HYPERCYCLIC WEIGHTED TRANSLATIONS ON GROUPS

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ABSTRACT. Let G be a locally compact group and let $1 \leq p < \infty$. We characterize hypercyclic weighted translation operators on the Lebesgue space $L^p(G)$ in terms of the weight, extending a recent result by the authors for discrete groups. Topologically mixing weighted translations are also characterized.

1. INTRODUCTION

Recently, we characterized in [6] hypercyclic weighted translations by aperiodic elements on coset spaces of discrete groups. In this paper, we complete our investigation by extending the characterization to all non-discrete coset spaces. In fact, we adopt an approach different from that in [6], where the Hypercyclic Criterion of Kitai [15], Gethner and Shapiro [10] was used. The proof in this paper does not depend on this criterion but makes use of a weaker condition of topological transitivity, and unifies both the discrete and non-discrete cases. This fresh approach entails two further consequences, one is the equivalence of hypercyclicity and the weakly mixing condition for weighted translations, the other is a characterization of topologically mixing weighted translations on coset spaces.

We recall that a continuous linear self-map T on a Fréchet space X is called a *hypercyclic operator* if it admits a hypercyclic vector $x \in X$, that is, if there is a vector $x \in X$ such that the orbit $\{x, Tx, \dots, T^n x, \dots\}$ is dense in X , where T^n denotes the n -th iterate of T . Hypercyclic operators have been studied by many authors; we refer to [2, 3, 5, 7, 9, 11, 12, 16, 17, 18] for some recent works.

Throughout, let G be a locally compact group with identity e and a right Haar measure λ . Since a Fréchet space admits a hypercyclic operator if, and only if, it is separable and infinite-dimensional, the question of hypercyclicity is meaningful for the complex Lebesgue space $L^p(G)$, with respect to λ , only when G is second countable and $1 \leq p < \infty$ which will be assumed henceforth. Given a compact subgroup H of G , the right coset space G/H admits a G -invariant measure $\nu = \lambda \circ q^{-1}$, where $q : G \rightarrow G/H$ is the quotient map. Convolution operators on the Lebesgue space $L^p(G/H)$ with respect to ν can be lifted naturally to $L^p(G)$ (cf. [8, p. 77]). Hence all hypercyclic results in this paper for $L^p(G)$ can be extended without difficulty to $L^p(G/H)$, as in [6]. For simplicity and to expose the essential ideas, we will confine our discussion to $L^p(G)$.

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A bounded continuous function $w : G \rightarrow (0, \infty)$ is called a *weight* on G . Let $a \in G$ and let δ_a be the unit point mass at a . A *weighted translation* on G is a weighted convolution operator $T_{a,w} : L^p(G) \rightarrow L^p(G)$ defined by

$$T_{a,w}(f) := wT_a(f) \quad (f \in L^p(G)),$$

where w is a weight on G and $T_a(f) = f * \delta_a \in L^p(G)$ is the convolution

$$(f * \delta_a)(x) := \int_G f(xy^{-1})d\delta_a(y) = f(xa^{-1}) \quad (x \in G),$$

which is just the right translation of f by a^{-1} .

A translation operator T_a is never hypercyclic since $\|T_a\| \leq 1$. However, a *weighted* translation operator $wT_a = T_{a,w}$ can be hypercyclic. Our objective is to address the natural question of hypercyclicity of $T_{a,w}$. For discrete groups G , we answer the question completely by giving a necessary and sufficient condition in terms of the weight w for $T_{a,w}$ to be hypercyclic. For non-discrete groups, the same condition also characterizes hypercyclicity of $T_{a,w}$ when the element a is aperiodic.

Before discussing the details, we first observe that certain weights and elements in G can be excluded from our consideration. Plainly, if $\|w\|_\infty \leq 1$, then $\|T_{a,w}\| \leq 1$ and $T_{a,w}$ is never hypercyclic. The following lemma reveals that we can ignore torsion elements in G , a fact not noted in [6].

An element a in a group G is called a *torsion element* if it is of finite order. In a locally compact group G , an element $a \in G$ is called *periodic* [13] (or *compact* [14, 9.9]) if the closed subgroup $G(a)$ generated by a is compact. We call an element in G *aperiodic* if it is not periodic. For discrete groups, periodic and torsion elements are identical; in other words, aperiodic elements are the non-torsion elements.

Lemma 1.1. *Let G be a locally compact group and let $a \in G$ be a torsion element. Then any weighted translation $T_{a,w} : L^p(G) \rightarrow L^p(G)$ is not hypercyclic, for $1 \leq p < \infty$.*

Proof. Let a have order d say, and consider the product ω_d of the first d translations of the weight w :

$$\omega_d(x) = w(x)w(xa^{-1}) \cdots w(xa^{-(d-1)}) \quad (x \in G).$$

Either $\|\omega_d\|_\infty \leq 1$ or $\|\omega_d\|_\infty > 1$. We show in both cases that $T_{a,w}$ is not hypercyclic.

In the first case, we make use of $a^d = e$ and deduce

$$T_{a,w}^n = \omega_d^{\lfloor \frac{n}{d} \rfloor} T_{a,w}^j \quad (n \in \mathbb{N}),$$

where $n = d\lfloor \frac{n}{d} \rfloor + j$ with $j \in \{0, 1, \dots, d-1\}$ and $T_{a,w}^0$ is the identity map. For each $f \in L^p(G)$, the orbit of f is given by

$$\{f, \omega_d f, \omega_d^2 f, \dots\} \cup \{f_1, \omega_d f_1, \omega_d^2 f_1, \dots\} \cup \cdots \cup \{f_{d-1}, \omega_d f_{d-1}, \omega_d^2 f_{d-1}, \dots\},$$

where the function f_j denotes the j -th iterate $T_{a,w}^j f$, for $j \in \{1, 2, \dots, d-1\}$. Since $\|\omega_d\|_\infty \leq 1$, the orbit of f is bounded and hence cannot be dense in $L^p(G)$. Therefore $T_{a,w}$ does not admit a hypercyclic vector in this case.

Next, suppose $T_{a,w}$ is hypercyclic for the case $\|\omega_d\|_\infty > 1$. We deduce a contradiction. By continuity of w and regularity of λ , there is a compact set $U \subset G$ such that $\lambda(U) > 2\varepsilon^p$ for some $\varepsilon > 0$ and $\omega_d(x) > 1$ for all $x \in U$. We note that the d -th iterate $T_{a,w}^d$ is also hypercyclic by [1, Theorem 1].

Let χ_U be the characteristic function of U . Then $\chi_U \in L^p(G)$. Since the hypercyclic vectors for $T_{a,w}^d$ are dense, there exist a hypercyclic vector $f \in L^p(G)$ and sufficiently large $n \in \mathbb{N}$ such that

$$\|f - 2\chi_U\|_p < \varepsilon \quad \text{and} \quad \|(T_{a,w}^d)^n f\|_p < \varepsilon.$$

Let $V = \{x \in U : |f(x)| < 1\}$. Then

$$\varepsilon^p > \|f - 2\chi_U\|_p^p \geq \int_V |f(x) - 2|^p d\lambda(x) \geq \lambda(V).$$

Let $W = U \setminus V$. Then we have

$$\begin{aligned} \varepsilon^p &> \|(T_{a,w}^d)^n f\|_p^p \\ &\geq \int_W |\omega_d^n(x) f(x)|^p d\lambda(x) \geq \lambda(W). \end{aligned}$$

Hence $\lambda(U) = \lambda(V) + \lambda(W) < 2\varepsilon^p$, which is a contradiction. Therefore $T_{a,w}$ is not hypercyclic in the second case either. $\square$

Remark 1.2. In the case of $\|\omega_d\|_\infty \leq 1$ above, one can also apply [1, Theorem 1] to deduce that $T_{a,w}$ is not hypercyclic by observing $T_{a,w}^d = T_{e,w_d}$.

2. CHARACTERIZATION OF HYPERCYCLICITY

We established in [6] a criterion of hypercyclicity for a weighted translation $T_{a,w}$ by an aperiodic element a in a discrete group G . We extend this result to non-discrete groups in this section. It has been shown in [6, Lemma 2.7] that an element a in a discrete group G is aperiodic if, and only if, for any finite subset $K \subset G$, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \emptyset$ (equivalently, $K \cap Ka^{-n} = \emptyset$) for $n > N$. It is the latter condition which was used to deduce the necessary weight condition for hypercyclicity. We first extend the above criterion of aperiodicity to non-discrete groups.

Lemma 2.1. *An element a in a second countable group G is aperiodic if, and only if, for each compact subset $K \subset G$, there exists $N \in \mathbb{N}$ such that $K \cap Ka^n = \emptyset$ for $n > N$.*

Proof. Let a be aperiodic. Then the closed subgroup $G(a)$ generated by a in G is not compact. By [14, Theorem 5.14], the element a is contained in a compactly generated closed subgroup of $G(a)$. It follows that $G(a)$ is a compactly generated abelian group and is therefore topologically isomorphic to a direct product $\mathbb{R}^n \times \mathbb{Z}^m \times F$ where F is a compact group, by [14, Theorem 9.8]. Moreover, a identifies with an element in $(\mathbb{R}^n \times \mathbb{Z}^m \times F) \setminus (\{0\} \times \{0\} \times F)$ since $G(a)$ is not compact. Hence for any sequence (n_k) in $\mathbb{N}$, the sequence (a^{n_k}) in G cannot converge to the identity e . Now let K be any non-empty compact subset of G . If $K \cap Ka^n \neq \emptyset$ frequently for n , then the sequence (a^n) lies in the compact set $K^{-1}K$ frequently, and we can find a subsequence (a^{n_k}) of (a^n) converging to some $b \in K^{-1}K$. Hence the sequence $(a^{n_{k+1}-n_k})$ converges to e , which is impossible. This proves $K \cap Ka^n = \emptyset$ from some n onwards.

Conversely, let a be periodic. By compactness of $G(a)$, we can find a subsequence (a^{n_k}) of (a^n) converging to some $b \in G(a)$, and hence $a^{n_{k+1}-n_k} \rightarrow e$. Therefore the compact set $K = \{e\} \cup \{a^{n_{k+1}-n_k} : k = 1, 2, \dots\}$ satisfies $K \cap Ka^{n_{k+1}-n_k} \neq \emptyset$ for each k . $\square$

Remark 2.2. The above proof also shows that an element a in a second countable group is periodic if, and only if, there is a sequence (n_k) in $\mathbb{N}$ such that $\lim_{k \rightarrow \infty} a^{n_k} = e$. In many familiar non-discrete groups, including the additive group $\mathbb{R}^n$, the Heisenberg group and the affine group, all elements except the identity are aperiodic. It should be noted, however, that non-torsion elements in a non-discrete group need not be aperiodic. For instance, the circle group $\mathbb{T}$ is compact and hence all elements in it are periodic, but an element $e^{i\theta}$ is non-torsion if $\frac{\theta}{2\pi}$ is irrational.

We are now ready to show how hypercyclicity of the weighted translation $T_{a,w}$ depends on the behavior of successive translations of w by a . We will make use of the equivalence of hypercyclicity and topological transitivity [11]; namely, an operator T on a Banach space X is hypercyclic if and only if for any non-empty open sets U and V in X , we have $T^n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. If T satisfies the stronger condition that $T^n(U) \cap V \neq \emptyset$ from some n onwards, then T is said to be *topologically mixing*.

Theorem 2.3. *Let G be a locally compact group and let a be an aperiodic element in G . Let $1 \leq p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. The following conditions are equivalent:*

- (i) $T_{a,w}$ is hypercyclic.
- (ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k \rightarrow \infty} \lambda(E_k)$ and both sequences*

$$w_n := \prod_{s=1}^n w * \delta_{a^{-1}}^s \quad \text{and} \quad \tilde{w}_n := \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit respectively subsequences (w_{n_k}) and $(\tilde{w}_{n_k})$ satisfying

$$\lim_{k \rightarrow \infty} \|w_{n_k}|_{E_k}\|_{\infty} = \lim_{k \rightarrow \infty} \|\tilde{w}_{n_k}|_{E_k}\|_{\infty} = 0.$$

Proof. (i) $\Rightarrow$ (ii). We extend the proof of [19, Theorem 2.1] to our setting. Let $T_{a,w}$ be hypercyclic. Let $K \subset G$ be a compact set with $\lambda(K) > 0$. Let $\varepsilon > 0$. By aperiodicity of a , there is some N such that $K \cap Ka^{-n} = \emptyset$ for $n > N$. Let $\chi_K \in L^p(G)$ be the characteristic function of K . Choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. By density of the hypercyclic vectors for $T_{a,w}$, there exist a hypercyclic vector $f \in L^p(G)$ and some $m > N$ such that

$$\|f - \chi_K\|_p < \delta^2 \quad \text{and} \quad \|T_{a,w}^m f - \chi_K\|_p < \delta^2.$$

Let $A_\delta = \{x \in K : |f(x) - 1| \geq \delta\}$. Then we have

$$|f(x)| > 1 - \delta \quad (x \in K \setminus A_\delta)$$

and $\lambda(A_\delta) < \delta^p$. Indeed, either $\lambda(A_\delta) = 0$ or

$$\begin{aligned} \delta^{2p} > \|f - \chi_K\|_p^p &= \int_G |f(x) - \chi_K(x)|^p d\lambda(x) \\ &\geq \int_K |f(x) - 1|^p d\lambda(x) \\ &\geq \int_{A_\delta} |f(x) - 1|^p d\lambda(x) \geq \delta^p \lambda(A_\delta), \end{aligned}$$

giving $\lambda(A_\delta) < \delta^p$.

Let $B_\delta = \{x \in G \setminus K : |f(x)| \geq \delta\}$. Then we have

$$|f(x)| < \delta \quad \text{for } x \in (G \setminus K) \setminus B_\delta$$

and as before, $0 \leq \lambda(B_\delta) < \delta^p$ since

$$\delta^{2p} > \int_{G \setminus K} |f(x) - \chi_K(x)|^p d\lambda(x) \geq \int_{B_\delta} |f(x)|^p d\lambda(x) \geq \delta^p \lambda(B_\delta).$$

Let $C_{m,\delta} = \{x \in K : |\tilde{w}_m(x)^{-1} f(xa^{-m}) - 1| \geq \delta\}$. Then we have

$$\tilde{w}_m(x)^{-1} |f(xa^{-m})| > 1 - \delta \quad (x \in K \setminus C_{m,\delta})$$

and also

$$\begin{aligned} \delta^{2p} &> \|T_{a,w}^m f - \chi_K\|_p^p = \int_G |T_{a,w}^m f(x) - \chi_K(x)|^p d\lambda(x) \\ &\geq \int_{C_{m,\delta}} |w(x)w(xa^{-1}) \cdots w(xa^{-(m-1)})f(xa^{-m}) - 1|^p d\lambda(x) \geq \delta^p \lambda(C_{m,\delta}) \end{aligned}$$

which implies $\lambda(C_{m,\delta}) < \delta^p$. Since $K \cap Ka^{-m} = \emptyset$, we have

$$\tilde{w}_m(x) < \frac{|f(xa^{-m})|}{1 - \delta} < \frac{\delta}{1 - \delta} < \varepsilon \quad \text{for } x \in K \setminus (C_{m,\delta} \cup B_\delta a^m).$$

Let $D_{m,\delta} = \{x \in K : |w_m(x)f(x)| \geq \delta\}$. Then we have

$$w_m(x)|f(x)| < \delta \quad (x \in K \setminus D_{m,\delta}),$$

and from $K \cap Ka^m = \emptyset$ we deduce

$$\begin{aligned} \delta^{2p} &> \int_G |w(x)w(xa^{-1}) \cdots w(xa^{-(m-1)})f(xa^{-m}) - \chi_K(x)|^p d\lambda(x) \\ &= \int_G |w(xa^m)w(xa^{m-1}) \cdots w(xa)f(x) - \chi_K(xa^m)|^p d\lambda(x) \\ &\geq \int_{D_{m,\delta}} |w(xa^m)w(xa^{m-1}) \cdots w(xa)f(x)|^p d\lambda(x) \geq \delta^p \lambda(D_{m,\delta}) \end{aligned}$$

which implies $\lambda(D_{m,\delta}) < \delta^p$. It follows that

$$w_m(x) < \frac{\delta}{|f(x)|} < \frac{\delta}{1 - \delta} < \varepsilon \quad \text{for } x \in K \setminus (D_{m,\delta} \cup A_\delta).$$

Let $E_{m,\delta} = K \setminus (A_\delta \cup B_\delta a^m \cup C_{m,\delta} \cup D_{m,\delta})$. Then we have $\lambda(K \setminus E_{m,\delta}) < 4\delta^p$, $\|w_m|_{E_{m,\delta}}\|_\infty < \varepsilon$ and $\|\tilde{w}_m|_{E_{m,\delta}}\|_\infty < \varepsilon$.

For $k = 1, 2, \dots$, the above arguments enable us to construct a Borel set $E_k \subset K$ and inductively $n_1 < \dots < n_k < \dots$ such that $\lambda(K \setminus E_k) < 4(\frac{1}{k})^p$, $\|w_{n_k}|_{E_k}\|_\infty < \frac{1}{k}$ and $\|\tilde{w}_{n_k}|_{E_k}\|_\infty < \frac{1}{k}$. Therefore the sequence (E_k) satisfies condition (ii).

(ii) $\Rightarrow$ (i). We show that $T_{a,w}$ is topologically transitive. Let U and V be non-empty open subsets of $L^p(G)$. Since the space $C_c(G)$ of continuous functions on G with compact support is dense in $L^p(G)$, we can pick $f, g \in C_c(G)$ with $f \in U$ and $g \in V$. Let K be the union of the compact supports of f and g . Let $E_k \subset K$ and the subsequences (w_{n_k}) and $(\tilde{w}_{n_k})$ satisfy condition (ii).

By aperiodicity of a , there exists $M \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > M$.

First, we show that $\|T_{a,w}^{n_k}(f\chi_{E_k})\|_p \rightarrow 0$ as $k \rightarrow \infty$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $n_k > M$ and $w_{n_k}^p < \frac{\varepsilon}{\|f\|_p^p}$ on E_k for $k > N$. Hence

$$\begin{aligned}\|T_{a,w}^{n_k}(f\chi_{E_k})\|_p^p &= \int_{E_k a^{n_k}} |w(x)w(xa^{-1}) \cdots w(xa^{-(n_k-1)})|^p |f(xa^{-n_k})|^p d\lambda(x) \\ &= \int_{E_k} |w(xa^{n_k})w(xa^{n_k-1}) \cdots w(xa)|^p |f(x)|^p d\lambda(x) \\ &= \int_{E_k} w_{n_k}^p(x) |f(x)|^p d\lambda(x) < \varepsilon\end{aligned}$$

for $k > N$.

We define a self-map $S_{a,w}$ on the subspace $L_c^p(G)$ of functions in $L^p(G)$ with compact support by

$$S_{a,w}(h) = \frac{h}{w} * \delta_{a^{-1}} \quad (h \in L_c^p(G))$$

so that

$$T_{a,w}S_{a,w}(h) = h \quad (h \in L_c^p(G)).$$

Applying similar arguments to the iterates $S_{a,w}^{n_k}$, using the sequence $(\tilde{w}_{n_k})$, yields

$$\begin{aligned}\lim_{k \rightarrow \infty} \|S_{a,w}^{n_k}(g\chi_{E_k})\|_p^p &= \lim_{k \rightarrow \infty} \int_{E_k a^{-n_k}} \frac{1}{|w(xa)w(xa^2) \cdots w(xa^{n_k})|^p} |g(xa^{n_k})|^p d\lambda(x) = 0.\end{aligned}$$

For each $k \in \mathbb{N}$, we let

$$v_k = f\chi_{E_k} + S_{a,w}^{n_k}(g\chi_{E_k}) \in L^p(G).$$

Then

$$\|v_k - f\|_p^p \leq \|f\|_\infty^p \lambda(K \setminus E_k) + \|S_{a,w}^{n_k}(g\chi_{E_k})\|_p^p$$

and

$$\|T_{a,w}^{n_k}v_k - g\|_p^p \leq \|T_{a,w}^{n_k}(f\chi_{E_k})\|_p^p + \|g\|_\infty^p \lambda(K \setminus E_k).$$

Hence $\lim_{k \rightarrow \infty} v_k = f$ and $\lim_{k \rightarrow \infty} T_{a,w}^{n_k}v_k = g$ which implies $T_{a,w}^{n_k}(U) \cap V \neq \emptyset$ for some k . $\square$

Remark 2.4. One can construct examples of weights w , as in [6, Example 2.5], satisfying condition (ii) above. This condition can be seen to be weaker than the sufficient condition for hypercyclicity in [6, Proposition 2.2], via Egoroff's theorem. If G is a discrete group, condition (ii) is equivalent to the fact that both sequences (w_n) and $(\tilde{w}_n)$ admit subsequences (w_{n_k}) and $(\tilde{w}_{n_k})$ which converge to 0 pointwise in G . Hence Theorem 2.3 is identical to [6, Theorem 2.8] for discrete groups. It also implies that if $T_{a,w} : L^p(G) \rightarrow L^p(G)$ is hypercyclic for some $p \in [1, \infty)$, then it is so for all $p \in [1, \infty)$.

It has been shown in [4] that an operator T satisfies the Hypercyclic Criterion if, and only if, the direct sum $T \oplus T$ is hypercyclic. Condition (ii) in Theorem 2.3 also implies that $T_{a,w} \oplus T_{a,w}$ is topologically transitive. Indeed, if one considers four non-empty open sets U_j and V_j ($j=1,2$) in $L^p(G)$ and pick $f_j, g_j \in C_c(G)$ with $f_j \in U_j$ and $g_j \in V_j$, then the same arguments in the proof of (ii) $\Rightarrow$ (i) in Theorem 2.3 can be applied to these functions to obtain two sequences $(v_{1,k})$ and $(v_{2,k})$ in $L^p(G)$ satisfying $\lim_{k \rightarrow \infty} v_{j,k} = f_j$ and $\lim_{k \rightarrow \infty} T_{a,w}^{n_k}v_{j,k} = g_j$ for $j = 1, 2$, yielding $T_{a,w}^{n_k}(U_j) \cap V_j \neq \emptyset$ for some k . Hence we can draw the following conclusion.

Corollary 2.5. *Let G be a locally compact group and let a be an aperiodic element in G . Let $1 \leq p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. The following conditions are equivalent:*

- (i) $T_{a,w}$ satisfies the Hypercyclic Criterion.
- (ii) $T_{a,w}$ is hypercyclic.
- (iii) $T_{a,w} \oplus T_{a,w}$ is hypercyclic.
- (iv) $T_{a,w}$ is weakly mixing.

Using similar arguments as in the proof of Theorem 2.3, we can also characterize topological mixing weighted translations for non-discrete groups.

Corollary 2.6. *Let a be an aperiodic element in G and let $T_{a,w}$ be a weighted translation on $L^p(G)$ for $1 \leq p < \infty$. The following conditions are equivalent:*

- (i) $T_{a,w}$ is topologically mixing.
- (ii) *For every compact subset K of G with $\lambda(K) > 0$, there is a sequence of Borel sets (E_n) in K such that $\lambda(K) = \lim_{n \rightarrow \infty} \lambda(E_n)$ and both sequences*

$$w_n := \prod_{s=1}^n w * \delta_{a^{-s}} \quad \text{and} \quad \tilde{w}_n := \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

satisfy

$$\lim_{n \rightarrow \infty} \|w_n|_{E_n}\|_\infty = \lim_{n \rightarrow \infty} \|\tilde{w}_n|_{E_n}\|_\infty = 0.$$

Proof. The proof of (ii) $\Rightarrow$ (i) is identical to that in Theorem 2.3, using the full sequences (w_n) and $(\tilde{w}_n)$ instead of subsequences. For the converse, let K be a compact set in G with $\lambda(K) > 0$. Let $\varepsilon > 0$ and $\chi_K \in L^p(G)$ be the characteristic function as defined in the proof of Theorem 2.3. Choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$ and let $U = \{f \in L^p(G) : \|f - \chi_K\|_p < \delta^2\}$. By the topologically mixing assumption, there exists $N \in \mathbb{N}$ such that $n > N$ implies $K \cap Ka^{-n} = \emptyset$ and

$$T_{a,w}^n(U) \cap U \neq \emptyset.$$

We can therefore pick, for each $n > N$, a function $f_n \in U$ with $T_{a,w}^n f_n \in U$ which gives

$$\|f_n - \chi_K\|_p < \delta^2 \quad \text{and} \quad \|T_{a,w}^n f_n - \chi_K\|_p < \delta^2.$$

Using this for each f_n and repeating the arguments in the proof of Theorem 2.3, we find Borel sets $E_n \subset K$ with $\lambda(K) = \lim_{n \rightarrow \infty} \lambda(E_n)$ such that $\|w_n|_{E_n}\|_\infty < \frac{1}{n}$ and $\|\tilde{w}_n|_{E_n}\|_\infty < \frac{1}{n}$, proving condition (ii). $\square$

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