ON SUBSPACE-HYPERCYCLIC OPERATORS

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ABSTRACT. In this paper we study an operator T on a Banach space E which is M -hypercyclic for some subspace M of E . We give a sufficient condition for such an operator to be M -hypercyclic and use it to answer negatively two questions asked by Madore and Martínez-Avendaño. We also give a sufficient condition for T to be M -hypercyclic for all finite co-dimensional subspaces M in E .

1. INTRODUCTION

A bounded linear operator T on a separable Banach space E is *hypercyclic* if there exists a vector $x \in E$ whose orbit under T , $\text{Orb}(T, x) = \{T^n x : n \geq 0\}$, is dense in E . Such a vector x is called *hypercyclic* for T . Hypercyclic operators have been actively studied for more than twenty years. One can refer to [1], [2], [3] for more information about the subject.

Recently, B. F. Madore and R. A. Martínez-Avendaño in [4] introduced the concept of subspace hypercyclicity. An operator T is *subspace-hypercyclic* for a subspace M of E if there exists a vector such that the intersection of its orbit and M is dense in M . The authors prove several results analogous to the hypercyclic case. For example there is a Kitai-like criterion: if T is subspace-hypercyclic, then its spectrum must intersect the unit circle; subspace-hypercyclicity is a strictly infinite-dimensional phenomenon; compact or hyponormal operators are not subspace-hypercyclic.

In this paper we give an improvement of the Kitai-like criterion and use it to answer negatively two questions asked by the authors in [4]. Then we show that if an operator T satisfies a stronger condition than the hypercyclicity criterion, it is subspace-hypercyclic for any finite co-dimensional subspace.

We use c_{00} to denote the subspace of all sequences of scalars $x = (x(n))_{n=0}^{\infty}$ such that $x(0) = 0$ except for finitely many n . If $x \in c_{00}$ let $|x| = \max \{n : x(n) \neq 0\}$.

2. BACKWARD SHIFT OPERATOR IN ℓ_p

Let us recall some definitions from [4].

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Definition 2.1. Let $T \in B(E)$ and M be a closed subspace of E . We say that T is *M-hypercyclic* if there exists $x \in E$ such that $\text{Orb}(T, x) \cap M$ is dense in M . Such a vector x is called an *M-hypercyclic* vector for T .

Definition 2.2. Let $T \in B(E)$ and M be a closed subspace of E . We say that T is *M-transitive* if for any nonempty open subsets U, V of M there exists a positive number n such that $U \cap T^{-n}V$ contains a nonempty open subset of M .

In [4] the authors prove that *M-transitivity* implies *M-hypercyclicity*. Moreover, there is the following criterion.

Theorem 2.3. Let $T \in B(E)$ and M be a nonzero subspace of E . Assume there exist X and Y , dense subsets of M , and an increasing sequence of positive integers $(n_k)_{k=1}^\infty$ such that

- (1) $T^{n_k}x \rightarrow 0$ for all $x \in X$,
- (2) for each $y \in Y$, there exist a sequence $(x_k)_{k=1}^\infty$ in X such that $x_k \rightarrow 0$ and $T^{n_k}x_k \rightarrow y$,
- (3) M is an invariant subspace for T^{n_k} for all k .

Then T is *M-transitive* and hence *M-hypercyclic*.

Remark 2.4. The authors in [4] raise two questions relating *M-transitivity* and *M-hypercyclicity*: namely, whether conditions (1) and (2) of Theorem 2.3 are sufficient for T to be *M-hypercyclic* and whether *M-hypercyclicity* implies *M-transitivity*.

In this section we answer these questions negatively by constructing counterexamples. We begin with a slight improvement of the previous theorem concerning *M-hypercyclicity*. The idea of the proof is well-known, and we include it here for the sake of completeness.

Theorem 2.5. Let $T \in B(E)$ and let $M \subset E$ be a closed subspace. Assume that there exist subsets X and Y of M where Y is dense in M and an increasing sequence $(n_k)_{k=1}^\infty \subset \mathbb{N}$ such that the following hold:

- (1) $T^{n_k}(x) \rightarrow 0$ for all $x \in X$,
- (2) for each $y \in Y$ there exists a sequence $(x_k)_{k=1}^\infty$ in X such that

$$x_k \rightarrow 0 \text{ and } T^{n_k}x_k \rightarrow y,$$

- (3) $X \subset \bigcap_{k=1}^\infty T^{-n_k}(M)$.

Then T is *M-hypercyclic*.

Proof. Let $(\varepsilon_j)_{j=1}^\infty$ be a sequence of positive numbers such that

$$\lim_{j \rightarrow \infty} (j\varepsilon_j + \sum_{i=j+1}^\infty \varepsilon_i) = 0.$$

Since E is separable, we can assume that $Y = (y_j)_{j=1}^\infty$ for some sequence $(y_j)_{j=1}^\infty$. We construct a sequence $(x_j)_{j=1}^\infty \subset X$ and a subsequence $(n_{k_j})_{j=1}^\infty$ of $(n_k)_{k=1}^\infty$ by induction. Let $x_1 \in X$ and n_{k_1} be such that $\|x_1\| + \|T^{n_{k_1}}x_1 - y_1\| < \varepsilon_1$. For each j choose n_{k_j} and $x_j \in X$ such that $\|x_j\| + \|T^{n_{k_j}}x_j - y_j\| + \|T^{n_{k_i}}x_j - y_j\| < \varepsilon_j$ for all $i < j$.

Let $x = \sum_{i=1}^{\infty} x_i$. Since $\sum_{i=1}^{\infty} \|x_i\| \leq \sum_{i=1}^{\infty} \varepsilon_i$, x is well defined. For every j , we have $T^{n_{k_j}} x \in M$ and

$$\begin{aligned} \|T^{n_{k_j}} x - y_j\| &= \left\| T^{n_{k_j}} x_j - y_j + \sum_{i=1}^{j-1} T^{n_{k_j}} x_i + \sum_{i=j+1}^{\infty} T^{n_{k_j}} x_i \right\| \\ &\leq \|T^{n_{k_j}} x_j - y_j\| + \sum_{i=1}^{j-1} \|T^{n_{k_j}} x_i\| + \sum_{i=j+1}^{\infty} \|T^{n_{k_j}} x_i\| \\ &\leq j\varepsilon_j + \sum_{i=j+1}^{\infty} \varepsilon_i. \end{aligned}$$

Thus, $\lim_{j \rightarrow \infty} \|T^{n_{k_j}} x - y_j\| = 0$ and T is M -hypercyclic. $\square$

Remark 2.6. Notice that in Theorem 2.5 the set X is not assumed to be dense in M . The following example shows that the first two conditions of Theorem 2.5 are not sufficient for T to be M -hypercyclic. Thus we answer one question asked in [4].

In ℓ_p with canonical basis $(e_n)_{n=1}^{\infty}$, let B be the backward shift operator $Be_n = e_{n-1}$ and let S be the linear operator such that $Se_n = \frac{1}{2}e_{n+1}$. Let $(n_k)_{k=1}^{\infty}$ be an increasing sequence of positive numbers such that $n_0 = 0$ and $n_{k+1} > 2\sum_{i=0}^k n_i$. Let $L_0 = M_0 = \text{Lin}\{e_0\}$ be the linear space generated by e_0 , $L_1 = S^{n_1}M_0$, $M_1 = M_0 \oplus L_1$, and in general let $L_{k+1} = S^{n_{k+1}}M_k$, $M_{k+1} = M_k \oplus L_{k+1}$. Define M to be the closure of $\bigcup_{k \geq 0} M_k$. It is easy to show by induction that $M_k \subset \text{Lin}\{e_j : j \leq \sum_{i=0}^k n_i\}$ and $L_{k+1} \subset \text{Lin}\{e_j : n_{k+1} \leq j \leq \sum_{i=0}^{k+1} n_i\}$. Thus, if $x \in M$, then $x(j) = 0$ for all $\sum_{i=0}^k n_i < j < n_{k+1}$.

Example 2.7. The operator $T = 2B$ satisfies the first two conditions of Theorem 2.5, but T is not M -hypercyclic.

Proof. Let $Y = (y_j)_{j=1}^{\infty} \subset c_{00} \cap M$ be a dense sequence in M . There exists an increasing sequence $(k_j)_{j=1}^{\infty}$ such that $y_i \in M_{k_j}$ for all j and $i \leq j$. Let $X = \bigcup_{j \geq 1} \{S^{n_{k_i}} y_j : i \geq j\}$. Obviously T with $X, Y, (n_{k_i})_{i=1}^{\infty}$ satisfies the first two conditions of the previous theorem. It remains to show that T is not M -hypercyclic.

Suppose that T is M -hypercyclic with the M -hypercyclic vector $x \in M$. For an arbitrary k there exists $m > \sum_{i=0}^k n_i$ such that $T^m x \in M$. One can choose $l > k$ such that $n_{l+1} - \sum_{i=0}^l n_i > 2m$. Since $\sum_{i=0}^l n_i < n_{l+1} - m < n_{l+1} - m + \sum_{i=0}^k n_i < n_{l+1}$, if $n_{l+1} - m \leq j \leq n_{l+1} - m + \sum_{i=0}^k n_i$, then $(T^m x)(j) = 0$. The fact that $(T^m x)(j) = 2^m x(j+m)$ implies that $x(j) = 0$ for all $n_{l+1} \leq j \leq n_{l+1} + \sum_{i=0}^k n_i$.

By the construction of M , it is easy to see that $M = \bigoplus_{q=0}^{\infty} S^{m_q} M_l$ for some increasing sequence $(m_q)_{q=0}^{\infty}$ such that $m_0 = 0$ and $m_{q+1} - m_q - \sum_{i=0}^l n_i > 2m$, $q \geq 0$. The same argument can be used to show that $x(j) = 0$ for all $m_q \leq j \leq m_q + \sum_{i=0}^k n_i$, $q \geq 1$. This contradicts the fact that x is M -hypercyclic. $\square$

The following two lemmas show that M -hypercyclicity does not imply M -transitivity.

Let $(n_k)_{k=1}^{\infty}$ and $(m_k)_{k=1}^{\infty}$ be increasing sequences of positive integers such that $n_k < m_k < n_{k+1}$ for all k . In ℓ_p , $p \geq 1$, consider the closed linear subspace M generated by the set $\{e_j : n_k \leq j \leq m_k, k \geq 1\}$.

Lemma 2.8. *If $\sup_{k \geq 1}(m_k - n_k) = \infty$, then $T = 2B$ is M -hypercyclic.*

Proof. Let $Y = (y_j)_{j=1}^\infty \subset c_{00} \cap M$ be a dense subset of M . Since $\sup_{k \geq 1}(m_k - n_k) = \infty$, for y_1 there exist k_1 and N_1 such that $|y_1| < n_{k_1} < N_1 < N_1 + |y_1| < m_{k_1}$. By induction, it is easy to see that there exist increasing sequences $(N_j)_{j=1}^\infty$, $(k_j)_{j=1}^\infty$ such that for every fixed $j > 1$ we have

- (1) $|y_i| < n_{k_j} < N_j < N_j + |y_i| < m_{k_j}$ for all $1 \leq i \leq j$,
- (2) $N_j - N_i > n_{k_j}$ for all $1 \leq i < j$.

Let $X = \bigcup_{j \geq 1} \{S^{N_i} y_j : i \geq j\}$. It is clear that $X \subset M$. To verify that T , X , Y and $(N_j)_{j=1}^\infty$ satisfy all conditions of Theorem 2.5, hence that T is M -hypercyclic, it is enough to check condition (3) that $X \subset \bigcap_{j \geq 1} T^{-N_j} M$.

Let $x = S^{N_i} y_j \in X$, where $i \geq j$. For every l consider $T^{N_l} x$. If $l > i$, then $N_l - N_i > n_{k_l} > |y_j|$, so $T^{N_l} x = T^{N_l - N_i} y_j = 0 \in M$. If $l < i$, then $N_i - N_l + |y_j| < N_i + |y_j| < m_{k_i}$ and $N_i - N_l > n_{k_i}$, so $T^{N_l} x = S^{N_i - N_l} y_j \in \text{Lin}\{e_q : n_{k_i} \leq q \leq m_{k_i}\} \subset M$. If $l = i$, then $T^{N_l} x = y_j \in M$. $\square$

Lemma 2.9. *If $\sup_{k \geq 1}(n_{k+1} - m_k) = \infty$, then $T = 2B$ is not M -transitive.*

Proof. Let U and V be nonempty open subsets of M and suppose that there exists a positive number m such that $U \cap T^{-m}V$ contains an open subset W of M . For an element $x \in W$ there exists $\varepsilon > 0$ so that if $y \in M$ and $\|x - y\| < \varepsilon$, then $y \in W$. Since $\sup_{k \geq 1}(n_{k+1} - m_k) = \infty$, one can find a positive number j such that $n_{j+1} - m_j > m$. Consider $y = x + \frac{1}{2}\varepsilon e_{n_{j+1}}$. Since $y \in M$ and $\|x - y\| < \varepsilon$, $y \in W \subset U \cap T^{-m}V$. However, $T^m y = T^m x + \frac{1}{2}\varepsilon T^m e_{n_{j+1}} \in V$ implies $T^m e_{n_{j+1}} = 2^m e_{n_{j+1}-m} \in M$, which contradicts the fact that $n_{j+1} - m > m_j$. $\square$

Proposition 2.10. *If $\sup_{k \geq 1}(m_k - n_k) = \sup_{k \geq 1}(n_{k+1} - m_k) = \infty$, then $T = 2B$ is M -hypercyclic but T is not M -transitive.*

3. OPERATORS WITH DENSE GENERALIZED KERNEL

If T is a bounded linear operator in a separable Banach space E , then the generalized kernel of T is defined to be $\ker^*(T) = \bigcup_{m=1}^\infty \ker(T^m)$. In this section we prove the following result.

Theorem 3.1. *Let T be a bounded linear operator on a separable Banach space E such that $\ker^*(T)$ is dense in E and there exists a map $A : \ker^*(T) \rightarrow \ker^*(T)$ satisfying*

- (1) $A^m x \rightarrow 0$ for every $x \in \ker^*(T)$,
- (2) $TA = I$ on $\ker^*(T)$.

Then T is M -hypercyclic for all finite co-dimensional subspaces M .

In [4] the authors raise another question concerning the relation between hypercyclicity and M -hypercyclicity: namely, if T is hypercyclic, is T M -hypercyclic for some subspace M ? Theorem 3.1 gives a partial answer to that question. Notice that operator T satisfying the conditions of Theorem 3.1 is clearly hypercyclic. In order to prove the theorem we need the following two elementary lemmas.

Lemma 3.2. *There exist a sequence of linearly independent vectors $(e_p)_{p=1}^\infty$ in $\ker^*(T)$ such that $L = \text{Lin}\{e_p, p \geq 1\}$ is dense in E , and a linear map $S : L \rightarrow L$ such that $S^m x \rightarrow 0$ for every $x \in L$ and $TS = I$ on L .*

Proof. Let $(x_n)_{n=1}^\infty \subset \ker^*(T)$ be a dense sequence in E and $(e_p)_{p=1}^\infty$ be a maximal subset of linearly independent vectors of the set $\{A^m x_n, m \geq 0, n \geq 1\}$. If $L = \text{Lin}\{e_p, p \geq 1\}$, then L is dense in E and the image of $(e_p)_{p=1}^\infty$ under A is in L . Since $(e_p)_{p=1}^\infty$ are linearly independent, we can extend A linearly to L . The extension S of A satisfies all requirements of the lemma. $\square$

Lemma 3.3. *If $(f_j)_{j=1}^n \subset E^*$, then $\bigcap_{j=1}^n (L \cap \ker(f_j))$ is dense in $\bigcap_{j=1}^n \ker(f_j)$.*

Proof. Since L is dense in E , if $x \in \bigcap_{j=1}^n \ker(f_j)$, then there exists a sequence $(x_k)_{k=1}^\infty \subset L$ such that $x = \sum_{k=1}^\infty x_k$. Consider the subspace

$$H = \text{Lin}\{(f_1(x_k), f_2(x_k), \dots, f_n(x_k)), k \geq 1\} \subset \mathbb{R}^n.$$

If $(x_k)_{k=1}^\infty \subset \bigcap_{j=1}^n \ker(f_j)$, then there is nothing to prove. If $(x_k)_{k=1}^\infty \not\subset \bigcap_{j=1}^n \ker(f_j)$, then $d = \dim H > 0$. There exist $(k_i)_{i=1}^d$ so that $y_i := (f_1(x_{k_i}), f_2(x_{k_i}), \dots, f_n(x_{k_i}))$, $1 \leq i \leq d$, is a basis of H . Let

$$u_m = \left(f_1\left(\sum_{k=1}^m x_k\right), \dots, f_n\left(\sum_{k=1}^m x_k\right) \right) \in \mathbb{R}^n.$$

For every m there exist scalars $(t_{m,i})_{i=1}^d$ so that $\sum_{i=1}^d t_{m,i} y_i = u_m$. Since $u_m \rightarrow 0$, it follows that $t_{m,i} \rightarrow 0$ as $m \rightarrow \infty$ for all $i \leq d$. Therefore x can be approximated by $\sum_{k=1}^m x_k - \sum_{i=1}^d t_{m,i} x_{k_i} \in \bigcap_{j=1}^n (L \cap \ker(f_j))$. $\square$

Proof of Theorem 3.1. For simplicity of notation we prove the case when M is a one-codimensional subspace of E . It will be clear that the argument still works for the case when M is of finite codimension. By Lemma 3.2 we can replace $\ker^*(T)$ and A with L and S respectively.

Let $M = \ker(f)$ for some $f \in E^*$. By Lemma 3.3 there exists a dense sequence $(x_k)_{k=1}^\infty \subset L$ in M . We construct by induction sequences $(y_k)_{k=1}^\infty \subset M$ and $(m_k)_{k=1}^\infty \subset \mathbb{N}$ with the following properties:

- (1) $T^{m_j} y_k \in M$ for all $j < k$ and $T^{m_j} y_k = 0$ for all $j > k$,
- (2) $T^{m_k} y_k = x_k$,
- (3) $\|y_k\| < 2^{-k}$ and $\|T^{m_j} y_k\| < 2^{-k}$ for all $j < k$.

If such sequences exist, then the same argument as in the proof of Theorem 2.5 can be used to show that T is M -hypercyclic with the M -hypercyclic vector $\sum_{k=1}^\infty y_k$.

For $k = 1$ consider the subspace $L_1 = \text{Lin}\{f(e_p), p \geq 1\}$ of $\mathbb{R}$. If $\dim L_1 = 0$, then $S^m x_1 \in M$ for all $m \geq 1$. Since $S^m x_1 \rightarrow 0$, choose m_1 large enough so that $\|S^{m_1} x_1\| < 2^{-1}$ and put $y_1 = S^{m_1} x_1$. If $\dim L_1 = 1$, there exists $p_{1,1} \geq 1$ so that $f(e_{p_{1,1}}) \neq 0$. Since $e_{p_{1,1}} \in L$ and $S^m x_1 \rightarrow 0$, one can choose m_1 large enough so that $T^{m_1} e_{p_{1,1}} = 0$ and $\|t_{1,1} e_{p_{1,1}} + S^{m_1} x_1\| < 2^{-1}$, where $t_{1,1} = -\frac{f(S^{m_1} x_1)}{f(e_{p_{1,1}})}$. Let $y_1 = t_{1,1} e_{p_{1,1}} + S^{m_1} x_1$. In either case $y_1 \in M$ by the choice of $t_{1,1}$, $\|y_1\| < 2^{-1}$ and $T^{m_1} y_1 = x_1$. Thus, conditions (1), (2), and (3) are satisfied.

Suppose that we have constructed $(y_k)_{k=1}^{n-1} \subset M$ satisfying (1), (2), and (3). Consider the subspace $L_n = \text{Lin}\{(f(e_p), f(T^{m_1} e_p), \dots, f(T^{m_{n-1}} e_p)), p \geq 1\}$ of $\mathbb{R}^n$ and let $d = \dim L_n$. If $d = 0$, then $T^{m_j} S^l x_n \in M$ for all $j < n$ and $l \geq 1$. Since $(y_k)_{k=1}^{n-1} \subset M$ and $S^m x_n \rightarrow 0$ as $m \rightarrow \infty$, one can choose m_n large enough so that $y_n = S^{m_n} x_n$ satisfies conditions (1), (2), and (3). If $d \neq 0$, then there exist $p_{n,1}, \dots, p_{n,d}$ so that $g_{n,i} := (f(e_{p_{n,i}}), f(T^{m_1} e_{p_{n,i}}), \dots, f(T^{m_{n-1}} e_{p_{n,i}})), 1 \leq i \leq d$, is a

basis of L_n . Since $S^m x_n \in L$, $u_{n,m} = (f(S^m x_n), f(T^{m_1} S^m x_n), \dots, f(T^{m_{n-1}} S^m x_n)) \in L_n$ for all m . Moreover, $u_{n,m} \rightarrow 0$ as $m \rightarrow 0$. Therefore one can find m_n large enough and scalars $(t_{n,i})_{i=1}^d$ so that

- (a) $u_{n,m_n} + \sum_{i=1}^d t_{n,i} g_{n,i} = 0$ and $T^{m_n} y_j = 0$ for all $j < n$,
- (b) $T^{m_n} e_{p_{n,i}} = 0$ for all $i \leq d$,
- (c) $\|y_n\| < 2^{-n}$ and $\|T^{m_j} y_n\| < 2^{-n}$ for all $j < n$,

where $y_n = S^{m_n} x_n + \sum_{i=1}^d t_{n,i} e_{p_{n,i}}$. The properties (a), (b), and (c) ensure that $(y_k)_{k=1}^n$ satisfy conditions (1), (2), and (3). The theorem is proved. $\square$

Remark 3.4. It is easy to see that the theorem is still true if E is an F -space or the scalar field is complex.

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