POSITIVE COMBINATIONS AND SUMS OF PROJECTIONS IN PURELY INFINITE SIMPLE C^* -ALGEBRAS AND THEIR MULTIPLIER ALGEBRAS

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ABSTRACT. Every positive element in a purely infinite simple σ -unital C^* -algebra $\mathcal{A}$ is a finite linear combination of projections with positive coefficients. Also, every positive a in the multiplier algebra $\mathcal{M}(\mathcal{A})$ of a purely infinite simple σ -unital C^* -algebra $\mathcal{A}$ is a finite linear combination of projections with positive coefficients. Furthermore, if the essential norm $\|a\|_{ess} > 1$, then a is a finite sum of projections in $\mathcal{M}(\mathcal{A})$. As a consequence, any positive element in the generalized Calkin Algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$ or in $\mathcal{M}(\mathcal{A})$ but not in $\mathcal{A}$ is a positive scalar multiple of a finite sum of projections.

1. INTRODUCTION

In [8], Fillmore showed that if $\mathcal{H}$ is an infinite dimensional Hilbert space, then every operator in $\mathbb{B}(\mathcal{H})$ is a linear combination of at most 257 projections. This estimate was subsequently improved by Percy and Topping [20], who showed that 16 projections sufficed. The best result in this direction is due to K. Matsumoto [19] who showed that any selfadjoint operator in $\mathbb{B}(\mathcal{H})$ is a linear combination of 5 projections, and hence, any operator in $\mathbb{B}(\mathcal{H})$ is a linear combination of 10 or fewer projections.

Subsequently, Marcoux and Murphy generalized the above results, showing that for large classes of C^* -algebras $\mathcal{C}$, every element in $\mathcal{C}$ is a finite linear combination of projections in $\mathcal{C}$ ([18], [16], [17]). In particular, this result covers all *UHF*-algebras and all simple unital purely infinite C^* -algebras. However, it remained difficult to answer the question whether a positive element is a finite linear combination of projections with positive coefficients in such C^* -algebras, a *positive combination of projections* for short. A result due to Fong and Murphy ([9, Theorem 11]) showed that a positive operator $c \in \mathbb{B}(\mathcal{H})$ can be written as a positive combination of projections if and only if c is not a compact operator of infinite rank.

In this paper, we will continue the efforts of Fong-Murphy and Marcoux-Murphy to consider two problems.

One is: under what conditions on a given C^* -algebra $\mathcal{C}$ are all positive elements c of $\mathcal{C}$ indeed positive combinations of projections?

The other is: for a given C^* -algebra $\mathcal{C}$ and a positive element $c \in \mathcal{C}$, under what conditions is c a finite sum of projections from $\mathcal{C}$?

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We note that this second problem is also motivated by ideas from frame theory [5]. The weaker problem when the sums are allowed to be infinite has been completely solved when $\mathcal{C} = \mathbb{B}(\mathcal{H})$, $\mathcal{C}$ is a type *III* factor and for diagonalizable operators when $\mathcal{C}$ is a type *II* factor (with convergence in the strong operator topology) [10], and also when $\mathcal{C}$ is the multiplier algebra of a simple purely infinite σ -unital but nonunital C^* -algebra (with convergence in the strict topology) [11].

Our main results are that every positive element of a simple purely infinite σ -unital C^* -algebra $\mathcal{A}$ is a positive combination of projections (Theorem 2.11). The same conclusion holds if $\mathcal{A}$ is the multiplier algebra of a simple purely infinite σ -unital C^* -algebra (Theorem 3.1). If $a \in \mathcal{M}(\mathcal{A})_+$, where $\mathcal{A}$ is a simple purely infinite σ -unital nonunital C^* -algebra and $\|a\|_{ess} > 1$, then a is a sum of finitely many projections (Theorem 3.3). As a consequence, every positive element of $\mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ is a positive scalar multiple of a finite sum of projections.

2. POSITIVE COMBINATIONS OF PROJECTIONS

We shall call a *positive combination of projections* any linear combination $\sum_1^n \lambda_j p_j$ of projections p_j where all the coefficients λ_j are positive. We will study the problem of whether all positive elements in a certain C^* -algebra can be written as positive combinations of projections. This property is stronger than the CP-property in the literature ([1]): all positive combinations of projections are norm dense in the positive cone of a C^* -algebra.

Although in this paper we will be mainly interested in purely infinite simple σ -unital C^* -algebras or in the multiplier algebras of purely infinite simple σ -unital nonunital C^* -algebras, it may be worthwhile to establish some preliminary results with more generality.

Recall that a C^* -algebra is said to have the LP property if each of its hereditary C^* -subalgebras is the closure of the linear span of its projections [1].

We need to consider the stronger property:

Definition 2.1. A C^* -algebra is said to have the *PLP property* if the positive cone of each of its hereditary C^* -subalgebras is the closure of the positive combinations of its projections.

Note that:

Remark 2.1. It is known that the following algebras have the PLP property:

- (i) All C^* -algebras of real rank zero ([3]).
- (ii) All purely infinite simple σ -unital C^* -algebras ([24]).
- (iii) Multiplier algebras of all stable σ -unital C^* -algebras of real rank zero ([23, Theorem 1.1]).

In [11, Lemma 2.3] we have proven for purely infinite simple σ -unital C^* -algebras that every positive element can not only be approximated by positive combinations of projections, but the approximation can be chosen from underneath. Notice that purely infinite simple σ -unital C^* -algebras have real rank zero ([24]), and hence they have the PLP property, and notice also that the proof of [11, Lemma 2.3] only relies on the PLP property of the algebra. Thus, we have the following general statement.

Lemma 2.2. *Let $\mathcal{A}$ be a C^* -algebra with the PLP property. Then for each nonzero positive element $x \in \mathcal{A}$ and any $\epsilon > 0$ there exists a positive combination $\sum_{i=1}^n \lambda_i p_i$*

of projections in $\mathcal{A}$ (and hence in $\text{her}(x) =: (xAx)^-$) such that

$$\|x - \sum_{i=1}^n \lambda_i p_i\| \leq \epsilon \quad \text{and} \quad \sum_{i=1}^n \lambda_i p_i \leq x.$$

It is well known that in an infinite simple and unital C^* -algebra and in a multiplier algebra of any stable C^* -algebra there always exist two projections p, q such that $p \sim q \sim 1$ and $pq = 0$, where 1 will denote the identity of the C^* -algebra.

Fack showed in [6, Theorem 2.1] that all selfadjoint elements of an algebra $\mathcal{A}$ with two such projections are sums of self-commutators. Contained in his proofs are the following norm bounds:

Lemma 2.3 ([6, Theorem 2.1]). *Let $\mathcal{A}$ be a unital C^* -algebra for which there exist two projections $p, q \in \mathcal{A}$ such that $p \sim q \sim 1$ and $pq = 0$, where 1 denotes the identity of $\mathcal{A}$. Then for every selfadjoint $a \in \mathcal{A}$,*

$$a = \sum_{i=1}^5 [x_i, x_i^*],$$

for some $x_i \in \mathcal{A}$ with $\|x_i\| \leq \frac{5}{2}\|a\|^{\frac{1}{2}}$, where $[x_i, x_i^*] = x_i x_i^* - x_i^* x_i$ denotes the commutator of x_i and x_i^* .

The norm bound for one of the elements x_i can be obtained immediately by inspection of the first step in Fack's proof, [6, Lemma 1.2] and the norms of the remaining elements x_i are bounded by $\|a\|^{\frac{1}{2}}$ as by [6, Lemmas 1.3, 1.4, 2.3].

Next we examine Marcoux's results [16, Theorem 3.8 and preceding lemmas]. Contained in his proofs are the following norm bounds:

Lemma 2.4 ([16, Theorem 3.8]). *Let $\mathcal{A}$ be a unital C^* -algebra for which there exist three mutually orthogonal projections p_1, p_2 and p_3 such that $1 = p_1 + p_2 + p_3$ and $p_i \preceq 1 - p_i$ for $1 \leq i \leq 3$. Then for all $x, y \in \mathcal{A}$ with $\|x\|, \|y\| \leq 1$, there exist $n \leq 84$ projections $q_1, q_2, \dots, q_n \in \mathcal{A}$ and real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ with $|\alpha_j| \leq 2\sqrt{2}$ such that*

$$[x, y] := xy - yx = \alpha_1 q_1 + \alpha_2 q_2 + \dots + \alpha_n q_n.$$

Proof. Step 1 [16, Lemma 3.4]: In the matrix representation with respect to the decomposition $1 = p_1 + p_2 + p_3$,

$$[a, b] = \begin{bmatrix} [a_{1,1}, b_{1,1}] & 0 & 0 \\ 0 & [a_{2,2}, b_{2,2}] & 0 \\ 0 & 0 & [a_{3,3}, b_{3,3}] \end{bmatrix} + \sum_{i=1}^9 c_i,$$

where $c_i \in \mathcal{A}$ are nilpotent of order 2 (i.e., $c_i^2 = 0$). Three of these nilpotent elements (m_1, m_2, m_3 in the notation of [16, Lemma 3.4]) have the form $qc(1-q)$ for some projection $q \in \mathcal{A}$ and some $c \in \mathcal{A}$ and it is easily seen that they have norms bounded by $4\sqrt{2}$. The remaining six of the nilpotent elements (the summands of m_o) are of the form $0 \oplus \begin{bmatrix} cd & cdc \\ -d & -dc \end{bmatrix}$, and it is also immediate that they have norms bounded by 2. Furthermore, by the proof of [16, Corollary 3.7 (iv)], they are also of the form $qc(1-q)$ for some projection $q \in \mathcal{A}$ and some $c \in \mathcal{A}$.

Step 2 [16, Lemma 3.2 (ii)]: The commutators $[a_{i,i}, b_{i,i}]$ are sums of 4 nilpotent elements of the form $qc(1-q)$ for some projection $q \in \mathcal{A}$ and some $c \in \mathcal{A}$; again, this follows from the proof of [16, Corollary 3.7 (iii) and (iv)]. It is routine that each of these nilpotent elements has norm bounded by 2.

Step 3 [16, Remark after Theorem 3.6, due to K. Davidson]: For every element of the form $qc(1-q)$ there are four projections q_j such that

$$qc(1-q) = \sum_{k=1}^4 \frac{\|c\|}{2} q_k.$$

Thus $[a, b]$ is the sum 21 nilpotent elements of order two of the form $qc(1-q)$, each with norm bounded by $4\sqrt{2}$, and hence, each a linear combination of 4 projections with coefficients bounded by $2\sqrt{2}$. $\square$

Remark 2.5. (i) In any nonelementary unital simple C^* -algebra of real rank zero there are three projections p_1, p_2, p_3 as required in Lemma 2.4. In fact, this is an easy consequence of [26, Theorem 1.1]. First, one writes $1 = r_1 \oplus r_2 \oplus r_0$, where $r_1 \sim r_2$. Halving r_0 , one can write $r_0 = r'_0 \oplus r''_0 \oplus r'$ such that $r'_0 \sim r''_0$ and r' is equivalent to a subprojection of r_1 . Set $p_1 = r_1 \oplus r'_0$, $p_2 = r_2 \oplus r''_0$, and $p_3 = r'$. Then $1 = p_1 + p_2 + p_3$ and $p_i \preceq 1 - p_i$ for $1 \leq i \leq 3$.

(ii) It is well-known that in unital purely infinite simple C^* -algebras and in the multiplier algebras of all stable C^* -algebras there always exist two projections $p, q \in \mathcal{A}$ such that $p \sim q \sim 1$ and $pq = 0$, as required by Lemma 2.3. It is trivial that the existence of two such projections p, q implies the existence of the three projections p_1, p_2, p_3 as required in Lemma 2.4.

Combining the two lemmas above, we obtain the following more general result that holds for all infinite simple C^* -algebras and for the multiplier of all stable C^* -algebras.

Proposition 2.6. *Let $\mathcal{A}$ be a unital C^* -algebra for which there exist two projections $p, q \in \mathcal{A}$ such that $p \sim q \sim 1$ and $pq = 0$. Then there exists a constant V_0 such that for every element $a \in \mathcal{A}$, there exist numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and there exist projections $p_1, p_2, \dots, p_n \in \mathcal{A}$ such that*

$$a = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n$$

and

$$|\alpha_1| + |\alpha_2| + \dots + |\alpha_n| \leq V_0 \|a\|.$$

If $a = a^*$ all the numbers α_j can be chosen to be real.

Proof. By Lemma 2.3,

$$a = \sum_{j=1}^5 [x_j, x_j^*] + i \sum_{j=6}^{10} [x_j, x_j^*]$$

for some $x_j \in \mathcal{A}$ with $\|x_j\| \leq \frac{5}{2} \|a\|^{1/2}$. Thus

$$a = \sum_{j=1}^{10} [x_j, y_j] \quad \text{with} \quad y_j = \begin{cases} x_j^* & 1 \leq j \leq 5 \\ ix_j^* & 5 \leq j \leq 10 \end{cases}$$

and hence $\|y_j\| \leq \frac{5}{2}\|a\|^{1/2}$. By Lemma 2.4, each of the commutators is a linear combination of not more than 84 projections, with coefficients bounded by $2\sqrt{2}(\frac{5}{2}\|a\|^{1/2})^2 = \frac{25}{\sqrt{2}}\|a\|$. This completes the proof. $\square$

Next, we show that if the algebra has a universal constant V_0 as in Proposition 2.6 and satisfies a weaker form of the PLP property, then all positive invertible elements are positive combinations of projections.

Proposition 2.7. *Let $\mathcal{A}$ be a unital C^* -algebra with the following two properties:*

- (1) *There is a universal constant V_0 such that for every $a \in \mathcal{A}$, there exist numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and projections $p_1, p_2, \dots, p_n \in \mathcal{A}$ such that*

$$a = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n$$

and

$$|\alpha_1| + |\alpha_2| + \dots + |\alpha_n| \leq V_0 \|a\|.$$

- (2) *Positive combinations of projections are norm dense in $\mathcal{A}_+$.*

Then every positive invertible element in $\mathcal{A}$ is a positive combination of projections.

Proof. This argument is a modification of the argument of [9, Lemma 8] (see also [12, Theorem 2.1 I]).

Let $a \in \mathcal{A}$ be a positive invertible element. We may assume that $\|a\| < 1$. Then since $a \geq \epsilon_1$ for some $\epsilon > 0$, it follows that also $\|1 - a\| < 1$. Let $\epsilon_1, \epsilon_2 > 0$ be such that $0 < \epsilon_1 < \epsilon_2 < 1/2$ and $\|a\|, \|1 - a\| < 1 - \epsilon_2$. Choose $\delta > 0$ such that $\delta < \frac{\epsilon_1}{V_0}$ and $\delta < \epsilon_2 - \epsilon_1$. Note that $a - \epsilon_1 1$ is also a positive element of $\mathcal{A}$. Hence, by the hypothesis (2), let $a_0 \in \mathcal{A}$ be a positive combination of projections such that

$$\|(a - \epsilon_1 1) - a_0\| < \delta < \frac{\epsilon_1}{V_0}.$$

By the hypothesis (1),

$$a_1 := (a - \epsilon_1 1) - a_0 = \alpha_1 p_1 + \alpha_2 p_2 + \dots + \alpha_n p_n,$$

where the α_i 's are complex numbers and $q_1, q_2, \dots, q_n$ are projections in $\mathcal{A}$. Since a_1 is selfadjoint, we can use the trivial relation $a_1 = \frac{1}{2}(a_1 + a_1^*)$ to assure that there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$a_1 = (a - \epsilon_1 1) - a_0 = \sum_{i=1}^n \alpha_i q_i$$

and

$$\sum_{i=1}^n |\alpha_i| \leq \delta V_0 < \epsilon_1.$$

Hence,

$$\begin{aligned} a &= a_0 + \epsilon_1 1 + a_1 \\ &= a_0 + \epsilon_1 1 + \sum_{i=1}^n \alpha_i q_i \\ &= a_0 + \sum_{\alpha_i > 0} \alpha_i q_i + \sum_{\alpha_i < 0} (-\alpha_i)(1 - q_i) + (\epsilon_1 + \sum_{\alpha_i < 0} \alpha_i)1, \end{aligned}$$

which is a finite linear combination of projections with positive coefficients, as required. $\square$

If a C^* -algebra $\mathcal{A}$ satisfies condition (1) of Proposition 2.7, we cannot conclude that the same holds for all its corners $r\mathcal{A}r$ of $\mathcal{A}$ for projections $r \in \mathcal{A}$. Moreover, in general, if all positive invertible elements in a C^* -algebra $\mathcal{A}$ are positive combinations of projections, we cannot conclude the same for all of its corners $r\mathcal{A}r$. However, this conclusion holds for all the corners of the algebras on which we focus in this paper, namely:

Corollary 2.8. (i) *If $\mathcal{A}$ is a purely infinite and simple C^* -algebra, then for every projection $r \in \mathcal{A}$, every positive invertible element in $r\mathcal{A}r$ is a positive combination of projections in $r\mathcal{A}r$.*
(ii) *If $\mathcal{B}$ is any σ -unital nonunital simple purely infinite C^* -algebra, then every positive invertible element in a corner of $\mathcal{M}(\mathcal{B})$ is a positive combination of projections in $\mathcal{M}(\mathcal{B})$.*

Proof. The hypothesis of Proposition 2.6 holds for these corner subalgebras, and condition (1) of Proposition 2.7 holds as well. Moreover, since all purely infinite and simple C^* -algebras and their multiplier algebras have the PLP property by Remark 2.1 (ii) and (iii), thus they satisfy the weaker condition (2) of Proposition 2.7. Then the conclusion follows immediately. $\square$

Now we can present our two key lemmas.

Lemma 2.9. *Let $\mathcal{A}$ be C^* -algebra with the property that for every projection $r \in \mathcal{A}$, every positive invertible element of $r\mathcal{A}r$ is a positive combination of projections in $r\mathcal{A}r$. Let $a := \alpha p \oplus b$ be a positive element in $\mathcal{A}$, where $\alpha > \|b\|$ and there exists a projection $q \in (1-p)\mathcal{A}(1-p)$ such that $b = qb = bq$ and $q \prec p$. Then a is a positive combination of projections.*

Proof. Replacing if necessary a by a positive multiple of a , we may assume that $\|b\| \leq 1 < \alpha$. Since $q \prec p$, let $v \in \mathcal{A}$ be a partial isometry such that $v^*v = q$ and $vv^* = p' \leq p$. Define two projections as follows:

$$\begin{aligned} r_1 &:= b + v\sqrt{b - b^2} + \sqrt{b - b^2}v^* + (p' - vbv^*), \\ r_2 &:= b - v\sqrt{b - b^2} - \sqrt{b - b^2}v^* + (p' - vbv^*). \end{aligned}$$

Then $\frac{1}{2}(r_1 + r_2) = b + p' - vbv^*$. Consequently,

$$a = \frac{1}{2}(r_1 + r_2) + \alpha(p - p') + (\alpha - 1)p' + vbv^*.$$

It follows from the inequality

$$h := (\alpha - 1)p' + vbv^* \geq (\alpha - 1)p'$$

that h is a positive invertible element in the C^* -subalgebra $p'\mathcal{A}p'$ of $\mathcal{A}$. By hypothesis, h is a positive combination of projections and hence so is a . $\square$

Lemma 2.10. *Let $\epsilon > 0$ and $\mathcal{A}$ be a C^* -algebra with the PLP property and let $a \in \mathcal{A}_+$ with $\|a\| > 1$.*

- (i) *If $\|a\|$ is an isolated point in $\sigma(a)$, then $a = \alpha p \oplus \left(\sum_1^n \lambda_j r_j + b\right)$ for some projections p and $r_j \in \mathcal{A}$, numbers $0 < \lambda_j \leq 1 < \alpha$, and a positive element $b \in \mathcal{A}$ with $\|b\| \leq \epsilon$.*

- (ii) If $\|a\|$ is not an isolated point in $\sigma(a)$, then a can be decomposed as $a = a_1 + a_2$, with $a_i = \alpha_i p_i \oplus \left(\sum_1^{n_i} \lambda_{ij} r_{ij} + b_i \right)$ having the same properties as in (i).

Proof. We prove case (ii) and leave to the reader the elementary case (i) which requires only the final step of the proof of case (ii). Choose $t \in \sigma(a) \cap (1, \|a\|)$ and four additional numbers $\{\alpha_1, s_1, s_2, \alpha_2\}$ such that

$$1 < \alpha_1 < t < s_1 < s_2 < \alpha_2 < \|a\|.$$

Define two continuous functions on $[0, \|a\|]$ as follows:

$$f_1(s) = \begin{cases} 0 & s \in [0, 1] \cup [s_2, \|a\|] \\ s & s \in [\alpha_1, s_1] \\ \text{linear} & \text{elsewhere.} \end{cases}$$

$$f_2(s) = \begin{cases} 0 & s \in [0, s_2] \\ s & s \in [\alpha_2, \|a\|] \\ \text{linear} & \text{elsewhere.} \end{cases}$$

Set $d_1 = f_1(a)$ and $d_2 = f_2(a)$. Since $f_i(s) \geq 0$, $f_1(s)f_2(s) = 0$ and $f_1(s) + f_2(s) \leq s$ on $[0, \|a\|]$. It follows from Gelfand's transformation that d_1 and d_2 are positive elements in $\mathcal{A}$ and that

$$d_1 d_2 = 0 \quad \text{and} \quad d_1 + d_2 \leq a.$$

Now set

$$r_1 = \chi_{(\alpha_1, s_1)}(a) \quad \text{and} \quad r_2 = \chi_{(\alpha_2, \|a\|+1)}(a).$$

Then r_1 and r_2 are open projections (in $\mathcal{A}^{**}$). Since

$$\sigma(a) \cap (\alpha_1, s_1) \neq \emptyset \quad \text{and} \quad \sigma(a) \cap (\alpha_2, \|a\|] \neq \emptyset,$$

it follows that

$$\text{her}(r_1) := r_1 \mathcal{A}^{**} r_1 \cap \mathcal{A} \quad \text{and} \quad \text{her}(r_2) := r_2 \mathcal{A}^{**} r_2 \cap \mathcal{A}$$

are nonzero and mutually orthogonal hereditary C^* -subalgebras of $\mathcal{A}$. By the assumption that $\mathcal{A}$ has the PLP property, the positive parts of both these hereditary C^* -subalgebras are the closure of the positive combinations of their projections. In particular, there are nonzero projections $p_1 \in \text{her}(r_1)$ and $p_2 \in \text{her}(r_2)$ such that $p_1 p_2 = 0$. Notice that

$$\chi_{(\alpha_1, s_1)}(s)(s - f_1(s) - f_2(s)) = 0 \quad \text{on } [0, \|a\|];$$

hence $r_1(a - d_1 - d_2) = 0$. Similarly, one can easily see that

$$\begin{aligned} \alpha_1 p_1 &\leq \alpha_1 r_1 \leq d_1, \\ \alpha_2 p_2 &\leq \alpha_2 r_2 \leq d_2, \\ p_1 d_2 &= p_1 r_1 d_2 = 0, \\ p_1(a - d_1 - d_2) &= p_1 r_1(a - d_1 - d_2) = 0, \\ p_2 d_1 &= p_2 r_2 d_1 = 0. \end{aligned}$$

Set

$$\begin{aligned} c_1 &= (d_2 - \alpha_2 p_2) + (a - d_1 - d_2), \\ c_2 &= d_1 - \alpha_1 p_1, \\ a_1 &= \alpha_1 p_1 + c_1, \\ a_2 &= \alpha_2 p_2 + c_2. \end{aligned}$$

Then

$$a = a_1 + a_2.$$

Notice that $c_1 \geq 0$ and $p_1 c_1 = 0$, and similarly, $c_2 \geq 0$ and $p_2 c_2 = 0$. Furthermore, by the assumption that $\mathcal{A}$ has the PLP property and by Lemma 2.2 there are projections $\{r_{1j}\}_1^{n_1}$ in $\text{her}(c_1)$ and $\{r_{2j}\}_1^{n_2}$ in $\text{her}(c_2)$ and hence in $\mathcal{A}$ and positive scalars $\{\lambda_{1j}\}_1^{n_1}$ and $\{\lambda_{2j}\}_1^{n_2}$ such that

$$\begin{aligned} \|c_1 - \sum_{i=1}^{n_1} \lambda_{1j} r_{1j}\| &\leq \epsilon, \quad \sum_{i=1}^{n_1} \lambda_{1j} r_{1j} \leq c_1, \\ \|c_2 - \sum_{i=1}^{n_2} \lambda_{2j} r_{2j}\| &\leq \epsilon, \quad \sum_{i=1}^{n_2} \lambda_{2j} r_{2j} \leq c_2. \end{aligned}$$

By further decomposing if necessary λ_{ij} into a sum of scalars ≤ 1 , assume without loss of generality that $\lambda_{ij} \leq 1$ for all i and j . Then set for $i = 1, 2$,

$$b_i := c_i - \sum_{i=1}^{n_i} \lambda_{ij} r_{ij}.$$

Thus

$$a_i = \alpha_i p_i \oplus \left(\sum_1^{n_i} \lambda_{ij} r_{ij} + b_i \right)$$

with the required properties, which completes the proof. $\square$

Theorem 2.11. *Let $\mathcal{A}$ be any purely infinite simple C^* -algebra. Then every positive element of $\mathcal{A}$ is a positive combination of projections.*

Proof. Let $a \in \mathcal{A}_+$. By scaling it if necessary, we may assume without loss of generality that $\|a\| > 1$.

Assume first that the hereditary C^* -subalgebra of $\mathcal{A}$ generated by a , denoted by $\text{her}(a) := (a\mathcal{A}a)^-$, is unital. By Remark 2.1(ii), $\mathcal{A}$ has the PLP property. Then a can be decomposed into the sum of one or two elements satisfying the condition (i) of Lemma 2.10. So, without loss of generality, assume that

$$a = \alpha p \oplus \left(\sum_1^n \lambda_j r_j + b \right)$$

for some projections p and $r_j \in \mathcal{A}$, some numbers $0 < \lambda_j \leq 1 < \alpha$ and a positive element $b \in \mathcal{A}$ with $\|b\| \leq 1$. Notice that $b = (I - p)b(I - p)$ and that $I - p \prec p$. By Corollary 2.8, $\mathcal{A}$ satisfies the hypotheses of Lemma 2.9. Thus $\alpha p + b$ is a positive combination $\sum_1^{n'} \lambda'_j r'_j$ of projections. Hence,

$$a = \sum_1^n \lambda_j r_j + \sum_1^{n'} \lambda'_j r'_j$$

is a positive combination of projections.

We now consider the case when $her(a) := (a\mathcal{A}a)^-$ is not unital. Notice that $her(a)$ is σ -unital; actually, a is a strictly positive element. It follows from [27, Theorem 1.2] that $her(a)$ is stable. Using the fact that any nonzero projection r of $\mathcal{A}$ is infinite, one can find a sequence of projections

$$r > r_1 > r_2 > r_3 > \cdots > r_n > r_{n+1} > \cdots$$

such that $r \sim r_i$ for all $i \geq 1$. Then one can easily show that $r_i - r_{i-1} \sim r_j - r_{j-1}$ for all $i, j \geq 1$. Define

$$r_0 := \bigoplus_{i=1}^{\infty} (r_{i+1} - r_i) \in \mathcal{A}^{**}.$$

Then r_0 is an open projection in $\mathcal{A}^{**}$ and $her(r_0)$ is a stable hereditary C^* -subalgebra of $r\mathcal{A}r$. By Brown's stabilization theorem ([2]) one has that $her(a)$ is $*$ -isomorphic to $her(r_0)$. Call this $*$ -isomorphism

$$\phi : her(a) \longrightarrow her(r_0).$$

It follows from the unital case considered above that $\phi(a)$, in the unital purely infinite simple C^* -algebra $r\mathcal{A}r$ (see [3] and [27]), can be written as a positive combination $\phi(a) = \sum_1^n \lambda_j q_j$ of projections $q_j \in r\mathcal{A}r$. Since $q_j \leq \frac{1}{\lambda_j} \phi(a) \in her(r_0)$, it follows that $q_j \in her(r_0)$ for all j . But then $a = \sum_1^n \lambda_j \phi^{-1}(q_j)$ is a positive combination of projections in $(a\mathcal{A}a)^-$. The proof is now complete. $\square$

3. FINITE SUMS OF PROJECTIONS IN MULTIPLIER ALGEBRAS

With some modifications of the construction in the previous section we can prove a stronger result in the multiplier algebras of purely infinite simple σ -unital but nonunital C^* -algebras (such a C^* -algebra is necessarily stable by [27, Theorem 1.2]), namely that every positive element in $\mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ is a positive multiple of a finite sum of projections. First, we prove that it is always a positive combination of projections.

If $x \in \mathcal{M}(\mathcal{A})$, by $\|x\|_{ess}$ we denote the norm of $\pi(x)$, where $\pi : \mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{A})/\mathcal{A}$ is the canonical quotient map.

Theorem 3.1. *Let $\mathcal{A}$ be a σ -unital purely infinite simple C^* -algebra. Then every nonzero positive element of $\mathcal{M}(\mathcal{A})$ is a positive combination of projections.*

Proof. If $\mathcal{A}$ is unital, then $\mathcal{M}(\mathcal{A}) = \mathcal{A}$ and then the conclusion follows from Theorem 2.11. Thus assume that $\mathcal{A}$ is nonunital, and that $a \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$. By scaling a if necessary, assume also that $\|a\|_{ess} > 1$. Since $\mathcal{M}(\mathcal{A})$ has the PLP property by Remark 2.1(iii), we can proceed with a similar construction as in Lemma 2.10. We consider the cases when $\|a\|_{ess}$ is or is not an isolated point in the essential spectrum $\sigma_{ess}(a) := \sigma(\pi(a))$. In the case when $\|a\|_{ess}$ is not isolated, we proceed with the same construction with

$$1 < \alpha_1 < t < s_1 < s_2 < \alpha_2 < \|a\|_{ess}$$

and $t \in \sigma_{ess}(a)$ and the ensuing positive elements d_1 and d_2 and open projections r_1 and r_2 . By the condition that t and $\|a\|_{ess}$ are in $\sigma_{ess}(a)$, it follows that for $i = 1, 2$, $her(r_i) \not\subset \mathcal{A}$. But then we can choose projections $p_i \in her(r_i) \setminus \mathcal{A}$. The rest

of the construction then follows unchanged, and hence we obtain that $a = a_1 + a_2$ with the a_i both having the form

$$\alpha p \oplus \left(\sum_1^n \lambda_j r_j + b \right)$$

for some projections $p \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$, $r_j \in \mathcal{M}(\mathcal{A})$, numbers $0 < \lambda_j \leq 1 < \alpha$ and a positive element $b \in \mathcal{A}$ with $\|b\| \leq 1$. Without loss of generality, thus assume that $a = \alpha p \oplus \left(\sum_1^n \lambda_j r_j + b \right)$.

Since $p \notin \mathcal{A}$, then $p \sim 1$, the identity of $\mathcal{M}(\mathcal{A})$ ([25, Theorem 3.1]). It can be decomposed into two mutually orthogonal projections $p = p_1 \oplus p_2$ such that $p_1 \sim p_2 \sim 1$. Reasoning as in the proof of Theorem 2.11, and invoking Lemma 2.8 (ii), by Lemma 2.9 we obtain that $\alpha p_1 + b$ is a positive combination $\sum_1^{n'} \lambda'_j r'_j$ of projections, necessarily all orthogonal to p_2 . Thus

$$a = \alpha p_2 \oplus \left(\sum_1^n \lambda_j r_j + \sum_1^{n'} \lambda'_j r'_j \right)$$

is a positive combination of projections. $\square$

Remark 3.2. For further reference, notice that we have proven that if $\|a\|_{ess} > 1$, then a is the sum of one or two elements of the form

$$\alpha p \oplus \left(\sum_1^n \lambda_j r_j \right)$$

for some projections $p \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$, $r_j \in \mathcal{M}(\mathcal{A})$ and numbers and $0 < \lambda_j \leq 1 < \alpha$.

Theorem 3.3. *Let $\mathcal{A}$ be a σ -unital but nonunital purely infinite simple C^* -algebra. Then every nonzero positive element of $\mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ with $\|a\|_{ess} > 1$ can be written as a finite sum of projections. Consequently, every nonzero positive element of $\mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ can be written as a positive scalar multiple of a sum of projections.*

Proof. By Remark 3.2, a is the sum of one or two elements of the form

$$\alpha p \oplus \left(\sum_1^n \lambda_j r_j \right)$$

for some projections $p \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$, $r_j \in \mathcal{M}(\mathcal{A})$ and numbers and $0 < \lambda_j \leq 1 < \alpha$. By decomposing $p = \sum_1^n p_j$ into a sum of mutually orthogonal projections $p_j \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$, we thus can assume, without loss of generality, that

$$a = \alpha p \oplus \lambda r$$

for some projections $p \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$, $0 \neq r \in \mathcal{M}(\mathcal{A})$ and numbers $\alpha > 1$ and $\lambda \geq 0$. Now decompose p into an infinite sum (converging in the strict topology) of mutually orthogonal projections $p_j \sim r$. This decomposition yields a unital embedding of $B(\ell^2)$ into $(p+r)\mathcal{M}(\mathcal{A})(p+r)$ that maps a rank one projection into r and $1-r$ into p . Thus a is the image of a positive operator with essential spectrum > 1 and a finite sum of projections by [22] and [4] (see also [12].) $\square$

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