ON LOCAL KHINTCHINE INEQUALITIES FOR SPACES OF EXPONENTIAL INTEGRABILITY

JAVIER CARRILLO-ALANÍS

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ABSTRACT. We prove a local version of the Khintchine inequality for the spaces $\text{Exp } L^p([0, 1])$ of functions having p -th exponential integrability, for $1 \leq p \leq 2$. The result also holds for the lacunary Walsh series.

The classical Khintchine inequality states that if $0 < p < \infty$ and $(a_n) \in \ell^2$, then

$$(1) \quad A_p \|(a_n)\|_{\ell^2} \leq \left\| \sum_{n \geq 1} a_n r_n \right\|_{L^p([0,1])} \leq B_p \|(a_n)\|_{\ell^2},$$

for some constants A_p and B_p depending only on p . Here r_n denotes the Rademacher functions, namely $r_n(t) := \text{sign} \sin(2^n \pi t)$, $t \in [0, 1]$, $n \geq 1$. Recall that the series $\sum a_n r_n$ converges a.e. if and only if $(a_n) \in \ell^2$. These results were proved by Rademacher [6], Khintchine [2], and Khintchine and Kolmogorov [3].

For any Banach function space X on $[0, 1]$, $\mathcal{R}(X)$ will stand for the closed linear space generated by the Rademacher functions in X . It follows from the Khintchine inequality that (r_n) is a basic sequence in $L^p([0, 1])$. Thus $\mathcal{R}(L^p([0, 1])) = \{\sum a_n r_n : (a_n) \in \ell^2\}$, and it is isomorphic to ℓ^2 , $0 < p < \infty$. The case $p = \infty$ is different, since $\|\sum a_n r_n\|_{L^\infty([0,1])} = \|(a_n)\|_{\ell^1}$.

The attempt to extend the Khintchine inequality to other function spaces led to a result of Rodin and Semenov, [7, Theorem 6], characterizing those rearrangement invariant function spaces where (1) holds. A rearrangement invariant space X on $[0, 1]$ is a Banach function space such that $\|f\|_X = \|g\|_X$ whenever $f \in X$ and f and g are equimeasurable functions. That is, $m_f(\lambda) = m_g(\lambda)$ for all $\lambda > 0$, where m is the Lebesgue measure and m_f denotes the distribution function of f , i.e., $m_f(\lambda) = m(\{x \in [0, 1] : |f(x)| > \lambda\})$. For any rearrangement invariant space X on $[0, 1]$ we have $L^\infty([0, 1]) \hookrightarrow X \hookrightarrow L^1([0, 1])$, where $X \hookrightarrow Y$ stands for the fact that X is continuously contained in Y . We will denote by X_0 the closure of $L^\infty([0, 1])$ in X . For further details on rearrangement invariant spaces, see [1], [5].

Recall that the Orlicz space L^Φ generated by a Young function Φ is the set of all measurable functions f on $[0, 1]$ such that the norm

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_0^1 \Phi(|f(t)|/\lambda) dt \leq 1 \right\}$$

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is finite. These spaces are rearrangement invariant (for further details on Orlicz spaces, see [1, §IV.8], [4]). Let L^M be the Orlicz space generated by the function $M(t) = \exp t^2 - 1$.

The result of Rodin and Semenov is the following.

Theorem 1 (Rodin and Semenov, 1975). *Let X be a rearrangement invariant space on $[0, 1]$. Then $\mathcal{R}(X)$ is isomorphic to ℓ^2 if and only if $(L^M)_0 \hookrightarrow X$.*

In this paper we focus our attention on local versions of the Khintchine inequality. The first result of this type, inspired by an analogous result for the trigonometric series, was proved by Zygmund and gives a local version of (1) on $L^2([0, 1])$, [11, Lemma V.8.3]. A later theorem by Sagher and Zhou extended the result to other $L^p([0, 1])$ spaces, [8].

Theorem 2 (Sagher and Zhou, 1990). *For any $0 < p < \infty$ there exist constants A'_p, B'_p so that for any measurable set $E \subset [0, 1]$ of positive measure there exists $N = N(E)$ such that*

$$(2) \quad A'_p \left(\sum_{n \geq N} a_n^2 \right)^{1/2} \leq \left(\int_E | \sum_{n \geq N} a_n r_n(t) |^p \frac{dt}{|E|} \right)^{1/p} \leq B'_p \left(\sum_{n \geq N} a_n^2 \right)^{1/2},$$

for any $(a_n) \in \ell^2$.

Noting that, for any set $E \subset [0, 1]$ of positive measure, we have

$$\left(\int_E |f(t)|^p \frac{dt}{|E|} \right)^{1/p} = \|f\|_{L^p(E, dm/|E|)},$$

it is natural to consider, for L^Φ an Orlicz space, the space

$$\|f\|_{L^\Phi(E, dm/|E|)} = \inf \left\{ \lambda > 0 : \int_E \Phi(|f(t)|/\lambda) \frac{dt}{|E|} \leq 1 \right\}.$$

A further result of Sagher and Zhou, [9], gives a local Khintchine inequality for the space $L_{\exp}$, which is the Orlicz space generated by the function $\Phi(t) = e^t - 1$.

Theorem 3 (Sagher and Zhou, 1996). *Let $E \subset [0, 1]$ be a set of positive measure. There exist $N = N(E)$ and constants A and B such that*

$$A \|(a_n)_{n \geq N}\|_2 \leq \left\| \sum_{n \geq N} a_n r_n \right\|_{L_{\exp}(E, dm/|E|)} \leq B \|(a_n)_{n \geq N}\|_2,$$

for any $(a_n) \in \ell^2$.

The proof of this theorem in [9] depends on providing an alternative expression for the dyadic BMO norm of a function of the form $\exp(\sum a_n r_n)$.

Rodin and Semenov's theorem suggests considering local Khintchine inequalities on rearrangement invariant spaces other than $L^p([0, 1])$ and $L_{\exp}$. Due to the inclusions

$$(3) \quad (L^M)_0 \hookrightarrow L^M \hookrightarrow L_{\exp} \hookrightarrow L^p([0, 1]),$$

for $0 < p < \infty$, it is natural to consider extending the results of Sagher and Zhou to L^M . In fact, we prove a local Khintchine inequality for the spaces $\text{Exp } L^p([0, 1])$, $1 \leq p \leq 2$, of functions having exponential p -th integrability, which are the Orlicz spaces corresponding to $\Phi(t) = \exp t^p - 1$. For $p = 2$, $\text{Exp } L^2([0, 1]) = L^M$, and for $p = 1$, $\text{Exp } L^1([0, 1]) = L_{\exp}$.

Let $1 \leq p \leq 2$.

Theorem 4. Let $E \subset [0, 1]$ be a set of positive measure. There exist $N = N(E)$ and constants A and B such that

$$A\|(a_n)_{n \geq N}\|_2 \leq \left\| \sum_{n \geq N} a_n r_n \right\|_{\text{Exp } L^p(E, dm/|E|)} \leq B\|(a_n)_{n \geq N}\|_2,$$

for any $(a_n) \in \ell^2$.

In order to prove the theorem we need an upper bound for the constant B'_p appearing in (2) which stems from [8] but is not explicitly computed there.

Lemma 5. For $k \geq 1$ and $1 \leq p \leq 2$, we have $B'_{pk} \leq 2^{1/2k}(1 + \sqrt{2})\sqrt{k}$.

Proof. Since $B'_{pk} \leq B'_{2k}$ for $1 \leq p \leq 2$, we can assume that $p = 2$. We follow the ideas from [8]. Denote the dyadic intervals by $I_k^n = ((k-1)2^{-n}, k2^{-n})$, $1 \leq k \leq 2^n$. For any set $E \subset [0, 1]$ of positive measure, there exists a pairwise disjoint family of dyadic intervals F such that $\chi_E \leq \chi_F$ a.e. and $|F| \leq 2|E|$. Set $F = F_1 \cup F_2$, where $F_1 \cap F_2 = \emptyset$, $|F_2| \leq |F|^2$ and $F_1 = \bigcup_{j=1}^s I_{k_j}^{n_j}$ is a union of finitely many intervals from F . Set $N = \max\{n_j : j = 1, \dots, s\} + 1$.

Then, we have

$$\begin{aligned} \left(\int_E \left| \sum_{n=N}^M a_n r_n(t) \right|^{2k} \frac{dt}{|E|} \right)^{1/2k} &\leq 2^{1/2k} \left(\int_F \left| \sum_{n=N}^M a_n r_n(t) \right|^{2k} \frac{dt}{|F|} \right)^{1/2k} \\ &\leq 2^{1/2k} \left(\int_{F_1} \left| \sum_{n=N}^M a_n r_n(t) \right|^{2k} \frac{dt}{|F_1|} \right)^{1/2k} \\ &\quad + 2^{1/2k} \left(\int_{F_2} \left| \sum_{n=N}^M a_n r_n(t) \right|^{2k} \frac{dt}{\sqrt{|F_2|}} \right)^{1/2k}. \end{aligned}$$

From the Khintchine inequality and the fact that, for $n \geq N \geq n_j$, r_n resembles on $I_{k_j}^{n_j}$ the behaviour of r_{n-N+1} on $[0, 1]$, it follows that

$$\begin{aligned} \left(\int_{F_1} \left| \sum_{n=N}^M a_n r_n(t) \right|^{2k} \frac{dt}{|F_1|} \right)^{1/2k} &= \left(\frac{1}{|F_1|} \sum_{j=1}^s \int_{I_{k_j}^{n_j}} \left| \sum_{n=N}^M a_n r_n(t) \right|^{2k} dt \right)^{1/2k} \\ &= \left(\frac{1}{|F_1|} \sum_{j=1}^s 2^{-n_j} \int_0^1 \left| \sum_{n=N}^M a_n r_{n-N+1}(t) \right|^{2k} dt \right)^{1/2k} \\ &\leq \left(\frac{1}{|F_1|} \sum_{j=1}^s 2^{-n_j} B_{2k}^{2k} \|(a_n)_{n=N}^M\|_2^{2k} \right)^{1/2k} \\ &= B_{2k} \|(a_n)_{n=N}^M\|_2. \end{aligned}$$

By means of the Khintchine and Cauchy-Schwarz inequalities, we have

$$\begin{aligned} \left(\frac{1}{\sqrt{|F_2|}} \int_{F_2} \left| \sum_{n=N}^M a_n r_n(t) \right|^{2k} dt \right)^{1/2k} &\leq \left(\int_0^1 \left| \sum_{n=N}^M a_n r_n(t) \right|^{4k} dt \right)^{1/4k} \\ &\leq B_{4k} \|(a_n)_{n=N}^M\|_2. \end{aligned}$$

Hence, $B'_{2k} \leq 2^{1/2k}(B_{2k} + B_{4k})$. Since we have the general inequality $B_{2k} \leq \sqrt{k}$ (see [11, Theorem V.8.4]), it follows that $B'_{2k} \leq 2^{1/2k}(1 + \sqrt{2})\sqrt{k}$. $\square$

Now we can proceed with the proof of the theorem.

Proof of Theorem 4. For any set $E \subset [0, 1]$ of positive measure, let $N = N(E)$ be as in (2). Define the operator

$$T_E : (a_n) \in \ell^2 \mapsto T_E(a_n) := f_N = \sum_{n \geq N} a_n r_n \in \text{Exp } L^p(E, dm/|E|).$$

We first show that T_E is well-defined. From the power series expansion of $\Phi(t) = \exp t^p - 1$ and Lemma 5 we have

$$\begin{aligned} \int_E (\exp |f_N(t)/\lambda|^p - 1) \frac{dt}{|E|} &= \sum_{k \geq 1} \frac{1}{k! \lambda^{kp}} \int_E \left| \sum_{n \geq N} a_n r_n(t) \right|^{kp} \frac{dt}{|E|} \\ &\leq \sum_{k \geq 1} \frac{1}{k! \lambda^{kp}} (2^{1/2k} (1 + \sqrt{2}) \sqrt{k})^{kp} \|(a_n)_{n \geq N}\|_{\ell^2}^{kp}. \end{aligned}$$

Applying the asymptotic equivalence $k! \sim (2\pi k)^{1/2} k^k e^{-k}$, given by Stirling's formula, there exists an absolute constant c such that

$$(4) \quad \int_E (\exp |f_N(t)/\lambda|^p - 1) \frac{dt}{|E|} \leq c \sum_{k \geq 1} k^{-1/2} 2^{p/2} \left(\frac{e(1 + \sqrt{2})^p k^{p/2-1} \|(a_n)_{n \geq N}\|_{\ell^2}^p}{\lambda^p} \right)^k.$$

Since the series converges for $\lambda > e^{1/p} (1 + \sqrt{2}) \|(a_n)_{n \geq N}\|_{\ell^2}$ and $p/2 - 1 \leq 0$, we have that T_E is well-defined.

The continuity of the operator T_E follows from the closed graph theorem. Assume that $x_n \rightarrow 0$ in ℓ^2 and $T_E x_n \rightarrow g$ in $\text{Exp } L^p(E, dm/|E|)$. Since $\text{Exp } L^p(E, dm/|E|) \hookrightarrow L^2(E, dm/|E|)$, it follows that $T_E x_n \rightarrow g$ in $L^2(E, dm/|E|)$. The local Khintchine inequality for $p = 2$ shows that

$$\|T_E x_n\|_{L^2(E, dm/|E|)} \leq B_2' \|x_n\|_{\ell^2} \rightarrow 0;$$

therefore $g = 0$.

In order to prove the right-hand side inequality, we show that the family $\{T_E : E \subset [0, 1], |E| > 0\}$ has a uniformly bounded norm. Let $(a_n) \in \ell^2$ such that $\|(a_n)\|_{\ell^2} \leq 1$. It follows from (4) that

$$\int_E (\exp (f_N(t)/\lambda)^p - 1) \frac{dt}{|E|} \leq c \sum_{k \geq 1} k^{-1/2} 2^{p/2} \left(\frac{e(1 + \sqrt{2})^p k^{p/2-1}}{\lambda^p} \right)^k.$$

Thus, $\|T_E(a_n)\|_{L^M(E, dm/|E|)} \leq B$, where $E \subset [0, 1]$ is any set of positive measure for some constant B not depending on E .

The left-hand side inequality follows immediately from the local Khintchine inequality (2) for $p = 1$ and from (3), which show that

$$A_1' \|(a_n)_N\|_2 \leq \left\| \sum_{n \geq N} a_n r_n \right\|_{L^1(E, dm/|E|)} \leq c' \left\| \sum_{n \geq N} a_n r_n \right\|_{\text{Exp } L^p(E, dm/|E|)}.$$

□

Remark 6. (i) Since $L_{\text{exp}} = \text{Exp } L^1([0, 1])$, the local Khintchine inequality for L_{exp} follows from the local results for $L^1([0, 1])$ and L^M . This gives an alternative proof of the result of Sagher and Zhou in [9].

(ii) For any Orlicz space such that $L^M \hookrightarrow L^\Phi$, it follows from the theorem just proved that the local Khintchine inequality holds on L^Φ since $L^\Phi \hookrightarrow L^1([0, 1])$.

Remark 7. The Walsh functions are defined as $w_0 = 1$ and $w_n = r_{a_1} \cdots r_{a_l}$ for $n = 2^{a_1} + \dots + 2^{a_l}$, with $a_1 > \dots > a_l \geq 0$.

Sagher and Zhou have extended the local Khintchine inequality in the spaces $L^p([0, 1])$, $0 < p < \infty$, to the lacunary Walsh series, that is, $\sum_{n \geq 1} a_n w_{i_n}$, where $i_{n+1}/i_n \geq q$ for some $q > 1$ [10, Theorem 1]. Following the arguments in Theorem 4 (and with the corresponding version of Lemma 5) we have the following result:

Let $1 \leq p \leq 2$ and $q > 1$, and $E \subset [0, 1]$ such that $|E| > 0$. There exist two constants $A_{p,q}$ and $B_{p,q}$ not depending on E such that

$$A_{p,q} \|(a_n)\|_2 \leq \left\| \sum_{n \geq 1} a_n w_{i_n} \right\|_{\text{Exp } L^p(E, dm/|E|)} \leq B_{p,q} \|(a_n)\|_2$$

for any Walsh series such that $(a_n) \in \ell^2$, $i_0 = 0$ and $i_{n+1}/i_n > q$ for $n \geq 1$.

Note that in order to apply the corresponding version of Lemma 5 one should use in [10, Theorems 1 and 2] the following bounds:

$$B(2k, q) \leq (1 + m)^{1/2k} k^{1/2},$$

$$B'(2k, q) \leq (1 + \sqrt{2})(2 + 2m)^{1/2k} k^{1/2},$$

where m is the least integer such that $q^m \geq 2$ for $1 < q < 2$ and $m = 0$ for $q \geq 2$.

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE SEVILLA, APDO. 1160, SEVILLA, 41080, SPAIN
E-mail address: fcarrillo@us.es