RANDOM DATA CAUCHY THEORY FOR THE INCOMPRESSIBLE THREE DIMENSIONAL NAVIER–STOKES EQUATIONS

TING ZHANG AND DAOYUAN FANG

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ABSTRACT. We study the existence and uniqueness of the strong solution for the incompressible Navier–Stokes equations with the L^2 initial data and the periodic space domain $\mathbb{T}^3$. After a suitable randomization, we are able to construct the local unique strong solution for a large set of initial data in L^2 . Furthermore, if $\|u_0\|_{L^2}$ is small, we show that the probability for the global existence and uniqueness of the solution is large.

1. INTRODUCTION

In this paper, we are going to study the incompressible periodic Navier–Stokes system, namely,

$$(1.1) \quad \begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla \Pi = 0, & x \in \mathbb{T}^3, \ t > 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where $u(t, x)$ and $\Pi(t, x)$ denote the fluid velocity and the pressure, respectively, the three dimensional torus $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$.

The existence of at least one weak solution of (1.1) has been well-known since the pioneering work of Leray [13] and Hopf [8]. To prove the uniqueness of the solution in some time interval $[0, T)$, we need besides the condition $u_0 \in L^2$ a further regularity property of the initial value u_0 . The first sufficient condition on the initial value for a bounded domain seems to have been described in [11]. Then many results on sufficient conditions on u_0 to guarantee the existence and uniqueness of the solution were proved; see, e.g., [1, 4, 6, 7, 9, 12]. In [5], Farwig and Sohr gave the weakest necessary and sufficient condition when the space is a bounded domain with $C^{2,1}$ boundary. These well-posedness theories are valid all the way down to the scaling regularity. In [3], Burq and Tzvetkov considered the local well-posedness theory for the cubic nonlinear wave equation with the initial data in the supercritical space. Using Burq and Tzvetkov’s method, we will obtain

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the existence and uniqueness of the local solution in “a large class of initial data in $L^2_\sigma(\mathbb{T}^3)$ ”.

Let us first explain what we mean by “a large class of initial data in $L^2_\sigma(\mathbb{T}^3)$ ”, where $L^2_\sigma(\mathbb{T}^3) = \overline{C^\infty_\sigma(\mathbb{T}^3)}^{\|\cdot\|_{L^2}}$, $C^\infty_\sigma(\mathbb{T}^3) = \{u \in C^\infty(\mathbb{T}^3) : \operatorname{div} u = 0, \int_{\mathbb{T}^3} u dx = 0\}$. Define $\mathbb{Z}^3_+ = \{k \in \mathbb{Z}^3; k_1 > 0\} \cup \{k \in \mathbb{Z}^3; k_1 = 0, k_2 > 0\} \cup \{k \in \mathbb{Z}^3; k_1 = 0, k_2 = 0, k_3 > 0\}$, $\mathbb{Z}^3_- = -\mathbb{Z}^3_+$ and $\mathbb{Z}^3_* = \mathbb{Z}^3_+ \cup \mathbb{Z}^3_-$. Let $k^\perp = \{\eta \in \mathbb{R}^3; k \cdot \eta = 0\}$, $\{e_{k,1}, e_{k,2}\}$ be an orthonormal basis of $k^\perp$. Let $e_k(x) = \cos(k \cdot x)$ if $k \in \mathbb{Z}^3_+$, $e_k(x) = \sin(k \cdot x)$ if $k \in \mathbb{Z}^3_-$ denote

$$e_k^i(x) = e_k(x)e_{k,i}, \quad \forall k \in \mathbb{Z}^3_*, \quad i = 1, 2.$$

Then, $\{e_k^i; k \in \mathbb{Z}^3_*, i = 1, 2\}$ is a Fourier basis of $L^2_\sigma(\mathbb{T}^3)$ (up to the constant $\frac{\sqrt{2}}{(2\pi)^{\frac{3}{2}}}$).

Let $\{h_k^i(\omega)\}_{k \in \mathbb{Z}^3_*, i=1,2}$ be a sequence of independent, 0 mean value, real random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that

$$(1.2) \quad \exists C > 0, \quad \forall k \in \mathbb{Z}^3_*, \quad i = 1, 2, \quad \int_\Omega |h_k^i(\omega)|^4 dP(\omega) \leq C.$$

For $u_0 \in L^2_\sigma(\mathbb{T}^3)$, i.e.,

$$(1.3) \quad u_0(x) = \sum_{k \in \mathbb{Z}^3_*} \sum_{i=1}^2 \alpha_k^i e_k^i, \quad \text{where} \quad \sum_{k \in \mathbb{Z}^3_*} \sum_{i=1}^2 (\alpha_k^i)^2 < \infty,$$

we consider the map

$$(1.4) \quad \omega \mapsto u_0^\omega$$

from probability space $(\Omega, \mathcal{F})$ to $L^2_\sigma(\mathbb{T}^3)$ equipped with Borel sigma algebra, defined by

$$(1.5) \quad u_0^\omega(x) = \sum_{k \in \mathbb{Z}^3_*} \sum_{i=1}^2 h_k^i(\omega) \alpha_k^i e_k^i.$$

One can check that the map $\omega \mapsto u_0^\omega$ is measurable and $u_0^\omega \in L^2(\Omega; L^2_\sigma(\mathbb{T}^3))$. Thus it defines a $L^2_\sigma(\mathbb{T}^3)$ valued random variable. From Appendix 2 in [3], one can obtain that if $u_0 \notin H^\varepsilon(\mathbb{T}^3)$, $\varepsilon > 0$, for almost all ω , u_0^ω does not belongs to $H^\varepsilon(\mathbb{T}^3)$; i.e., the randomization $\omega \mapsto u_0^\omega$ does not give a regularization in the scale of the Sobolev spaces. But, the L^p regularization of this randomization will play an important role in this paper. Our main results read as follows.

Theorem 1.1 (Local existence and uniqueness). *Let us fix $u_0 \in L^2_\sigma(\mathbb{T}^3)$, and let u_0^ω be defined by the randomization (1.4)–(1.5). Then for almost all $\omega \in \Omega$, there exist $T_\omega^* > 0$ and a unique solution to (1.1) with the initial data u_0^ω in the space*

$$(1.6) \quad \{u : t^{\frac{1}{2}-\frac{3}{2p}} u(t) \in C([0, T_\omega^*]; L^p(\mathbb{T}^3)), \quad p \in [3, \infty], \\ t^{1-\frac{3}{2q}} \nabla u(t) \in C([0, T_\omega^*]; L^q(\mathbb{T}^3)), \quad q \in [3, \infty)\}.$$

In the proof of Theorem 1.1, we use Lemma 2.3 (obtained from Lemmas 3.1 and 4.2 in [3] or the Khinchin inequality [10]) to show that $u_0^\omega \in L^3$ for almost all $\omega \in \Omega$ in Proposition 2.4, and we use Kato’s result [9] to get the local existence and uniqueness of the local solution. In the following theorem, we will estimate the probability of the event $\{\omega \in \Omega : T_\omega^* = +\infty\}$.

Theorem 1.2 (Global existence and uniqueness). *Under the conditions in Theorem 1.1, there exist $C > 0$ and an event Ω_∞ such that*

$$(1.7) \quad P(\Omega_\infty) \geq 1 - C\|u_0\|_{L^2}^3,$$

and for every $\omega \in \Omega_\infty$ there exists a unique global solution u^ω of (1.1) with the initial data u_0^ω in the space (1.6) with $T_\omega^* = +\infty$.

Moreover, if the random variables h_k^i are standard real Gaussian or Bernoulli variables, then (1.7) can be improved to

$$(1.8) \quad P(\Omega_\infty) \geq 1 - Ce^{-(C\|u_0\|_{L^2})^{-2}}, \quad C > 0.$$

Remark 1.3. This theorem means that if $\|u_0\|_{L^2}$ is small, the probability for the global existence and uniqueness of the solution is large.

By using Kato's result [9], we cannot estimate $\{\omega \in \Omega : T_\omega^* > T\}$ directly. Thus, we need an additional condition

$$(1.9) \quad \exists C > 0, \forall k \in \mathbb{Z}_*^3, i = 1, 2, \int_\Omega |h_k^i(\omega)|^6 dP(\omega) \leq C,$$

and use the smallness of the time T to control the smallness of the linear solution $\|e^{t\Delta}u_0^\omega\|_{L^6(\Omega;L^4([0,T];L^6(\mathbb{T}^3)))}$ and $\|\nabla e^{t\Delta}u_0^\omega\|_{L^6(\Omega;L^2([0,T];L^3(\mathbb{T}^3)))}$ in Propositions 3.1 and 3.2. Finally, we will estimate the probability of the event $\{\omega \in \Omega : T_\omega^* > T\}$ in the following theorem.

Theorem 1.4. *Under the conditions in Theorem 1.1, if the random variables h_k^i satisfy (1.9), then for almost all $\omega \in \Omega$ there exist $T_\omega > 0$ and a unique solution to (1.1) with the initial data u_0^ω in a space continuously embedded in*

$$X_\omega = e^{t\Delta}u_0^\omega + C([0, T_\omega]; H^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2([0, T_\omega]; H^{\frac{3}{2}}(\mathbb{T}^3)).$$

More precisely, there exist $C > 0$, and for every $0 < T \leq 1$ an event Ω_T such that

$$(1.10) \quad P(\Omega_T) \geq 1 - C \left(\|u_0\|_{L^2(\mathbb{T}^3)} T^{\frac{1}{4}} + \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2} \right)^6,$$

and for every $\omega \in \Omega_T$ there exists a unique solution u^ω of (1.1) with the initial data u_0^ω in a space continuously embedded in $C([0, T]; L^2(\mathbb{T}^3)) \cap L^2([0, T]; H^1(\mathbb{T}^3))$.

Moreover, if the random variables h_k^i are standard real Gaussian or Bernoulli variables, then (1.10) can be improved to

$$(1.11) \quad P(\Omega_T) \geq 1 - C \exp \left(- \frac{c}{\|u_0\|_{L^2(\mathbb{T}^3)}^2 T^{\frac{1}{2}} + \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2}^2} \right), \quad C, c > 0.$$

Remark 1.5. Since $u_0 \in L_\sigma^2(\mathbb{T}^3)$, it is easy to see that

$$\lim_{T \rightarrow 0} P(\Omega_T) = 1.$$

Our results show that in some sense the problem (1.1) is local well-posed for almost all initial data in $L_\sigma^2(\mathbb{T}^3)$, exhibiting a gain of $\frac{1}{2}$ derivatives with respect to the critical Hilbert space $H^{\frac{1}{2}}(\mathbb{T}^3)$.

Remark 1.6. As that in [3], the method of proof consists in using the fact that although the initial data have low regularity, their L^p properties are (almost surely) much better than expected, allowing the use of a fixed point method after having singled out the linear evolution. Such L^p regularization phenomena have been well-known in the context of Fourier series since the work of Paley-Zygmund [14], and Ayache and Tzvetkov studied similar phenomena in [2].

Remark 1.7. From the estimate of $\|e^{t\Delta}u_0^\omega\|_{L_t^4 L_x^6}$ in Proposition 3.1 and similar arguments as that in [5], one can also obtain that there exist $T_\omega > 0$ and a unique solution to (1.1) with the initial data u_0^ω in a space continuously embedded in $L^4([0, T_\omega]; L^6(\mathbb{T}^3))$. Then (1.10) can be improved to

$$(1.12) \quad P(\Omega_T) \geq 1 - C \left(\|u_0\|_{L^2(\mathbb{T}^3)} T^{\frac{1}{4}} \right)^6,$$

and (1.11) can be improved to

$$(1.13) \quad P(\Omega_T) \geq 1 - C \exp \left(- \frac{c}{\|u_0\|_{L^2(\mathbb{T}^3)}^2 T^{\frac{1}{2}}} \right), \quad C, c > 0.$$

This means that considering the different solution spaces, we would have different estimates of $P(\Omega_T)$.

2. EXISTENCE AND UNIQUENESS

In this section, we prove the existence and uniqueness of the solution for the 3D incompressible Navier–Stokes equations. The following lemmas were proved in [3].

Lemma 2.1 ([3], Lemma 4.2). *Assume that $\{l_n(\omega)\}_{n=1}^\infty$ is a sequence of independent, 0-mean value, complex random variables satisfying for some $k \in \mathbb{N}^*$,*

$$\exists C > 0, \forall n \geq 1, \int_{\Omega} |l_n(\omega)|^{2k} dP(\omega) \leq C.$$

Then

$$(2.1) \quad \forall 2 \leq p \leq 2k, \exists C > 0, \forall \{c_n\}_{n \in \mathbb{N}^*} \in l^2(\mathbb{N}^*, \mathbb{C}), \left\| \sum_n c_n l_n \right\|_{L^p(\Omega)} \leq C \|c_n\|_{l^2}.$$

Lemma 2.2 ([3], Lemma 3.1). *Let $\{l_n(\omega)\}_{n=1}^\infty$ be a sequence of independent, 0-mean value, real random variables with associated sequence of distributions $\{\mu_n\}_{n=1}^\infty$. Assume that μ_n satisfy the property*

$$(2.2) \quad \exists c > 0 : \forall \gamma \in \mathbb{R}, \forall n \geq 1, \left| \int_{-\infty}^{\infty} e^{\gamma x} d\mu_n(x) \right| \leq e^{c\gamma^2}.$$

Then there exists $\alpha > 0$ such that for every $\lambda > 0$ and every sequence $\{c_n\}_{n=1}^\infty \in l^2$ of real numbers,

$$(2.3) \quad P \left(\omega : \left| \sum_{n=1}^{\infty} c_n l_n(\omega) \right| > \lambda \right) \leq 2e^{-\frac{\alpha \lambda^2}{\sum c_n^2}}.$$

As a consequence there exists $C > 0$ such that for every $p \geq 2$ and every sequence $\{c_n\}_{n=1}^\infty \in l^2$,

$$(2.4) \quad \left\| \sum_{n=1}^{\infty} c_n l_n(\omega) \right\|_{L^p(\Omega)} \leq C \sqrt{p} \left(\sum_{n=1}^{\infty} c_n^2 \right)^{\frac{1}{2}}.$$

Then, from Lemmas 2.1–2.2 (or the Khinchin inequality [10]), we can immediately obtain the following lemma.

Lemma 2.3. *Assume that $\{h_k^i\}_{k \in \mathbb{Z}_*^3, i=1,2}$ is a sequence of independent, 0-mean value, real random variables satisfying*

$$(2.5) \quad \exists C > 0, \forall k \in \mathbb{Z}_*^3, i = 1, 2, \int_{\Omega} |h_k^i(\omega)|^2 k dP(\omega) \leq C,$$

for some $k \in \mathbb{N}^*$. Then there exists $C > 0$ such that for every sequence $\{c_k^i\}_{k \in \mathbb{Z}_*^3, i=1,2} \in l_{k,i}^2$,

$$(2.6) \quad \left\| \sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 c_k^i h_k^i(\omega) \right\|_{L^q(\Omega)} \leq C \|c_k^i\|_{l_{k,i}^2}, \quad q \in [2, 2k].$$

Moreover, if h_k^i are standard real Gaussian or Bernoulli variables, then there exists $C > 0$ such that for every $p \geq 2$ and every sequence $\{c_k^i\}_{k \in \mathbb{Z}_*^3, i=1,2} \in l_{k,i}^2$,

$$(2.7) \quad \left\| \sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 c_k^i h_k^i(\omega) \right\|_{L^p(\Omega)} \leq C \sqrt{p} \|c_k^i\|_{l_{k,i}^2}.$$

Proposition 2.4. *Assume that $u_0 \in L_{\sigma}^2(\mathbb{T}^3)$, $\{h_k^i\}_{k \in \mathbb{Z}_*^3, i=1,2}$ is a sequence of independent, 0-mean value, real random variables satisfying (1.2). Then there exists $C > 0$ such that for every $u_0 \in L_{\sigma}^2(\mathbb{T}^3)$ we have*

$$(2.8) \quad \|u_0^{\omega}\|_{L^3(\Omega; L^3(\mathbb{T}^3))} \leq C \|u_0\|_{L^2(\mathbb{T}^3)}.$$

In particular, if we set

$$D_{\lambda, u_0} = \{\omega \in \Omega : \|u_0^{\omega}\|_{L^3(\mathbb{T}^3)} \geq \lambda\},$$

from the Bienaymé-Tchebichev inequality we obtain that there exists $C > 0$ such that for every $\lambda > 0$ and every $u_0 \in L_{\sigma}^2(\mathbb{T}^3)$,

$$(2.9) \quad P(D_{\lambda, u_0}) \leq C \lambda^{-3} \|u_0\|_{L^2}^3.$$

Proof. Using the Minkowski inequality and Lemma 2.3, we have

$$\begin{aligned} \|u_0^{\omega}\|_{L^3(\Omega; L^3(\mathbb{T}^3))} &= \left\| \sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 h_k^i(\omega) \alpha_k^i e_k^i \right\|_{L^3(\mathbb{T}^3; L^3(\Omega))} \\ &\leq C \left\| \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 |\alpha_k^i e_k^i|^2 \right)^{\frac{1}{2}} \right\|_{L^3(\mathbb{T}^3)} \\ &\leq C \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 \|\alpha_k^i e_k^i\|_{L^3(\mathbb{T}^3)}^2 \right)^{\frac{1}{2}} \\ &\leq C \|\alpha_k^i\|_{l_{k,i}^2} = C \|u_0\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

Using the Bienaymé-Tchebichev inequality, we obtain

$$(2.10) \quad P(E_{\lambda, u_0}) \leq \lambda^{-3} (C \|u_0\|_{L^2})^3. \quad \square$$

Theorem 2.5 (T. Kato, 1972, [9]). *If $u_0 \in L^3$, then there exists $T^* > 0$ such that there exists a unique solution to (1.1) with the initial data u_0 in the space*

$$\{u : t^{\frac{1}{2}-\frac{3}{2p}}u(u) \in C([0, T_\omega^*]; L^p(\mathbb{T}^3)), p \in [3, \infty], \\ t^{1-\frac{3}{2q}}\nabla u(t) \in C([0, T_\omega^*]; L^q(\mathbb{T}^3)), q \in [3, \infty)\}.$$

Furthermore, there exists a constant $c_1 > 0$ such that

$$\|u_0\|_{L^3} \leq c_1 \Rightarrow T^* = +\infty.$$

Proof of Theorem 1.1. From Proposition 2.4, we have that $u_0^\omega \in L^3(\mathbb{T}^3)$ for almost all $\omega \in \Omega$. Then, from Theorem 2.5, one can obtain the local existence and uniqueness of the solution and finish the proof of Theorem 1.1. $\square$

Proposition 2.6. *Under the assumptions of Proposition 3.1, if h_k^i are standard real Gaussian or Bernoulli variables, then there exists $C > 0$ such that for every $u_0 \in L_\sigma^2(\mathbb{T}^3)$, $p \geq 3$, we have*

$$(2.11) \quad \|u_0^\omega\|_{L^p(\Omega; L^3(\mathbb{T}^3))} \leq C\sqrt{p}\|u_0\|_{L^2(\mathbb{T}^3)}.$$

In particular, if we set

$$\bar{D}_{\lambda, u_0} = \{\omega \in \Omega : \|u_0^\omega\|_{L^3(\mathbb{T}^3)} \geq \lambda\}$$

from the Bienaymé-Tchebichev inequality, we obtain that there exist $C > 0$ and $c > 0$ such that for every $\lambda > 0$ and every $u_0 \in L_\sigma^2(\mathbb{T}^3)$,

$$(2.12) \quad P(\bar{D}_{\lambda, u_0}) \leq Ce^{-\left(\frac{c\lambda}{\|u_0\|_{L^2}}\right)^2}.$$

Proof. Using a similar argument as that in the proof of (2.8), we can easily obtain (2.11).

Then, using the Bienaymé-Tchebichev inequality, we obtain that there exists $\alpha > 0$ such that for every $p \geq 3$ and every $u_0 \in L^2(\mathbb{T}^3)$,

$$(2.13) \quad P(\bar{D}_{\lambda, u_0}) \leq \lambda^{-p} (\alpha\sqrt{p}\|u_0\|_{L^2})^p.$$

If λ satisfies

$$(2.14) \quad \frac{\lambda}{\|u_0\|_{L^2}} \leq 3\alpha e,$$

then

$$Ce^{-\left(\frac{c\lambda}{\|u_0\|_{L^2}}\right)^2} \geq Ce^{-(3c\alpha e)^2},$$

and we can find some $C > 0$ and $c > 0$ such that

$$Ce^{-(3c\alpha e)^2} \geq 1 \geq P(\bar{D}_{\lambda, u_0}).$$

Thus inequality (2.12) holds if λ satisfies (2.14). If (2.14) does not hold, we set

$$p := \left(\frac{\lambda}{\alpha e\|u_0\|_{L^2}}\right)^2 \geq 9,$$

and from (2.13) we obtain

$$P(\bar{D}_{\lambda, u_0}) \leq \left(\frac{\alpha\|u_0\|_{L^2}}{\lambda} \cdot \frac{\lambda}{\alpha e\|u_0\|_{L^2}}\right)^{\left(\frac{\lambda}{\alpha e\|u_0\|_{L^2}}\right)^2} = e^{-\left(\frac{\lambda}{\alpha e\|u_0\|_{L^2}}\right)^2}. \quad \square$$

Proof of Theorem 1.2. Let

$$\Omega_\infty = D_{c_1, u_0}^c.$$

From (2.9) we have (1.7). For any $\omega \in \Omega_\infty$, we have

$$\|u_0^\omega\|_{L^3(\mathbb{T}^3)} < c_1.$$

From Theorem 2.5 we have $T_\omega^* = +\infty$.

Moreover, if the random variables h_k^i are standard real Gaussian or Bernoulli variables, then from (2.12) we can let

$$\Omega_\infty = \bar{D}_{c_1, u_0}^c,$$

obtain (1.8), and finish the proof of Theorem 1.2. $\square$

3. THE ESTIMATE OF $P(\Omega_T)$

In this section, we will prove Theorem 1.4. Although the proof of the existence and uniqueness of the local solution is classical, for the convenience of the reader, we also give the proofs. First we estimate the linear solution $e^{t\Delta}u_0^\omega$.

Proposition 3.1. *For $u_0 \in L_\sigma^2(\mathbb{T}^3)$, we consider the free evolution with the initial data u_0^ω given by*

$$\bar{u}^\omega(t, x) = e^{t\Delta}u_0^\omega.$$

Assume that $\{h_k^i\}_{k \in \mathbb{Z}_^3, i=1,2}$ is a sequence of independent, 0-mean value, real random variables satisfying (1.9). Then, for any $0 < T \leq 1$, we have*

$$(3.1) \quad \|\bar{u}^\omega\|_{L^6(\Omega; L^4([0, T]; L^6(\mathbb{T}^3)))} \leq C \|u_0\|_{L^2(\mathbb{T}^3)} T^{\frac{1}{4}},$$

$$(3.2) \quad \|\nabla \bar{u}^\omega\|_{L^6(\Omega; L^2([0, T]; L^3(\mathbb{T}^3)))} \leq C \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2},$$

$$(3.3) \quad P(E_{\lambda, T, u_0}) \leq C \lambda^{-6} \left(\|u_0\|_{L^2(\mathbb{T}^3)} T^{\frac{1}{4}} + \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2} \right)^6,$$

where $\lambda > 0$, $C > 0$ is independent of T , and

$$E_{\lambda, T, u_0} = \left\{ \omega \in \Omega : \|\bar{u}^\omega\|_{L^4([0, T]; L^6(\mathbb{T}^3))} + \|\nabla \bar{u}^\omega\|_{L^2([0, T]; L^3(\mathbb{T}^3))} \geq \lambda \right\}.$$

Proof. Using the Minkowski inequality and Lemma 2.3, we have, for any $T \in (0, 1]$,

$$\begin{aligned} \|\bar{u}^\omega\|_{L^6(\Omega; L^4([0, T]; L^6(\mathbb{T}^3)))} &\leq \left\| \sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 h_k^i(\omega) e^{-|k|^2 t} \alpha_k^i e_k^i \right\|_{L^4([0, T]; L^6(\mathbb{T}^3; L^6(\Omega)))} \\ &\leq C \left\| \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 e^{-2|k|^2 t} |\alpha_k^i e_k^i|^2 \right)^{\frac{1}{2}} \right\|_{L^4([0, T]; L^6(\mathbb{T}^3))} \\ &\leq C \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 \left\| e^{-|k|^2 t} |\alpha_k^i e_k^i|^2 \right\|_{L^4([0, T]; L^6(\mathbb{T}^3))} \right)^{\frac{1}{2}} \\ &\leq C T^{\frac{1}{4}} \|\alpha_k^i\|_{l_{k,i}^2} = C \|u_0\|_{L^2(\mathbb{T}^3)} T^{\frac{1}{4}} \end{aligned}$$

and

$$\begin{aligned}
\|\nabla \bar{u}^\omega\|_{L^6(\Omega; L^2([0,T]; L^3(\mathbb{T}^3)))} &\leq \left\| \sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 h_k^i(\omega) e^{-|k|^2 t} k \alpha_k^i e_k^i \right\|_{L^2([0,T]; L^3(\mathbb{T}^3; L^6(\Omega)))} \\
&\leq C \left\| \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 e^{-2|k|^2 t} |k|^2 |\alpha_k^i e_k^i|^2 \right)^{\frac{1}{2}} \right\|_{L^2([0,T]; L^3(\mathbb{T}^3))} \\
&\leq C \left(\sum_{k \in \mathbb{Z}_*^3} \sum_{i=1}^2 \left\| e^{-|k|^2 t} |k| |\alpha_k^i e_k^i| \right\|_{L^2([0,T]; L^3(\mathbb{T}^3))}^2 \right)^{\frac{1}{2}} \\
&\leq C \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2}.
\end{aligned}$$

Using the Bienaymé-Tchebichev inequality, we can obtain (3.3). $\square$

Proposition 3.2. *Under the assumptions of Proposition 3.1, if h_k^i are standard real Gaussian or Bernoulli variables, then there exists $C > 0$ such that for every $u_0 \in L_\sigma^2(\mathbb{T}^3)$, $p \geq 6$, $0 < T \leq 1$, we have*

$$(3.4) \quad \|\bar{u}^\omega\|_{L^p(\Omega; L^4([0,T]; L^6(\mathbb{T}^3)))} \leq C\sqrt{p}\|u_0\|_{L^2(\mathbb{T}^3)} T^{\frac{1}{4}},$$

$$(3.5) \quad \|\nabla \bar{u}^\omega\|_{L^p(\Omega; L^2([0,T]; L^3(\mathbb{T}^3)))} \leq C\sqrt{p} \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2},$$

$$(3.6) \quad P(F_{\lambda,T,u_0}) \leq C \exp \left(- \frac{c\lambda^2}{\|u_0\|_{L^2(\mathbb{T}^3)}^2 T^{\frac{1}{2}} + \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2}^2} \right),$$

where $\lambda > 0$, $C > 0$ and $c > 0$ are independent of T , and

$$F_{\lambda,T,u_0} = \left\{ \omega \in \Omega : \|\bar{u}^\omega\|_{L^4([0,T]; L^6(\mathbb{T}^3))} + \|\nabla \bar{u}^\omega\|_{L^2([0,T]; L^3(\mathbb{T}^3))} \geq \lambda \right\}.$$

Proof. Using a similar argument as that in the proof of (3.1)–(3.2), we can easily obtain (3.4)–(3.5).

Then, using the Bienaymé-Tchebichev inequality, we obtain that there exists $\alpha > 0$ such that for every $p \geq 6$ and every $u_0 \in L^2(\mathbb{T}^3)$,

$$(3.7) \quad P(F_{\lambda,T,u_0}) \leq \lambda^{-p} (\alpha\sqrt{p}G(u_0))^p,$$

where $G(u_0) = \|u_0\|_{L^2(\mathbb{T}^3)} T^{\frac{1}{4}} + \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2 T}} \right\|_{l_{k,i}^2}$. If λ satisfies

$$(3.8) \quad \frac{\lambda}{G(u_0)} \leq 6\alpha e,$$

then

$$C e^{-(\frac{c\lambda}{G(u_0)})^2} \geq C e^{-(6\alpha e)^2},$$

and we can find some $C > 0$ and $c > 0$ such that

$$C e^{-(6\alpha e)^2} \geq 1 \geq P(E_{\lambda,u_0}).$$

Thus inequality (3.6) holds if λ satisfies (3.8). If (3.8) does not hold, we set

$$p := \left(\frac{\lambda}{\alpha e G(u_0)} \right)^2 \geq 36,$$

and from (3.7), we obtain

$$P(F_{\lambda,T,u_0}) \leq \left(\frac{\alpha G(u_0)}{\lambda} \cdot \frac{\lambda}{\alpha e G(u_0)} \right)^{\left(\frac{\lambda}{\alpha e G(u_0)} \right)^2} = e^{-\left(\frac{\lambda}{\alpha e G(u_0)} \right)^2}. \quad \square$$

The following lemma is a classical result of the operator $e^{t\Delta}$ in the three dimensional case.

Lemma 3.3. *For any α and β satisfying $\frac{2}{\alpha} + \frac{3}{\beta} = 3$ and $\beta \in (\frac{3}{2}, 2]$, we have*

$$\left\| \int_0^t e^{(t-s)\Delta} f(s, x) ds \right\|_{L^\infty(0,T; H^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2(0,T; H^{\frac{3}{2}}(\mathbb{T}^3))} \leq C \|f\|_{L^\alpha(0,T; L^\beta(\mathbb{T}^3))},$$

where the constant C is independent of T .

Proof of Theorem 1.4. If we wish to solve (1.1) with the initial data u_0^ω by writing $u = \bar{u}^\omega + v$, we get that v solves

$$(3.9) \quad \begin{cases} v_t + v \cdot \nabla v - \Delta v + \nabla \Pi = -v \cdot \nabla \bar{u}^\omega - \bar{u}^\omega \cdot \nabla v - \bar{u}^\omega \cdot \nabla \bar{u}^\omega, \\ \operatorname{div} v = 0, \quad x \in \mathbb{T}^3, t > 0, \\ v|_{t=0} = 0. \end{cases}$$

Write this equation as

$$v(t, \cdot) = - \int_0^t e^{(t-s)\Delta} \mathbb{P}(v \cdot \nabla v + v \cdot \nabla \bar{u}^\omega + \bar{u}^\omega \cdot \nabla v + \bar{u}^\omega \cdot \nabla \bar{u}^\omega) ds,$$

where $\mathbb{P} = \operatorname{Id} + \nabla(-\Delta)^{-1} \operatorname{div}$ is the Leray-Hopf projector. Define the map

$$K^\omega : v \mapsto - \int_0^t e^{(t-s)\Delta} \mathbb{P}(v \cdot \nabla v + v \cdot \nabla \bar{u}^\omega + \bar{u}^\omega \cdot \nabla v + \bar{u}^\omega \cdot \nabla \bar{u}^\omega) ds.$$

Proposition 3.4. *There exists $C > 0$ such that for every $T \in (0, 1]$, every $u_0 \in L^2(\mathbb{T}^3)$ and every $\lambda > 0$, for $\omega \in E_{\lambda,T,u_0}^c$, the map K^ω satisfies*

$$(3.10) \quad \|K^\omega(v)\|_{X_T^{\frac{1}{2}}} \leq C(\lambda^2 + \|v\|_{X_T^{\frac{1}{2}}}^2),$$

$$(3.11) \quad \|K^\omega(v_1) - K^\omega(v_2)\|_{X_T^{\frac{1}{2}}} \leq C\|v_1 - v_2\|_{X_T^{\frac{1}{2}}}(\lambda + \|v_1\|_{X_T^{\frac{1}{2}}} + \|v_2\|_{X_T^{\frac{1}{2}}}),$$

where

$$(3.12) \quad \|f\|_{X_T^{\frac{1}{2}}} = \|f\|_{L^\infty([0,T]; H^{\frac{1}{2}}(\mathbb{T}^3))} + \|\nabla f\|_{L^2([0,T]; H^{\frac{1}{2}}(\mathbb{T}^3))}.$$

Proof. For $\omega \in E_{\lambda,T,u_0}^c$, $T \in (0, 1]$, we have

$$(3.13) \quad \|\bar{u}^\omega\|_{L^4([0,T]; L^6(\mathbb{T}^3))} + \|\nabla \bar{u}^\omega\|_{L^2([0,T]; L^3(\mathbb{T}^3))} < \lambda.$$

Then, we obtain

$$(3.14) \quad \|v \cdot \nabla v\|_{L^{\frac{4}{3}}([0,T]; L^2)} \leq C\|v\|_{L^4([0,T]; L^6)} \|\nabla v\|_{L^2([0,T]; L^3)} \leq C\|v\|_{X_T^{\frac{1}{2}}}^2,$$

$$(3.15) \quad \|v \cdot \nabla \bar{u}^\omega\|_{L^{\frac{4}{3}}([0,T]; L^2)} \leq C\|v\|_{L^4([0,T]; L^6)} \|\nabla \bar{u}^\omega\|_{L^2([0,T]; L^3)} \leq C\lambda\|v\|_{X_T^{\frac{1}{2}}},$$

$$(3.16) \quad \|\bar{u}^\omega \cdot \nabla v\|_{L^{\frac{4}{3}}([0,T]; L^2)} \leq C\|\bar{u}^\omega\|_{L^4([0,T]; L^6)} \|\nabla v\|_{L^2([0,T]; L^3)} \leq C\lambda\|v\|_{X_T^{\frac{1}{2}}},$$

$$(3.17) \quad \|\bar{u}^\omega \cdot \nabla \bar{u}^\omega\|_{L^{\frac{4}{3}}([0,T]; L^2)} \leq C\|\bar{u}^\omega\|_{L^4([0,T]; L^6)} \|\nabla \bar{u}^\omega\|_{L^2([0,T]; L^3)} \leq C\lambda^2.$$

Then, using Lemma 3.3, we can easily obtain that

$$\begin{aligned} & \|K^\omega(v)\|_{X_T^{\frac{1}{2}}} \\ & \leq C(\|v \cdot \nabla v\|_{L_T^{\frac{4}{3}} L_x^2} + \|v \cdot \nabla \bar{u}^\omega\|_{L_T^{\frac{4}{3}} L_x^2} + \|\bar{u}^\omega \cdot \nabla v\|_{L_T^{\frac{4}{3}} L_x^2} + \|\bar{u}^\omega \cdot \nabla \bar{u}^\omega\|_{L_T^{\frac{4}{3}} L_x^2}) \\ & \leq C(\lambda^2 + \|v\|_{X_T^{\frac{1}{2}}}^2). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \|K^\omega(v_1) - K^\omega(v_2)\|_{X_T^{\frac{1}{2}}} \\ & \leq C(\|(v_1 - v_2) \cdot \nabla v_1\|_{L_T^{\frac{4}{3}} L_x^2} + \|v_2 \cdot \nabla(v_1 - v_2)\|_{L_T^{\frac{4}{3}} L_x^2} \\ & \quad + \|(v_1 - v_2) \cdot \nabla \bar{u}^\omega\|_{L_T^{\frac{4}{3}} L_x^2} + \|\bar{u}^\omega \cdot \nabla(v_1 - v_2)\|_{L_T^{\frac{4}{3}} L_x^2}) \\ & \leq C\|v_1 - v_2\|_{X_T^{\frac{1}{2}}}(\lambda + \|v_1\|_{X_T^{\frac{1}{2}}} + \|v_2\|_{X_T^{\frac{1}{2}}}). \end{aligned} \quad \square$$

Let us now complete the proof of Theorem 1.4. Define

$$\Sigma = \bigcup_{n=1}^{\infty} E_{\lambda_1, \frac{1}{n}, u_0}^c,$$

where λ_1 is a positive constant satisfying

$$(3.18) \quad C\lambda_1^2 + C(2C\lambda_1^2)^2 \leq 2C\lambda_1^2 \quad \text{and} \quad C\lambda_1 + 2C(2C\lambda_1^2) \leq \frac{1}{2}.$$

From (3.3) we have

$$P(E_{\lambda_1, \frac{1}{n}, u_0}^c) \geq 1 - C\lambda_1^{-6} \left(\|u_0\|_{L^2(\mathbb{T}^3)} n^{-\frac{1}{4}} + \left\| \alpha_k^i \sqrt{1 - e^{-2|k|^2/n}} \right\|_{l_{k,i}^2} \right)^6 \rightarrow 1, \quad \text{as } n \rightarrow \infty.$$

Thus, we have

$$P(\Sigma) = 1.$$

For any $\omega \in \Sigma$, there exists n such that $\omega \in E_{\lambda_1, \frac{1}{n}, u_0}^c$ and

$$\|\bar{u}^\omega\|_{L^4([0, \frac{1}{n}]; L^6(\mathbb{T}^3))} + \|\nabla \bar{u}^\omega\|_{L^2([0, \frac{1}{n}]; L^3(\mathbb{T}^3))} < \lambda_1.$$

From Proposition 3.4 we have that the map K^ω is a contraction on the ball of radius $2C\lambda_1^2$ of $X_{\frac{1}{n}}^{\frac{1}{2}}$. Thus, for almost all $\omega \in \Omega$, there exist $T_\omega > 0$ and a unique solution $v \in L^\infty([0, T_\omega]; H^{\frac{1}{2}}(\mathbb{T}^3)) \cap L^2([0, T_\omega]; H^{\frac{3}{2}}(\mathbb{T}^3))$ to (3.9). Since it is easy to obtain that $v \in C([0, T_\omega]; H^{\frac{1}{2}}(\mathbb{T}^3))$, we obtain the first part in Theorem 1.4.

When $\{h_k^i\}_{k \in \mathbb{Z}_+^3, i=1,2}$ satisfies (1.9), let

$$\Omega_T = E_{\lambda_1, T, u_0}^c.$$

From (3.3) and a similar argument as above, we have (1.10) and the fact that there exists a unique solution $u^\omega \in X_\omega$ of (1.1) with the initial data u_0^ω in the time interval $[0, T]$.

When h_k^i are standard real Gaussian or Bernoulli variables, let

$$\Omega_T = F_{\lambda_1, T, u_0}^c.$$

From (3.6) and a similar argument as above, we have (1.11) and the fact that there exists a unique solution $u^\omega \in X_\omega$ of (1.1) with the initial data u_0^ω in the time interval $[0, T]$. If there is a solution $u_2 \in C([0, T]; L^2(\mathbb{T}^3)) \cap L^2([0, T]; H^1(\mathbb{T}^3))$ of

(1.1) with the initial data u_0^ω , using the energy method, one can easily obtain that $u^\omega = u_2$ a.e. and finish the proof of Theorem 1.4. $\square$

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DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, PEOPLE’S REPUBLIC OF CHINA

E-mail address: zhangting79@zju.edu.cn

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, PEOPLE’S REPUBLIC OF CHINA

E-mail address: dyf@zju.edu.cn