ON EXTREMAL QUASICONFORMAL MAPPINGS OF NON-LANDSLIDE TYPE

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ABSTRACT. Let $S = \{(x, y) : 0 < x < 1, 0 < y < 1\}$, and let f be a quasiconformal mapping on S . It is proved that there is at least one extremal quasiconformal mapping of non-landslide type in the Teichmüller equivalence class $[f]$. This gives a positive answer to the problem proposed by Z. Li in a recent paper.

1. INTRODUCTION

Let Δ be the open unit disk in the complex plane. For a quasiconformal mapping f on Δ , we let $\mu_f = \frac{f_{\bar{z}}}{f_z}$ be the Beltrami coefficient of f and let $K(f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}$ be the maximal dilatation of f . A quasiconformal mapping f on Δ induces a homomorphism $f|_{\partial\Delta}$ on $\partial\Delta$. Two quasiconformal mappings f and g on Δ are Teichmüller equivalent if $f|_{\partial\Delta} = g|_{\partial\Delta}$, and we let $[f]$ be the Teichmüller equivalence class of f .

A quasiconformal mapping f on Δ is extremal when $K(f) \leq K(g)$ for any $g \in [f]$. If $[f]$ contains infinitely many extremal quasiconformal mappings, then there is an extremal quasiconformal mapping $g \in [f]$ such that the set

$$E_g(\delta) = \{z \in \Delta : \frac{1 + |\mu_g(z)|}{1 - |\mu_g(z)|} < K(f) - \delta\}$$

has a positive measure for some $\delta > 0$ (refer to [6], [11] for the proof). For the details of extremal quasiconformal mappings theory, we refer to [1], [2], [5], [7], [8], [9].

When the set $E_f(\delta)$ has interior points for some $\delta > 0$, we say the extremal quasiconformal mapping f is of *landslide type*. Otherwise, an extremal quasiconformal mapping f is of *non-landslide type* when the set $E_f(\delta)$ has no interior points for any $\delta > 0$. So it is natural to ask the following question (refer to [4], [11]).

Problem 1. Suppose that f is a quasiconformal mapping on Δ and $[f]$ contains infinitely many extremal quasiconformal mappings. Is it true that $[f]$ always contains an extremal quasiconformal mapping of landslide type?

In [4], Li proved the following theorem.

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Theorem 1.1 ([4]). *There exists a quasiconformal mapping f on Δ such that there are infinitely many extremal quasiconformal mappings of non-landslide type in $[f]$.*

The motivation of Theorem 1.1 was to give a negative answer to Problem 1. However, we still do not know the answer to Problem 1. In [4], Li proposed the following problem.

Problem 2. For any given quasiconformal mappings f of Δ , is there always an extremal quasiconformal mapping $g \in [f]$ of non-landslide type?

The purpose of this paper is to give Problem 2 a positive answer. For convenience of statement, we choose $S = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ as our consideration and prove the following theorem.

Theorem 1.2. *Let f be a quasiconformal mapping on S . Then there exists at least one extremal quasiconformal mapping $g \in [f]$ of non-landslide type.*

Furthermore, using the methods in the proof of Theorem 1.2, as a generalization of Theorem 1.1, we get the following theorem.

Theorem 1.3. *Let f be a quasiconformal mapping on S . If $[f]$ contains infinitely many extremal quasiconformal mappings, then there always exist infinitely many extremal quasiconformal mappings of non-landslide type in $[f]$.*

2. PROOF OF THEOREMS

In this section, by some lemmas, we will prove Theorems 1.2 and 1.3.

Lemma 2.1 ([3]). *Let $\{f_n\}$ be a sequence of K -quasiconformal mappings of Ω in the complex plane which converges locally to a quasiconformal mapping f with Beltrami coefficient μ . If the Beltrami coefficients μ_n of f_n tend to a limit a.e., then $\lim_{n \rightarrow \infty} \mu_n(z) = \mu(z)$ a.e.*

Lemma 2.2 ([10]). *Let f be a quasiconformal mapping of the unit disk Δ onto itself. Choose a fixed point $z_0 \in \Delta$ and, for $w \in \Delta$, let g be an extremal quasiconformal mapping Δ onto itself which is equal to f on $\partial\Delta$ and takes z_0 into w . Let K be the maximal dilatation of g . Then K is, for fixed z_0 , a continuous function of w which tends to infinity for $|w| \rightarrow 1$.*

Let f be a quasiconformal mapping on Δ . We call

$$V[z_0] = \{w = g(z_0), g \text{ is extremal in } [f]\}$$

the variability set of z_0 .

Lemma 2.3 ([10]). *Let f be an extremal quasiconformal mapping of the unit disk Δ onto itself with maximal dilatation K_0 , and let w be a point outside the variability set $V[z_0]$. Let g be an extremal quasiconformal mapping of Δ onto itself with the boundary values $f|_{\partial\Delta}$ and $g(z_0) = w$. Then $\mu_g = k \frac{\varphi}{|\varphi|}$, where $\frac{1+k}{1-k} > K_0$ and φ is analytic in $\Delta \setminus z_0$.*

Using Lemmas 2.2 and 2.3, we have the following lemma.

Lemma 2.4. *Let f be a quasiconformal mapping of the unit disk Δ onto itself with maximal dilatation K_0 . Then, for any $K > K_0$, there exists a quasiconformal $g \in [f]$ such that $|\mu_g(z)| = k, z \in \Delta$, where $k = \frac{K-1}{K+1}$.*

Proof. Let $\tilde{f} \in [f]$ be an extremal quasiconformal mapping. Then $K(\tilde{f}) \leq K_0$. For $z_0 \in \Delta$ and $K > K_0 \geq K(\tilde{f})$, by Lemma 2.2, we can choose a point $w \notin V[z_0]$ and $g \in [f]$ such that $K(g) = K$ and $g(z_0) = w$. For $w \notin V[z_0]$, by Lemma 2.3, we have $|\mu_g(z)| = k, z \in \Delta$. $\square$

Proof of Theorem 1.2. Let $S = \{(x, y) : 0 < x < 1, 0 < y < 1\}$, and let f be a quasiconformal mapping on S . We will construct an extremal quasiconformal mapping $g \in [f]$ of non-landslide type. With no loss of generality, we suppose f is extremal and set $K = K(f), k = \frac{K-1}{K+1}$. If $[f]$ contains a uniquely extremal quasiconformal mapping f , then the theorem holds for f of non-landslide type. So we suppose that $[f]$ contains infinitely many extremal quasiconformal mappings. We will prove the theorem by the following four steps.

Step 1. A sequence of open squares $\{S_j^{(n)}\}$ and a sequence of extremal quasiconformal mappings $\{f^{(n)} \in [f]\}$ are constructed.

$n = 1$. Dividing S by the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$, we get 4^1 open squares with side length $\frac{1}{2}$ denoted by

$$S_1^{(1)}, S_2^{(1)}, S_3^{(1)}, S_4^{(1)}.$$

For each $S_j^{(1)} (j = 1, 2, 3, 4)$, we define a quasiconformal mapping $f_j^{(1)}$ on $S_j^{(1)}$ as follows. If $\exists \delta > 0$ such that $K(f|_{S_j^{(1)}}) < K - \delta$, applying Lemma 2.4 for $f|_{S_j^{(1)}}$, we can define a quasiconformal mapping $f_j^{(1)}$ on $S_j^{(1)}$ such that $f_j^{(1)}|_{\partial S_j^{(1)}} = f|_{\partial S_j^{(1)}}$ and $|\mu_{f_j^{(1)}}(z)| = k$ for all $z \in S_j^{(1)}$. Otherwise, there is no $\delta > 0$ such that $K(f|_{S_j^{(1)}}) < K - \delta$, and we define the quasiconformal mapping $f_j^{(1)} = f|_{S_j^{(1)}}$.

Then we define $f^{(1)}$ on S as

$$f^{(1)}(z) = f_j^{(1)}(z), \quad z \in \overline{S_j^{(1)}}, (j = 1, 2, 3, 4).$$

By the definition of $f^{(1)}$, we have that $f^{(1)} \in [f]$ is extremal and $K(f^{(1)}|_{S_j^{(1)}}) = K(f_j^{(1)}) = K$.

$n - 1 \rightarrow n$. For each square $S_j^{(n-1)} (j = 1, 2, \dots, 4^{n-1})$, as was done to S , we divide $S_j^{(n-1)}$ into four congruent squares. Then we get 4^n open squares with side length $\frac{1}{2^n}$ denoted by

$$S_1^{(n)}, S_2^{(n)}, \dots, S_j^{(n)}, \dots, S_{4^n}^{(n)}.$$

Similar to the definition of $f_j^{(1)}$ on $S_j^{(1)}$, we define $f_j^{(n)}$ on $S_j^{(n)}$ as follows. If $\exists \delta > 0$ such that $K(f^{(n-1)}|_{S_j^{(n)}}) < K - \delta$, applying Lemma 2.4 for $f^{(n-1)}|_{S_j^{(n)}}$, we can define a quasiconformal mapping $f_j^{(n)}$ on $S_j^{(n)}$ such that $f_j^{(n)}|_{\partial S_j^{(n)}} = f^{(n-1)}|_{\partial S_j^{(n)}}$ and $|\mu_{f_j^{(n)}}(z)| = k$ for all $z \in S_j^{(n)}$. Otherwise, there is no $\delta > 0$ such that $K(f^{(n-1)}|_{S_j^{(n)}}) < K - \delta$, and we define the quasiconformal mapping $f_j^{(n)} = f^{(n-1)}|_{S_j^{(n)}}$. Then we define $f^{(n)}$ on S as

$$f^{(n)}(z) = f_j^{(n)}(z), \quad z \in \overline{S_j^{(n)}}, (j = 1, 2, \dots, 4^n).$$

By the definition of $f^{(n)}$, we have that $f^{(n)} \in [f]$ is extremal and $K(f^{(n)}|_{S_j^{(n)}}) = K(f_j^{(n)}) = K$, ($j = 1, 2, \dots, 4^n$).

So we have constructed a sequence of open squares $\{S_j^{(n)}, j = 1, 2, \dots, 4^n\}$ and a sequence of extremal quasiconformal mappings $\{f^{(n)} \in [f]\}$ such that

$$(2.1) \quad K(f^{(n)}|_{S_j^{(n)}}) = K(f_j^{(n)}) = K.$$

Step 2. For the normality of quasiconformal mappings, there is a subsequence of $\{f^{(n)}\}$ (we denote it also by $\{f^{(n)}\}$) that converges locally to a quasiconformal mapping g . It is obvious that $g \in [f]$ is extremal.

Step 3. Let μ be the Beltrami coefficient of g and let μ_n be the Beltrami coefficient of $f^{(n)}$. We claim that $\lim_{n \rightarrow \infty} \mu_n(z) = \mu(z)$ a.e. $z \in S$. By Lemma 2.1, it is sufficient to prove that μ_n a.e. tends to a limit. When the two-dimensional measure of $\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{4^n} \partial S_j^n$ is 0, let $z \in S \setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{4^n} \partial S_j^n$. We will prove that $\mu_n(z)$ tends to a limit.

By the definition of $\{f^{(n)}\}$, we have the following properties of μ_n . If $K(f^{(n)}|_{S_{j_{n+1}}^{(n+1)}}) = K$ for all n and squares $S_{j_{n+1}}^{(n+1)} \ni z$, then $f^{(n+1)}|_{S_{j_{n+1}}^{(n+1)}} = f|_{S_{j_{n+1}}^{(n+1)}}$, so $\lim_{n \rightarrow \infty} \mu_n(z) = \mu_f(z)$. Otherwise, if there exists $\delta > 0, m$ and $S_{j_{m+1}}^{(m+1)} \ni z$ such that $K(f^{(m)}|_{S_{j_{m+1}}^{(m+1)}}) < K - \delta$, then $f^{(n)}|_{S_{j_n}^{(n)}} = f^{(m+1)}|_{S_{j_n}^{(n)}}$ for all $n \geq m+1$ and $S_{j_n}^{(n)} \ni z$, so $\lim_{n \rightarrow \infty} \mu_n(z) = \mu_{f^{(m+1)}}(z)$.

Step 4. We claim that $g \in [f]$ obtained in Step 2 is of non-landslide type. If, on the contrary, g is of landslide type, then there is a $\delta > 0$ and an open subset $E \in S$ such that $K(g|_E) < K - \delta$. For E an open set, we fix m and j_m such that $S_{j_m}^{(m)} \subset E$. Then

$$(2.2) \quad K(g|_{S_{j_m}^{(m)}}) \leq K(g|_E) < K - \delta.$$

If $K(f^{(n-1)}|_{S_{j_n}^{(n)}}) = K$ for all $n > m$ and $S_{j_n}^{(n)} \subset S_{j_m}^{(m)}$, by the argument in Step 3, we have $f^{(n)}|_{S_{j_n}^{(n)}} = f^{(m)}|_{S_{j_n}^{(n)}}$ for all $n > m$, so

$$(2.3) \quad \mu|_{S_{j_m}^{(m)}} = \lim_{n \rightarrow \infty} \mu_n|_{S_{j_m}^{(m)}} = \mu_{f^{(m)}}|_{S_{j_m}^{(m)}}.$$

It follows from (2.1) and (2.3) that

$$(2.4) \quad K(g|_{S_{j_m}^{(m)}}) = K.$$

Otherwise, if there exists $\delta > 0, n$ and $S_{j_n}^{(n)} \subset S_{j_m}^{(m)}$ such that $K(f^{(n-1)}|_{S_{j_n}^{(n)}}) < K - \delta$, we have $f^{(l)}|_{S_{j_n}^{(n)}} = f^{(n)}|_{S_{j_n}^{(n)}}$ for all $l \geq n$, so

$$(2.5) \quad \mu|_{S_{j_n}^{(n)}} = \lim_{l \rightarrow \infty} \mu_l|_{S_{j_n}^{(n)}} = \mu_{f^{(n)}}|_{S_{j_n}^{(n)}}.$$

It follows from (2.1) and (2.5) that

$$(2.6) \quad K(g|_{S_{j_m}^{(m)}}) \geq K(g|_{S_{j_n}^{(n)}}) = K.$$

However, (2.2) contradicts (2.4) and (2.6), so $g \in [f]$ is an extremal quasiconformal mapping of non-landslide type. This completes the proof. $\square$

Proof of Theorem 1.3. Let $K(f) = K$ and $k = \frac{K-1}{K+1}$. If any extremal quasiconformal mapping in $[f]$ is of non-landslide type, the theorem holds when $[f]$ contains infinitely many extremal quasiconformal mappings. If there is an extremal quasiconformal mapping $g \in [f]$ of landslide type, this means that there is a $\delta > 0$ and an open subset $E \in S$ such that $K(g|_E) < K - \delta$. Let $z_0 \in E$, $w_0 = g(z_0)$ and $L_\theta = \{w, \arg(w - w_0) = \theta\}$ ($\theta \in (-\pi, \pi]$). Applying Lemma 2.4 for $g|_E$, we get $w_\theta \in L_\theta$ and a quasiconformal mapping $g_\theta \in [f]$ such that

$$(2.7) \quad g_\theta|_{\partial E} = g|_{\partial E}, |\mu_{g_\theta}(z)| = k \text{ for all } z \in E, \text{ and } g_\theta(z_0) = w_\theta.$$

Starting with the extremal quasiconformal mapping g_θ , following the proof of Theorem 1.2, we can construct an extremal quasiconformal mapping $\tilde{g}_\theta \in [f]$ of non-landslide type. By the construction of $\tilde{g}_\theta$ and $|\mu_{g_\theta}(z)| = k$ for all $z \in E$, we have $\tilde{g}_\theta|_E = g_\theta|_E$. Furthermore, by (2.7), if $\theta_1 \neq \theta_2$,

$$\tilde{g}_{\theta_1}(z_0) = g_{\theta_1}(z_0) = w_{\theta_1} \neq w_{\theta_2} = g_{\theta_2}(z_0) = \tilde{g}_{\theta_2}(z_0),$$

so $\tilde{g}_{\theta_1} \neq \tilde{g}_{\theta_2}$. Varying $\theta \in (-\pi, \pi]$, we get infinitely many extremal quasiconformal mappings $\tilde{g}_\theta \in [f]$ of non-landslide type, which completes the proof. $\square$

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REFERENCES

1. V. Bozin, N. Lakic, V. Markovic, M. Mateljevic, *Unique extremality*, J. Anal. Math. **75** (1998), 299–337. MR1655836 (2000a:30045)
2. R. S. Hamilton, *Extremal quasiconformal mappings with prescribed boundary values*, Trans. Amer. Math. Soc. **138** (1969), 399–406. MR0245787 (39:7093)
3. O. Lehto, *Univalent functions and Teichmüller spaces*, GTM 109, Springer-Verlag, New York, 1987. MR867407 (88f:30073)
4. Z. Li, *A note on extremal quasi-conformal mappings*, Sci. China Ser. A **53** (2010), 63–70. MR2594747
5. S. L. Krushkal, *Extremal quasiconformal mappings*, Siberian Math. J. **10** (1969), 411–418.
6. E. Reich, *Non-uniquely extremal quasiconformal mappings*, Libertas Math. **20** (2000), 33–38. MR1801108 (2002a:30033)
7. E. Reich, *The unique extremality counterexample*, J. Anal. Math. **75** (1998), 339–347. MR1655837 (99k:30037)
8. E. Reich, *Extremal quasiconformal mappings of the disk*, in Handbook of Complex Analysis: Geometry Function Theory, vol. 1, North-Holland, 2002, pp. 75–136. MR1966190 (2004c:30036)
9. E. Reich, K. Strebel, *Extremal quasiconformal mappings with given boundary values*, in Contribution to Analysis, Academic Press, New York, 1974, pp. 375–392. MR0361065 (50:13511)
10. K. Strebel, *Point shift differentials and extremal quasiconformal mappings*, Ann. Acad. Sci. Fenn. Math. **23** (1998), 475–494. MR1642146 (99j:30022)
11. Z. Zhou, J. Chen, Z. Yang, *On the extremal sets of extremal quasiconformal mappings*, Sci. China Ser. A **46** (2003), 552–561. MR2014487 (2004j:30036)

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