that $\text{trace}(U) = \text{trace}(V) = kc$. One clearly has

$$\begin{aligned} c\alpha(I : D \rightarrow F) &= \alpha(V : D \rightarrow F) \leq \int_G \alpha(g^{-1}Ug : D \rightarrow F) d\mu(g) \\ &\leq \alpha(U : D \rightarrow F) \int_G \|g^{-1}\|_{F \rightarrow F} \|g\|_{D \rightarrow D} d\mu(g). \end{aligned}$$

It follows that

$$\begin{aligned} &\alpha(I : D \rightarrow F) \alpha^*(I^{-1} : F \rightarrow D) \\ &\leq \frac{\alpha(U : D \rightarrow F) \alpha^*(I^{-1} : F \rightarrow D)}{c} \int_G \|g\|_{D \rightarrow D} \|g^{-1}\|_{F \rightarrow F} d\mu(g) \\ &= k \int_G \|g\|_{D \rightarrow D} \|g^{-1}\|_{F \rightarrow F} d\mu(g). \end{aligned}$$

This proves (18). We define a normed space D by setting $\|x\|_D = \|S^{-1}x\|_E$. Then $S^{-1} : D \rightarrow E$ and $S : E \rightarrow D$ are isometries, and one has

$$\begin{aligned} \alpha(I : D \rightarrow F) &= \alpha(S : E \rightarrow F), \quad \alpha^*(I^{-1} : F \rightarrow D) = \alpha^*(S^{-1} : F \rightarrow E) \\ &\text{and } \|g\|_{D \rightarrow D} = \|S^{-1}gS\|_{E \rightarrow E}. \end{aligned}$$

Together with (18) this implies (16). $\square$

Proof of Theorem 7. We shall use Lemma 8 with $S = I : Z_2^n \rightarrow \ell_2^{2n}$, where I is the identity sending the basis e_j, f_j $1 \leq j \leq n$ on the canonical basis of ℓ_2^{2n} . Here we present two proofs: **A** and **B**.

A. Let $\alpha = \Pi_1$ and $\alpha^* = \gamma_\infty$, the ideal norm of factorization through L_∞ .

a. We claim that

$$\Pi_1(I : Z_2^n \rightarrow \ell_2^{2n}) \geq c\sqrt{n} \log n.$$

In fact, define, for $1 \leq j \leq n$, $x_j \in Z_2^n$ by

$$x_j = \frac{1}{\sqrt{n}} \left(\log \frac{1}{\sqrt{n}} \right) e_j + \frac{1}{\sqrt{n}} f_j.$$

Then $\|I(x_j)\|_2 \sim \frac{\log n}{\sqrt{n}}$, and for every choice of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, one has

$$\left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_{Z_2^n} = 1.$$

Hence

$$\Pi_1(I : Z_2^n \rightarrow \ell_2^{2n}) \geq \frac{\sum_{j=1}^n \|x_j\|_2}{\sup_{\varepsilon} \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|_{Z_2^n}} \geq c\sqrt{n} \log n.$$

b. Define $P : Z_2^n \rightarrow \ell_2^n$ by $P(a, b) = b$. Then $\|P\|_{Z_2^n \rightarrow \ell_2^n} = 1$, and if $W : \ell_2^n \rightarrow \ell_2^{2n}$ is the embedding defined by $W(b) = (0, b)$, then

$$\begin{aligned} \gamma_{\infty}(I^{-1} : \ell_2^{2n} \rightarrow Z_2^n) &= \|P\|_{Z_2^n \rightarrow \ell_2^n} \gamma_{\infty}(I^{-1} : \ell_2^{2n} \rightarrow Z_2^n) \|W\|_{\ell_2^n \rightarrow \ell_2^{2n}} \\ &\geq \gamma_{\infty}(PI^{-1}W : \ell_2^n \rightarrow \ell_2^n) \sim \sqrt{n}, \end{aligned}$$

because $PI^{-1}W : \ell_2^n \rightarrow \ell_2^n$ is actually the identity mapping on ℓ_2^n .

c. It follows that

$$\Pi_1(I : Z_2^n \rightarrow \ell_2^{2n}) \gamma_{\infty}(I^{-1} : \ell_2^{2n} \rightarrow Z_2^n) \geq cn \log n.$$

B. Let α be the operator norm; then $\alpha^* = i_1$, the integral norm. It is easy to see that

$$\|I\|_{Z_2^n \rightarrow \ell_2^{2n}} \sim \log n$$

and

$$\begin{aligned} i_1(I^{-1} : \ell_2^{2n} \rightarrow Z_2^n) &= i_1(I^{-1} : \ell_2^{2n} \rightarrow Z_2^n) \|P\|_{Z_2^n \rightarrow \ell_2^n} \\ &\geq i_1(PI^{-1} : \ell_2^{2n} \rightarrow \ell_2^n) = \sup_{S \neq 0} \frac{\text{trace}(PI^{-1}S)}{\|S\|_{\ell_2^n \rightarrow \ell_2^{2n}}} \geq n. \end{aligned}$$

It follows that

$$\|I\|_{Z_2^n \rightarrow \ell_2^{2n}} i_1(I^{-1} : \ell_2^{2n} \rightarrow Z_2^n) \geq cn \log n.$$

Using either **A** or **B**, together with Lemma 8 with $F = \ell_2^{2n}$, $E = Z_2^n$ and $S = I$, we get that

$$\int_G \|g\|_{Z_2^n \rightarrow Z_2^n} d\mu(g) \geq c \log n.$$

It follows that

$$\log n \leq \frac{1}{c} \int_G \|g\|_{Z_2^n \rightarrow Z_2^n} d\mu(g) \leq \frac{1}{c} \max_{g \in G} \|g\|_{Z_2^n \rightarrow Z_2^n} \leq d(Z_2^n, \ell_2^{2n}) \sim \log n,$$

where here d denotes the Banach-Mazur distance. This proves (15). $\square$

Remark. One can prove in the same way as **A.a.** in the proof of the last theorem that if $I : Z_2^n \rightarrow \ell_{\infty}^{2n}$ is the identity mapping sending the basis $(e_j, f_j), 1 \leq j \leq n$, on the canonical basis of ℓ_{∞}^{2n} , then

$$\Pi_1(I : Z_2^n \rightarrow \ell_{\infty}^{2n}) \geq c\sqrt{n} \log n.$$

We also have that

$$\gamma_{\infty}(I^{-1} : \ell_{\infty}^{2n} \rightarrow Z_2^n) = \|I^{-1}\|_{\ell_{\infty}^{2n} \rightarrow Z_2^n} \sim \sqrt{n} \log n.$$

It follows from Lemma 8 that for every rich subgroup G of $GL_{2n}(\mathbb{R})$, one has

$$\begin{aligned} cn(\log n)^2 &\leq \Pi_1(I : Z_2^n \rightarrow \ell_\infty^{2n})\gamma_\infty(I^{-1} : \ell_\infty^{2n} \rightarrow Z_2^n) \\ &\leq n \int_G \|g^{-1}\|_{\ell_\infty^{2n} \rightarrow \ell_\infty^{2n}} \|g\|_{Z_2^n \rightarrow Z_2^n} d\mu(g). \end{aligned}$$

The first inequality is actually an equivalence, in view of the fact that if one takes for G the group of changes of signs and permutaions of indices of the basis, one has $\|g^{-1}\|_{\ell_\infty^{2n} \rightarrow \ell_\infty^{2n}} = 1$ for all $g \in G$ and

$$\int_G \|g\|_{Z_2^n \rightarrow Z_2^n} d\mu(g) \leq \|I : Z_2^n \rightarrow \ell_\infty^{2n}\| \|I^{-1} : \ell_\infty^{2n} \rightarrow Z_2^n\| \leq c(\log n)^2,$$

as $\|I : Z_2^n \rightarrow \ell_\infty^{2n}\| \sim \log n$.

Added in proof. After the paper was processed, we realized that answers to some of the questions posed at the end of the introduction had already been given in the paper [KT]. In particular, examples of n -dimensional Banach spaces with bounded GL-constant and l.u.st constant greater than cn^α (with α as large as $1/6$) are given there.

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