A RESTRICTION THEOREM FOR THE H-TYPE GROUPS

HEPING LIU AND YINGZHAN WANG

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ABSTRACT. We prove that the restriction operator for the H-type groups is bounded from L^p to $L^{p'}$ for p near to 1 when the dimension of the center is larger than one, and the range of p depends on the dimension of the center. This is different from the Heisenberg group, on which the restriction operator is not bounded from L^p to $L^{p'}$ unless $p = 1$.

1. INTRODUCTION

The restriction theorems for the Fourier transform play an important role in harmonic analysis as well as in the theory of partial differential equations. We refer the reader to [7] for the basic results on restriction theorems. Further developments can be found in [8] and the references therein. To generalize the restriction theorems to the Heisenberg group, D. Müller [5] investigated the restriction operator for the sublaplacian. He established the boundedness of the restriction operator with respect to the mixed L^p -norms and also gave a counterexample to show that the restriction operator is not bounded from L^p to $L^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, unless $p = 1$.

Müller's result is due to the following key fact. The center of the Heisenberg group is of dimension 1. One can expect $L^p \rightarrow L^{p'}$ restriction theorems for some values $p > 1$ on a nilpotent Lie group with the center of dimension larger than one. This has been done by S. Thangavelu [10] on products of Heisenberg groups. In [4] we prove that the restriction operator for the sublaplacian on the quaternion Heisenberg group is bounded from L^p to $L^{p'}$ if $1 \leq p \leq \frac{4}{3}$. In this article we will show that the restriction operator for the sublaplacian on the H-type groups is bounded from L^p to $L^{p'}$ for p near to 1 when the dimension of the center is larger than one, and the range of p depends on the dimension of the center.

First we state some facts of H-type groups. We refer the reader to [1], [2] and [13] for the basic references of H-type groups. Let $\mathfrak{g}$ be a two step nilpotent Lie algebra endowed with an inner product $\langle \cdot, \cdot \rangle$. Its center is denoted by $\mathfrak{Z}$. $\mathfrak{g}$ is called an H-type algebra if $[\mathfrak{Z}^\perp, \mathfrak{Z}^\perp] = \mathfrak{Z}$ and for every $t \in \mathfrak{Z}$, the map $J_t : \mathfrak{Z}^\perp \rightarrow \mathfrak{Z}^\perp$ defined by

$$\langle J_t u, w \rangle = \langle t, [u, w] \rangle, \quad \text{for any } u, w \in \mathfrak{Z}^\perp,$$

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is an orthogonal map whenever $|t| = 1$. The corresponding connected and simply connected Lie group G is called an H-type group. For a given $0 \neq a \in \mathfrak{Z}^*$, the dual of $\mathfrak{Z}$, we can define a skew-symmetric linear mapping $B(a)$ on $\mathfrak{Z}^\perp$ by

$$\langle B(a)u, w \rangle = a([u, w]), \quad \text{for any } u, w \in \mathfrak{Z}^\perp.$$

$B(a)$ is non-degenerate and we have

$$\dim \mathfrak{Z}^\perp = 2n.$$

Thus we can choose an orthonormal basis

$$\{E_1(a), E_2(a), \dots, E_n(a), \overline{E}_1(a), \overline{E}_2(a), \dots, \overline{E}_n(a)\}$$

of $\mathfrak{Z}^\perp$ such that

$$B(a)E_i(a) = -|a|\overline{E}_i(a), \quad B(a)\overline{E}_i(a) = |a|E_i(a).$$

We also choose an orthonormal basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$ of $\mathfrak{Z}$ such that $a(\varepsilon_1) = |a|$, $a(\varepsilon_j) = 0$, $j = 2, \dots, m$. Throughout this paper we will assume that $m > 1$. Then the elements of $\mathfrak{g}$ can be written as $(z, t) = (x, y, t) = \sum_{i=1}^n (x_i E_i + y_i \overline{E}_i) + \sum_{j=1}^m t_j \varepsilon_j$. We identify G with its Lie algebra $\mathfrak{g}$ by the exponential map. The group structure is given by

$$(z, t)(z', t') = (z + z', t + t' + \frac{1}{2}[z, z']),$$

where $[z, z']_j = \langle z, U^j z' \rangle$, and U^j satisfies the following conditions:

- (i) U^j is a $2n \times 2n$ skew-symmetric and orthogonal matrix, $j = 1, 2, \dots, m$,
- (ii) $U^i U^j + U^j U^i = 0$, $i, j = 1, 2, \dots, m$.

In particular, $\langle z, U^1 z' \rangle = \sum_{j=1}^n (x'_j y_j - x_j y'_j)$. We remark that all the above expressions depend on a given $0 \neq a \in \mathfrak{Z}^*$, but we omit it. We will keep this simplification.

The left invariant vector fields which agree respectively with $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, j = 1, 2, \dots, n$, at the origin are given by

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^m \left(\sum_{l=1}^{2n} z_l U_{l,j}^k \right) \frac{\partial}{\partial t_k}, \\ Y_j &= \frac{\partial}{\partial y_j} + \frac{1}{2} \sum_{k=1}^m \left(\sum_{l=1}^{2n} z_l U_{l,j+n}^k \right) \frac{\partial}{\partial t_k}, \end{aligned}$$

where $z_l = x_l, z_{l+n} = y_l, l = 1, 2, \dots, n$. Then the sublaplacian is defined by

$$(1.1) \quad \mathcal{L} = - \sum_{j=1}^n (X_j^2 + Y_j^2) = -\Delta_z - \frac{1}{4}|z|^2 \Delta_t - \sum_{k=1}^m \langle z, U^k \nabla_z \rangle \frac{\partial}{\partial t_k},$$

where

$$\Delta_z = \sum_{j=1}^{2n} \frac{\partial^2}{\partial z_j^2}, \quad \Delta_t = \sum_{j=1}^m \frac{\partial^2}{\partial t_j^2}, \quad \nabla_z = \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_{2n}} \right)^T.$$

It is well known that the sublaplacian $\mathcal{L}$ is positive and essentially self-adjoint. Let

$$\mathcal{L} = \int_0^\infty \lambda dE(\lambda)$$

be the spectral decomposition of $\mathcal{L}$. Then operator

$$P_\lambda f = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_\lambda^{\lambda+\varepsilon} dE(\lambda) f$$

is well defined for $f \in \mathcal{S}(G)$. Thus

$$f = \int_0^\infty P_\lambda f d\lambda,$$

where $P_\lambda f$ is an eigenfunction of $\mathcal{L}$ with the eigenvalue λ . P_λ is called the restriction operator for the sublaplacian $\mathcal{L}$. Our main result is the following theorem.

Theorem 1. *Let G be an H -type group with the underlining manifold $\mathbb{R}^{2n+m}$, where $m > 1$ is the dimension of the center. Then for $1 \leq p \leq \frac{2m+2}{m+3}$,*

$$(1.2) \quad \|P_\lambda f\|_{p'} \leq C \lambda^{2(n+m)(\frac{1}{p}-\frac{1}{2})-1} \|f\|_p.$$

This article is organized as follows. In the next section we give an explicit expression for the restriction operator. Theorem 1 will be proved in the last section.

2. RESTRICTION OPERATOR

Let us recall some results about the special Hermite expansion. We refer to [11, 12] for details. The Hermite polynomials $H_k(x)$ are defined on $\mathbb{R}$ by the formula

$$H_k(x) = (-1)^k \frac{d^k}{dx^k} (e^{-x^2}) e^{x^2}, \quad k = 0, 1, 2, \dots$$

Set

$$h_k(x) = (2^k k! \sqrt{\pi})^{-\frac{1}{2}} H_k(x) e^{-\frac{1}{2}x^2}, \quad k = 0, 1, 2, \dots$$

Let α be a multiindex and $x \in \mathbb{R}^n$. We define Hermite functions Φ_α by

$$\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j).$$

For each pair of multiindices α, β , and $z = x + iy \in \mathbb{C}^n$, the special Hermite function $\Phi_{\alpha\beta}$ is defined in terms of the Fourier-Wigner transform of Φ_α and Φ_β , which is essentially the matrix coefficient of the Schrödinger representation denoted by π , i.e.,

$$\Phi_{\alpha\beta}(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \Phi_\alpha\left(\xi + \frac{y}{2}\right) \Phi_\beta\left(\xi - \frac{y}{2}\right) d\xi.$$

The special Hermite functions form a complete orthonormal system for $L^2(\mathbb{C}^n)$. Thus for $f \in L^2(\mathbb{C}^n)$, we have the special Hermite expansion

$$f = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^n} \langle f, \Phi_{\alpha\beta} \rangle \Phi_{\alpha\beta}.$$

The special Hermite expansion can be put in a compact form. For $f, g \in L^1(\mathbb{C}^n)$, we define the twisted convolution $f \times g$ by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w) g(w) e^{\frac{i}{2} \text{Im} z \cdot \bar{w}} dw.$$

Then

$$\sum_{\beta \in \mathbb{N}^n} \langle f, \Phi_{\beta\alpha} \rangle \Phi_{\beta\alpha}(z) = (2\pi)^{-\frac{n}{2}} f \times \Phi_{\alpha\alpha}(z).$$

Let $L_k^\nu(x)$ be the Laguerre polynomial of type ν and degree k defined by

$$L_k^\nu(x) = \frac{1}{k!} \frac{d^k}{dx^k} (e^{-x} x^{k+\nu}) e^x x^{-\nu}, \quad k = 0, 1, 2, \dots, \nu > -1.$$

Set

$$\varphi_k(z) = L_k^{n-1} \left(\frac{1}{2} |z|^2 \right) e^{-\frac{1}{4} |z|^2}, \quad k = 0, 1, 2, \dots.$$

Then

$$\varphi_k(z) = (2\pi)^{\frac{n}{2}} \sum_{|\alpha|=k} \Phi_{\alpha\alpha}(z).$$

Thus we can rewrite the special Hermite expansion as

$$f = (2\pi)^{-n} \sum_{k=0}^{\infty} f \times \varphi_k.$$

Let L be the special Hermite operator defined by

$$L = -\Delta_z + \frac{|z|^2}{4} - i \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right),$$

where we have identified $z = x + iy \in \mathbb{C}^n$ with $z = (x, y) \in \mathbb{R}^{2n}$. For $1 \leq p < \frac{6n+2}{3n+4}$, we have

$$\|f \times \varphi_k\|_2 \leq C(k+1)^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_p$$

(see [9] or [12, p. 72]).

We need to consider the scaled special Hermite expansion. Let $\lambda > 0$. The scaled special Hermite operator is defined by

$$L_\lambda = -\Delta_z + \frac{\lambda^2 |z|^2}{4} - i\lambda \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

Set $\varphi_k^\lambda(z) = \varphi_k(\sqrt{\lambda} z)$ and define the λ -twisted convolution by

$$f \times_\lambda g(z) = \int_{\mathbb{C}^n} f(z-w) g(w) e^{\frac{i\lambda}{2} \operatorname{Im} z \cdot \overline{w}} dw.$$

By a dilation argument, for $f \in \mathcal{S}(\mathbb{C}^n)$, we have

$$(2.1) \quad f(z) = \left(\frac{\lambda}{2\pi} \right)^n \sum_{k=0}^{\infty} f \times_\lambda \varphi_k^\lambda(z)$$

and

$$(2.2) \quad \|f\|_2^2 = \left(\frac{\lambda}{2\pi} \right)^{2n} \sum_{k=0}^{\infty} \|f \times_\lambda \varphi_k^\lambda\|_2^2.$$

Moreover, $f \times_\lambda \varphi_k^\lambda$ is an eigenfunction of L_λ with the eigenvalue $(2k+n)\lambda$, and

$$(2.3) \quad \|f \times_\lambda \varphi_k^\lambda\|_2 \leq C(k+1)^{n(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \lambda^{n(\frac{1}{p}-\frac{3}{2})} \|f\|_p \quad \text{for } 1 \leq p < \frac{6n+2}{3n+4}.$$

Now we turn to the expression for the restriction operator. We may identify $\mathfrak{Z}^*$ with $\mathfrak{Z}$. Thus we will write $\langle a, t \rangle$ instead of $a(t)$ for $a \in \mathfrak{Z}^*$ and $t \in \mathfrak{Z}$.

Lemma 1. *Let $0 \neq a \in \mathfrak{Z}^*$. If $f(z, t) = e^{-i\langle a, t \rangle} \varphi(z)$, then*

$$\mathcal{L}f(z, t) = e^{-i\langle a, t \rangle} L_{|a|} \varphi(z).$$

Proof. Because $\langle a, t \rangle = |a|t_1$ and

$$\langle z, U^1 \nabla_z \rangle = \sum_{j=1}^n \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right),$$

Lemma 1 is easily deduced from the expression (1.1). $\square$

Set $e_k^a(z, t) = e^{-i\langle a, t \rangle} \varphi_k^{|a|}(z)$. For $f \in \mathcal{S}(G)$, let

$$f^a(z) = \int_{\mathbb{R}^m} f(z, t) e^{i\langle a, t \rangle} dt$$

be the Fourier transform of f with respect to the central variable t . It is easy to get

$$(2.4) \quad f * e_k^a(z, t) = e^{-i\langle a, t \rangle} (f^a \times_{|a|} \varphi_k^{|a|})(z).$$

It follows from Lemma 1 that $f * e_k^a$ is an eigenfunction of $\mathcal{L}$ with the eigenvalue $(2k + n)|a|$.

Let $\tilde{e}_k^a = e_k^{\frac{a}{2k+n}}$. Using the inversion formula for the Fourier transform together with (2.1) and (2.4), we get

$$\begin{aligned} f(z, t) &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} f^a(z) e^{-i\langle a, t \rangle} da \\ &= \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \left(\frac{|a|^n}{(2\pi)^n} \sum_{k=0}^{\infty} f^a \times_{|a|} \varphi_k^{|a|}(z) \right) e^{-i\langle a, t \rangle} da \\ &= \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \int_{\mathbb{R}^m} f * e_k^a(z, t) |a|^n da \\ &= \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \frac{1}{(2k + n)^{n+m}} \int_{\mathbb{R}^m} f * \tilde{e}_k^a(z, t) |a|^n da \\ &= \int_0^{\infty} \left(\sum_{k=0}^{\infty} \frac{\lambda^{n+m-1}}{(2\pi(2k + n))^{n+m}} \int_{S^{m-1}} f * \tilde{e}_k^{\lambda \tilde{a}}(z, t) d\sigma(\tilde{a}) \right) d\lambda, \end{aligned}$$

where $\tilde{a} = \frac{a}{|a|}$ and $d\sigma(\tilde{a})$ is the induced Euclidean measure on the unit sphere S^{m-1} . Thus

$$P_\lambda f(z, t) = \sum_{k=0}^{\infty} \frac{\lambda^{n+m-1}}{(2\pi(2k + n))^{n+m}} \int_{S^{m-1}} f * \tilde{e}_k^{\lambda \tilde{a}}(z, t) d\sigma(\tilde{a}).$$

We also have, by the Plancherel formula and (2.2),

$$\begin{aligned} \|f\|_{L^2(G)}^2 &= (2\pi)^{-m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} |f^a(z)|^2 dz da \\ &= \sum_{k=0}^{\infty} (2\pi)^{-m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^{2n}} \frac{|a|^{2n}}{(2\pi)^{2n}} |f^a \times_{|a|} \varphi_k^{|a|}(z)|^2 dz da \\ &= \sum_{k=0}^{\infty} \frac{1}{(2\pi)^{2n+m}} \int_{\mathbb{R}^m} |a|^{2n} \int_{\mathbb{R}^{2n}} |f * e_k^a(z, 0)|^2 dz da \\ &= \sum_{k=0}^{\infty} \frac{1}{(2\pi(2k + n))^{2n+m}} \int_{\mathbb{R}^m} |a|^{2n} \int_{\mathbb{R}^{2n}} |f * \tilde{e}_k^a(z, 0)|^2 dz da. \end{aligned}$$

Set

$$Q(a, f)^2 = \sum_{k=0}^{\infty} \frac{|a|^{2n}}{(2\pi(2k+n))^{2n+m}} \int_{\mathbb{R}^{2n}} |f * \tilde{e}_k^a([z, 0])|^2 dz.$$

Then

$$\|f\|_2^2 = \int_{\mathbb{R}^m} Q(a, f)^2 da.$$

3. RESTRICTION THEOREM

In this section we prove Theorem 1.

Lemma 2. *Letting $1 \leq p \leq \frac{2m+2}{m+3}$, we have*

$$(3.1) \quad \left(\int_{S^{m-1}} Q(\lambda \tilde{a}, f)^q d\sigma(\tilde{a}) \right)^{\frac{1}{q}} \leq C \lambda^{\frac{n+m}{p} - \frac{n}{2} - m} \|f\|_p, \quad q = \frac{m-1}{m+1} p'.$$

Because $\frac{m-1}{m+1} p' \geq 2$, it follows that

$$(3.2) \quad \left(\int_{S^{m-1}} Q(\lambda \tilde{a}, f)^2 d\sigma(\tilde{a}) \right)^{\frac{1}{2}} \leq C \lambda^{\frac{n+m}{p} - \frac{n}{2} - m} \|f\|_p.$$

Proof. Note that

$$Q(\lambda \tilde{a}, f)^2 = \sum_{k=0}^{\infty} \frac{\lambda^{2n}}{(2\pi(2k+n))^{2n+m}} \int_{\mathbb{R}^{2n}} |f^{\frac{\lambda \tilde{a}}{2k+n}} \times_{\frac{\lambda}{2k+n}} \varphi_k^{\frac{\lambda}{2k+n}}(z)|^2 dz.$$

It is well known that H-type algebras are closely related to Clifford modules (cf. [6]). H-type algebras can be classified by the standard theory of Clifford algebras. Specifically, the relation of the dimensions of the center to its orthogonal complement space is clear. In any case, we have $m < 2n$ (cf. [3]). Thus $\frac{2m+2}{m+3} < \frac{6n+2}{3n+4}$. By (2.3),

$$(3.3) \quad \|f^{\frac{\lambda \tilde{a}}{2k+n}} \times_{\frac{\lambda}{2k+n}} \varphi_k^{\frac{\lambda}{2k+n}}\|_2^2 \leq C \lambda^{-3n+\frac{2n}{p}} (k+1)^{2n-1} \|f^{\frac{\lambda \tilde{a}}{2k+n}}\|_p^2.$$

Because $q = \frac{m-1}{m+1} p' \geq 2 > p$, by (3.3) and the Minkowski inequality, we get

$$\begin{aligned} & \left(\int_{S^{m-1}} Q(\lambda \tilde{a}, f)^q d\sigma(\tilde{a}) \right)^{\frac{1}{q}} \\ & \leq C \sum_{k=0}^{\infty} \frac{\lambda^n}{(k+1)^{\frac{2n+m}{2}}} \left(\int_{S^{m-1}} \left(\int_{\mathbb{R}^{2n}} |f^{\frac{\lambda \tilde{a}}{2k+n}} \times_{\frac{\lambda}{2k+n}} \varphi_k^{\frac{\lambda}{2k+n}}(z)|^2 dz \right)^{\frac{q}{2}} d\sigma(\tilde{a}) \right)^{\frac{1}{q}} \\ & \leq C \sum_{k=0}^{\infty} \lambda^{n(\frac{1}{p}-\frac{1}{2})} (k+1)^{-\frac{m+1}{2}} \left(\int_{S^{m-1}} \left(\int_{\mathbb{R}^{2n}} |f^{\frac{\lambda \tilde{a}}{2k+n}}(z)|^p dz \right)^{\frac{q}{p}} d\sigma(\tilde{a}) \right)^{\frac{1}{q}} \\ & \leq C \sum_{k=0}^{\infty} \lambda^{n(\frac{1}{p}-\frac{1}{2})} (k+1)^{-\frac{m+1}{2}} \left(\int_{\mathbb{R}^{2n}} \left(\int_{S^{m-1}} |f^{\frac{\lambda \tilde{a}}{2k+n}}(z)|^q d\sigma(\tilde{a}) \right)^{\frac{p}{q}} dz \right)^{\frac{1}{p}} \\ & \leq C \sum_{k=0}^{\infty} \lambda^{n(\frac{1}{p}-\frac{1}{2})-m} (k+1)^{\frac{m-1}{2}} \left(\int_{\mathbb{R}^{2n}} \left(\int_{S^{m-1}} |\tilde{f}_k^{\tilde{a}}(z)|^q d\sigma(\tilde{a}) \right)^{\frac{p}{q}} dz \right)^{\frac{1}{p}}, \end{aligned}$$

where $\tilde{f}_k(z, t) = f(z, \frac{2k+n}{\lambda}t)$. It follows from the restriction estimate of the Fourier transform on S^{m-1} (see [7, p. 386]) that

$$\left(\int_{S^{m-1}} |\tilde{f}_k^{\tilde{a}}(z)|^q d\sigma(\tilde{a}) \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^m} |\tilde{f}_k(z, t)|^p dt \right)^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned} & \left(\int_{S^{m-1}} Q(\lambda \tilde{a}, f)^q d\sigma(\tilde{a}) \right)^{\frac{1}{q}} \\ & \leq C \sum_{k=0}^{\infty} \lambda^{n(\frac{1}{p}-\frac{1}{2})-m} (k+1)^{\frac{m-1}{2}} \left(\int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^m} |\tilde{f}_k(z, t)|^p dt dz \right)^{\frac{1}{p}} \\ & \leq C \sum_{k=0}^{\infty} \lambda^{\frac{n}{p}-\frac{n}{2}-m+\frac{m}{p}} k^{\frac{m-1}{2}-\frac{m}{p}} \|f\|_p \\ & \leq C \lambda^{\frac{n}{p}-\frac{n}{2}-m+\frac{m}{p}} \|f\|_p. \end{aligned}$$

□

Now we give the proof of Theorem 1.

Proof of Theorem 1. Because the special Hermite expansion is an orthogonal expansion, we have

$$\begin{aligned} \langle P_\lambda f, g \rangle &= \int_G P_\lambda f(z, t) \overline{g(z, t)} dz dt \\ &= \int_G \left(\sum_{k=0}^{\infty} \frac{\lambda^{n+m-1}}{(2\pi(2k+n))^{n+m}} \int_{S^{m-1}} f * \tilde{e}_k^{\lambda \tilde{a}}(z, t) d\sigma(\tilde{a}) \right) \overline{g(z, t)} dz dt \\ &= \int_{S^{m-1}} \sum_{k=0}^{\infty} \int_{\mathbb{R}^{2n}} \frac{\lambda^{n+m-1}}{(2\pi(2k+n))^{n+m}} f^{\frac{\lambda \tilde{a}}{2k+n}} \times_{\frac{\lambda}{2k+n}} \varphi_k^{\frac{\lambda}{2k+n}}(z) \\ & \quad \cdot \overline{g^{\frac{\lambda \tilde{a}}{2k+n}}(z)} dz d\sigma(\tilde{a}) \\ &= \int_{S^{m-1}} \sum_{k=0}^{\infty} \int_{\mathbb{R}^{2n}} \frac{\lambda^{2n+m-1}}{(2\pi(2k+n))^{2n+m}} f^{\frac{\lambda \tilde{a}}{2k+n}} \times_{\frac{\lambda}{2k+n}} \varphi_k^{\frac{\lambda}{2k+n}}(z) \\ & \quad \cdot \overline{g^{\frac{\lambda \tilde{a}}{2k+n}} \times_{\frac{\lambda}{2k+n}} \varphi_k^{\frac{\lambda}{2k+n}}(z)} dz d\sigma(\tilde{a}) \\ &\leq C \lambda^{m-1} \|Q(\lambda \tilde{a}, f)\|_{L^2(S^{m-1})} \|Q(\lambda \tilde{a}, g)\|_{L^2(S^{m-1})}. \end{aligned}$$

By (3.2), when $1 \leq p \leq \frac{2m+2}{m+3}$, we have

$$\langle P_\lambda f, g \rangle \leq C \lambda^{2(n+m)(\frac{1}{p}-\frac{1}{2})-1} \|f\|_p \|g\|_p.$$

So

$$\|P_\lambda f\|_{p'} = \sup_{\|g\|_p=1} |\langle P_\lambda f, g \rangle| \leq C \lambda^{2(n+m)(\frac{1}{p}-\frac{1}{2})-1} \|f\|_p.$$

Theorem 1 is proved. □

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LMAM, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, PEOPLE'S REPUBLIC OF CHINA

E-mail address: hpliu@pku.edu.cn

COLLEGE OF MATHEMATICS AND INFORMATION SCIENCES, GUANGZHOU UNIVERSITY, GUANGZHOU 510006, PEOPLE'S REPUBLIC OF CHINA

E-mail address: wzyde@tom.com