GLOBAL EXISTENCE OF SOLUTIONS TO A COUPLED PARABOLIC-HYPERBOLIC SYSTEM WITH MOVING BOUNDARY

Y. S. CHOI AND CRAIG MILLER

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ABSTRACT. A cell motility study leads to a moving boundary problem governed by a system of parabolic-hyperbolic equations. Establishing the parabolicity of one of the governing equations requires a priori bound analysis. Such bounds also exclude the formation of shock in the hyperbolic equation. Speeds of the moving boundaries can then be controlled, which eventually leads to the global existence of solutions.

1. INTRODUCTION

Let $\eta > 0$, $\gamma \geq 0$, $b_0 > 0$, $\ell_0 > 0$ and $V_d > 0$ be given constants. We consider the following coupled system of equations arising from a study in cell motility:

$$(1.1) \quad \begin{cases} w_t + \eta w_x \sigma_x = 0, \\ \sigma_t = (w - \sigma)(\eta \sigma_x)_x - \eta \sigma_x^2 + \gamma(w - \sigma), \end{cases}$$

with boundary conditions

$$(1.2) \quad \begin{cases} \sigma = 0 \text{ at } x = f(t) \text{ and } x = r(t), \\ w = b_0 \text{ at } x = f(t). \end{cases}$$

Here $x = r(t)$ and $x = f(t)$ are the rear and the front boundaries of a moving cell, respectively, with $f(t) > r(t)$. Without loss of generality we let $r(0) = 0$ and $f(0) = \ell_0$ (see Figure 1). Both boundaries, whose locations are yet to be determined, move with speeds depending on the solution to the equations, as described by

$$(1.3) \quad \begin{cases} \frac{df}{dt} = \eta \sigma_x \Big|_{x=f(t)} + \frac{1}{f(t) - r(t)}, \\ \frac{dr}{dt} = \eta \sigma_x \Big|_{x=r(t)} + V_d. \end{cases}$$

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FIGURE 1. A one-dimensional moving cell.

Let $0 < \beta < 1$. We assume the following initial conditions:

$$(1.4) \quad \begin{aligned} w_0(x) = w(x, 0) &\in C^1[0, \ell_0], \quad \sigma_0(x) = \sigma(x, 0) \in C^{2+\beta}[0, \ell_0], \\ &\text{with } w_0(x) > \max\{\sigma_0(x), 0\} \text{ for } x \in [0, \ell_0]. \end{aligned}$$

The moving boundary problem, which is being studied in this paper, then consists of (1.1), (1.2), (1.3) and (1.4). The above condition $w_0 > \sigma_0$ guarantees that (1.1b) is parabolic near $t = 0$, while (1.1a) is hyperbolic. We note that system (1.1) is the same as equations (1.9) in [2] if we let $\eta = 1/\xi$ and $w = b + \sigma$ in [2]. (w as defined here differs from that in [2]. In fact w in this paper corresponds to $w/c = (b + p)/c$ in [2], as can be seen in equation (1.7) of [2]. Since $1/c$ represents the distance between two adjacent cytoskeletal nodes and b and p are the length densities of bundled and free filaments, respectively, therefore w in this paper represents the total length of both types of filaments in between two adjacent cytoskeletal nodes, while σ is the stress due to both types of filaments.)

Provided that the first-order compatibility conditions between the initial and the boundary conditions are satisfied at $t = 0$ (see [2] or [1]), there exists a maximal time $T > 0$ such that the moving boundary problem has unique smooth solutions $w \in C^{1,1}$ and $\sigma \in C^{2+\beta, (2+\beta)/2}$ in the domain $Q_T \equiv \{(x, t) : 0 \leq t < T, r(t) \leq x \leq f(t)\}$. For details and how such a moving boundary problem models cell motility, see [8], Theorem 1.1 in [2] or Theorem 1.1 in [1]. Recent review literature on biochemistry, biomechanics and modeling on cell motility can be found, for example, in [9], [11], [7]. The goal in this paper is to establish the global existence of a solution for this moving boundary problem when $\sigma_0 \geq 0$.

Theorem 1.1. *Assume that the first-order compatibility conditions between the initial and the boundary conditions are satisfied at $t = 0$ and the initial conditions $\sigma_0 \geq 0$. Then the moving boundary problem (1.1), (1.2), (1.3) and (1.4) has a unique smooth global solution in $Q_T \equiv \{(x, t) \mid r(t) \leq x \leq f(t), t < T\}$ for any*

$T > 0$. Furthermore there exists a positive constant b_{min} , which may depend on T and the initial conditions σ_0 and w_0 , such that $w - \sigma \geq b_{min} > 0$ in Q_T .

We now explain the choice of the boundary condition (1.2b) for w . Let the characteristics associated with the hyperbolic equations (1.1a) through a point (x_0, t_0) be denoted by $x = \tilde{x}(x_0, t_0, t)$. Then $\frac{\partial \tilde{x}}{\partial t} = \eta \sigma_x(\tilde{x}, t)$. Since $\frac{df}{dt} - \eta \sigma_x \Big|_{x=f(t)} = \frac{1}{f(t) - r(t)} > 0$, the characteristics starting at $x = f(t)$ goes into the domain $[r(t), f(t)]$. Hence a boundary condition needs to be prescribed at the front. On the other hand,

$$\frac{dr}{dt} - \eta \sigma_x \Big|_{x=r(t)} = V_d > 0.$$

This implies that the characteristics at the rear goes outside $[r(t), f(t)]$. No boundary condition can be imposed at the rear end.

In case we have the special initial condition $w(x, 0) = b_0$ so that (1.1a) dictates that $w(x, t) = b_0$ for all (x, t) as long as the solution exists, system (1.1) will be reduced to a scalar second-order parabolic equation

$$\sigma_t = (b_0 - \sigma)(\eta \sigma_x)_x - \eta \sigma_x^2 + \gamma(b_0 - \sigma).$$

With the definition $\hat{b} \equiv b_0 - \sigma$, the above equation is transformed into

$$(1.5) \quad \hat{b}_t = (\eta \hat{b} \hat{b}_x)_x - \gamma \hat{b}$$

with boundary conditions $\hat{b} = b_0$ at both $x = f(t)$ and $x = r(t)$, and initial condition $\hat{b}(x, 0) = b_0 - \sigma_0(x)$ on $[0, \ell_0]$. Since $\hat{b}(x, 0) > 0$, parabolicity of (1.5) is ensured near $t = 0$. With the governing equation being scalar, one can construct a comparison function to show that $\hat{b}$ is bounded away from zero as time evolves. This gives parabolicity of (1.5) as long as a solution exists. This observation ultimately leads to the global existence of a solution (see [2]). Moreover it has been shown that a traveling domain solution to (1.5) exists (see [3]), and the linearized stability of such a solution has been studied in [4]. An analogous two-dimensional result on the traveling domain solution has also been established in [5], in which the shape of the domain, the speed of a traveling cell, and the protein concentration $\hat{b}$ are all unknowns to be determined.

In this paper we consider an initial condition w_0 that is not identically b_0 . Hence the moving boundary problem is the coupled hyperbolic-parabolic system (1.1). Let Γ_0 be the characteristics coming out from $(x_0, t_0) = (\ell_0, 0)$. On the right-hand side of Γ_0 in the (x, t) plane, we see that $w(x, t) = b_0$ due to (1.1a) and (1.2b). It is the left-hand side of Γ_0 that requires careful analysis.

Remark 1. Since (1.1a) implies that w is constant along any characteristics, at first sight it seems that a solution to the hyperbolic equation can be easily estimated so that the governing hyperbolic-parabolic system can be controlled. It turns out that one needs to control the first and the second spatial derivatives of w in order to maintain parabolicity of (1.1b). By differentiating (1.1a) with respect to x twice, one can see that the growth rate of w_{xx} along a characteristic depends on a bound on σ_{xxx} . In order to break such interdependence on successively higher derivatives of w and σ , one needs to further exploit the structure of (1.1). This will be done in section 2.

In section 2, parabolicity of (1.1b) will be established. In so doing a priori bounds on derivatives of w need to be studied. Once this crucial step is bridged, techniques that apply to a scalar parabolic equation can be employed, which lead to a modification of the procedures in [2]. In section 3, properties of the solution to an ordinary differential equation are studied. Such a solution is employed in section 4 to construct barrier functions in order to control the speeds $r'(t)$ and $f'(t)$ of the moving boundaries. High norm bounds on w and σ then follow from further regularity estimates, which eventually lead to a global existence of solutions.

2. PARABOLICITY

The assumption that $\sigma_0 \geq 0$ in Theorem 1.1 will not be used in this section and the next. We need it only in section 4 to construct a lower comparison function.

Assume that smooth solutions $w \in C^{1,1}$ and $\sigma \in C^{2+\beta, (2+\beta)/2}$ for the moving boundary problem exist in the domain Q_T . Without loss of generality we let $T < \infty$, as otherwise we have the global existence of smooth solutions already. We like to make sure that $w - \sigma > 0$ for all $(x, t) \in Q_T$ in order to ensure that equation (1.1b) remains parabolic.

Let $w - \sigma > 0$ for all $(x, t) \in Q_\tau$ for some $0 < \tau < T$ with $w - \sigma = 0$ at $(x, t) = (x_1, \tau)$ for some x_1 that lies in the range $[r(\tau), f(\tau)]$. Since $w - \sigma = b_0$ at $x = f(t)$, it is clear that $x_1 \neq f(t)$. Next we know that w remains constant along a characteristic. The boundary $x = r(t)$ will be hit by a characteristic that originates from either the initial time $t = 0$ or the front boundary starting at an earlier time than t . In the former case since $\sigma = 0$ at $x = r(t)$ and $w \geq \min_{x \in [0, \ell_0]} w_0 > 0$, we have $w - \sigma \geq \min_{x \in [0, \ell_0]} w_0 > 0$ at $x = r(t)$. Similarly in the latter case $w - \sigma = b_0$ on the rear boundary. With $b_0 = w_0(\ell_0, 0) \geq \min_{x \in [0, \ell_0]} w_0$, hence in both cases, $w - \sigma \geq \min_{x \in [0, \ell_0]} w_0 > 0$ at $x = r(t)$. Consequently $x_1 \in (r(t), f(t))$. It is important to note that although we do not know the boundary condition of $w - \sigma$ at $x = r(t)$, the above argument establishes that it is positive and uniformly bounded away from 0.

If we can show that there exists a $\delta_\tau > 0$, which can depend on τ and the given initial conditions, such that $w - \sigma \geq \delta_\tau > 0$ on Q_τ , then this contradicts the definition of Q_τ . This contradiction leads to parabolicity of (1.1b) in Q_T .

We start this contradiction argument by introducing the following.

Lemma 2.1. *Let the initial condition $\sigma_0 \geq -M_0$ for some constant $M_0 \geq 0$. Then $\sigma \geq -M_0$ on Q_τ .*

Proof. From the definition of Q_τ , smooth solutions w and σ satisfy the parabolicity condition $w - \sigma > 0$ for (1.1b). Define $\underline{\sigma} \equiv -M_0$. We claim that $\sigma \geq \underline{\sigma}$ in Q_τ . For if not, then $\sigma < \underline{\sigma}$ at some (x_2, τ_2) for some $\tau_2 < \tau$ and $r(\tau) < x_2 < f(\tau)$. Hence

$$(w - \sigma) \{ \eta(\sigma_x - \underline{\sigma}_x) \}_x - \eta \sigma_x (\sigma - \underline{\sigma})_x - \gamma(\sigma - \underline{\sigma}) - (\sigma - \underline{\sigma})_t = -\gamma(w - \underline{\sigma}) < 0.$$

With $\sigma - \underline{\sigma} \geq 0$ at the parabolic boundaries $x = r(t)$, $x = f(t)$ and $t = 0$, then $\sigma - \underline{\sigma}$ achieves a negative minimum in the interior of Q_{τ_2} . This contradicts the maximum principle. Thus $\sigma \geq \underline{\sigma}$ in Q_τ . $\square$

Remark 2. In case we invoke the assumption that $\sigma_0 \geq 0$, then $M_0 = 0$.

Let $x_0 \in [0, \ell_0]$ and the characteristics through $(x, t) = (x_0, 0)$ associated with the hyperbolic equation (1.1a) be represented by $x = \tilde{x}(x_0, t)$. Hence

$$(2.1) \quad \begin{cases} \frac{\partial \tilde{x}}{\partial t} = \eta \sigma_x(\tilde{x}, t), \\ \tilde{x}(x_0, 0) = x_0. \end{cases}$$

Let (x, t) lie on this characteristic so that $\tilde{x}(x_0, t) = x$. If we define $\tilde{w}(x_0, t) \equiv w(\tilde{x}(x_0, t), t)$, then (1.1a) becomes $\tilde{w}_t = 0$. Thus $w(x, t) = w(\tilde{x}(x_0, t), t) = \tilde{w}(x_0, t) = \tilde{w}(x_0, 0) = w_0(x_0)$. In other words w is constant along a characteristic. Define $b = w - \sigma$. Then $\sigma + b \leq \max_{0 \leq x_0 \leq \ell_0} w_0(x_0)$. By Lemma 2.1, we have

$$(2.2) \quad b \leq \max_{0 \leq x_0 \leq \ell_0} w_0(x_0) + M_0,$$

which gives a uniform upper bound for b . Next from (2.1) we have

$$(2.3) \quad \begin{cases} \frac{\partial}{\partial t} \left(\frac{\partial \tilde{x}}{\partial x_0} \right) = \eta \sigma_{xx}(\tilde{x}, t) \frac{\partial \tilde{x}}{\partial x_0}, \\ \frac{\partial \tilde{x}}{\partial x_0}(x_0, 0) = 1, \end{cases}$$

which implies

$$(2.4) \quad \frac{\partial}{\partial t} \left[\log \frac{\partial \tilde{x}}{\partial x_0} \right] = \eta \sigma_{xx}(\tilde{x}, t).$$

Using $b = w - \sigma$ in (1.1b), we obtain

$$(2.5) \quad b_t = -(\eta b \sigma_x)_x - \gamma b$$

after some simplification. We now define $\tilde{b}(x_0, t) = b(\tilde{x}(x_0, t), t)$ and differentiate with respect to t on both sides. After employing (2.5) to simplify the calculation, we have

$$(2.6) \quad \tilde{b}_t = -(\gamma + \eta \sigma_{xx}) \tilde{b}.$$

Eliminating $\eta \sigma_{xx}$ using (2.4) and (2.6), we see that

$$(2.7) \quad \frac{\partial}{\partial t} \left[\log \frac{\partial \tilde{x}}{\partial x_0} \right] = - \left(\frac{\partial}{\partial t} \log \tilde{b} + \gamma \right),$$

which can be integrated to yield

$$(2.8) \quad \frac{\partial \tilde{x}}{\partial x_0} = \frac{b(x_0, 0) e^{-\gamma t}}{\tilde{b}}.$$

Recall that $w(x, t) = w_0(x_0)$ for $x = \tilde{x}(x_0, t)$. We now look at its derivative:

$$(2.9) \quad w_x = \frac{\partial w_0}{\partial x_0} \frac{\tilde{b} e^{\gamma t}}{b(x_0, 0)}.$$

From the definition of Q_τ , it is immediate that $b \geq 0$. Hence (2.2) allows us to conclude that $\tilde{b}$ is bounded. Moreover $e^{\gamma t} \leq e^{\gamma \tau}$ in Q_τ ; therefore we have a uniform bound on $|w_x|$ as long as parabolicity holds and solutions remain smooth.

Taking the second derivative we obtain

$$(2.10) \quad w_{xx} = \frac{\frac{\partial^2 w_0}{\partial x_0^2}}{\left(\frac{\partial \tilde{x}}{\partial x_0}\right)^2} - w_x \frac{\frac{\partial^2 \tilde{x}}{\partial x_0^2}}{\left(\frac{\partial \tilde{x}}{\partial x_0}\right)^2} = \frac{w_{0x_0x_0} e^{2\gamma t} \tilde{b}^2}{(b(x_0, 0))^2} - \frac{w_{0x_0}}{\left(\frac{\partial \tilde{x}}{\partial x_0}\right)^3} \frac{\partial^2 \tilde{x}}{\partial x_0^2}.$$

From (2.8), a direct computation gives

$$\begin{aligned} \frac{\partial^2 \tilde{x}}{\partial x_0^2} &= e^{-\gamma t} \frac{\partial}{\partial x_0} \left\{ \frac{b(x_0, 0)}{\tilde{b}} \right\} \\ &= \frac{e^{-\gamma t}}{\tilde{b}^2} \{ \tilde{b} b_x(x_0, 0) - b(x_0, 0) b_x \frac{\partial \tilde{x}}{\partial x_0} \}. \end{aligned}$$

Therefore the last term on the right-hand side of (2.10) becomes

$$\begin{aligned} \frac{w_{0x_0}}{\left(\frac{\partial \tilde{x}}{\partial x_0}\right)^3} \frac{\partial^2 \tilde{x}}{\partial x_0^2} &= \frac{w_{0x_0}}{\left(\frac{\partial \tilde{x}}{\partial x_0}\right)^3} \frac{e^{-\gamma t}}{\tilde{b}^2} \{ \tilde{b} b_x(x_0, 0) - b(x_0, 0) b_x \frac{\partial \tilde{x}}{\partial x_0} \} \\ &= \frac{w_{0x_0} e^{-\gamma t}}{\tilde{b}^2} \frac{e^{3\gamma t} \tilde{b}^4}{(b(x_0, 0))^3} b_x(x_0, 0) - \frac{w_{0x_0} b(x_0, 0)}{\tilde{b}^2} b_x \frac{1}{\left(\frac{\partial \tilde{x}}{\partial x_0}\right)^2} e^{-\gamma t} \\ &= w_{0x_0} \frac{e^{2\gamma t} \tilde{b}^2}{(b(x_0, 0))^3} b_x(x_0, 0) - \frac{w_{0x_0} b(x_0, 0)}{\tilde{b}^2} b_x \frac{e^{2\gamma t} \tilde{b}^2}{(b(x_0, 0))^2} e^{-\gamma t} \\ &= w_{0x_0} \frac{e^{2\gamma t} \tilde{b}^2}{(b(x_0, 0))^3} b_x(x_0, 0) - w_{0x_0} b_x \frac{e^{\gamma t}}{(b(x_0, 0))}. \end{aligned}$$

The last equation shows that w_{xx} as given by (2.10) can in fact be expressed in the form of $f_1(x, t) + f_2(x, t)b_x$, where f_1 and f_2 are bounded continuous functions on the left-hand side of Γ_0 in $\overline{Q_\tau}$. We now return to (2.5) and, given that $\sigma = w - b$, derive

$$(2.11) \quad b_t = (\eta b b_x)_x - \gamma b - \eta b w_{xx} - \eta b_x w_x.$$

We have bounds on $|w_x|$ and can express w_{xx} in terms of b_x . Combining like terms we arrive at

$$(2.12) \quad b_t = (\eta b b_x)_x - \gamma b + f_3 b_x + f_4 b,$$

where f_3 and f_4 are bounded continuous functions of (x, t) on the left-hand side of Γ_0 in $\overline{Q_\tau}$. On the right-hand side of Γ_0 , equation (2.11) will be reduced to (1.5); i.e., we can set $f_3 = f_4 = 0$. Hence we can regard (2.12) as being valid in all of Q_τ .

Lemma 2.2. *Assume initial conditions $b(x, 0) - \sigma_0(x) > 0$ for $0 \leq x \leq \ell_0$. Then $\tau = T$, where T is the maximal time of existence of smooth solution to (1.1). Assume $T < \infty$. Then there exist positive constants $b_{\min}$ and $b_{\max}$, which can depend on T and initial conditions σ_0 and w_0 , such that $0 < b_{\min} \leq b \leq b_{\max} \equiv \max_{0 \leq x_0 \leq \ell_0} w_0(x_0) + M_0$.*

In particular uniform parabolicity in (1.1b) holds for $0 \leq t < T$.

Proof. The upper bound is given by (2.2). For the lower bound we first define $\underline{b} = \delta e^{-(\gamma + \|f_4\|_\infty)t}$. Then

$$\begin{aligned} & -\underline{b}_t + (\eta \underline{b} b_x)_x - \gamma \underline{b} + f_3 \underline{b}_x + f_4 \underline{b} \\ &= (\gamma + \|f_4\|_\infty) \underline{b} - \gamma \underline{b} + f_4 \underline{b} \\ &= (\|f_4\|_\infty + f_4) \underline{b} \\ &\geq 0 \end{aligned}$$

so that $\underline{b}$ is a lower solution.

Next we need to compare the boundary conditions. At $x = f(t)$ we have $b = b_0 > 0$. On the other hand at $x = r(t)$, we do not know the value of b there. However

$$(2.13) \quad b(r(t), t) = w(r(t), t) - \sigma(r(t), t) \geq \min_{0 \leq x_0 \leq \ell_0} w_0(x_0),$$

because $\sigma(r(t), t) = 0$. Take $\delta = \min\{b_0, \min_{0 \leq x_0 \leq \ell_0} w_0(x_0), \min_{0 \leq x_0 \leq \ell_0} b(x_0, 0)\}$. Hence $b \geq \underline{b}$ at the parabolic boundaries $t = 0$, $x = r(t)$ and $x = f(t)$. An application of the maximum principle yields $b \geq \underline{b}$ in $\overline{Q_\tau}$. Define $b_{min} \equiv \delta e^{-(\gamma + \|f_4\|_\infty)\tau}$. Then $b \geq b_{min}$ in $\overline{Q_\tau}$.

From the discussion before Lemma 2.1, we see that $\tau = T$ and uniform parabolicity in (1.1b) holds for $0 \leq t < T$, where T is the maximal time of existence of smooth solutions to (1.1). $\square$

Parabolicity of (1.1b) is equivalent to $b = w - \sigma > 0$ in Q_T . Once this is established in the above lemma, it further deduces that $b \geq b_{min} > 0$. This crucial estimate allows further analysis to be carried out in the following sections.

3. PRELIMINARY LEMMAS

Theorem 3.1. *Let $T < \infty$ be the maximal time of existence of a smooth solution and $\ell(t) \equiv f(t) - r(t)$. Then there exist positive constants ℓ_{min} and ℓ_{max} , which can depend on T and the initial conditions σ_0 and w_0 , such that $0 < \ell_{min} \leq \ell(t) \leq \ell_{max}$ in the domain Q_T .*

Proof. Define $z(t) = \int_{r(t)}^{f(t)} b(x, t) dx$. Then from (2.5),

$$\begin{aligned} z' &= \int_{r(t)}^{f(t)} b_t dx + b_0 f' - b(r(t), t) r' \\ &= b_0 \{f' - \eta \sigma_x \big|_{x=f(t)}\} - b(r(t), t) \{r' - \eta \sigma_x \big|_{x=r(t)}\} - \gamma \int_{r(t)}^{f(t)} b \\ (3.1) \quad &= \frac{b_0}{\ell} - b(r(t), t) V_d - \gamma z. \end{aligned}$$

Using Lemma 2.2 it is clear that $b_{min} \ell(t) \leq \int_{r(t)}^{f(t)} b \leq b_{max} \ell(t)$, which is equivalent to

$$\frac{b_{min}}{z} \leq \frac{1}{\ell} \leq \frac{b_{max}}{z}.$$

Using the above inequalities to eliminate $\frac{1}{\ell}$ from (3.1), we can derive

$$(3.2) \quad \frac{b_0 b_{min}}{z} - b_{max} V_d - \gamma z \leq z' \leq \frac{b_0 b_{max}}{z} - b_{min} V_d - \gamma z.$$

Take a positive constant $M_1 \geq z(0)$. By increasing M_1 if necessary, the last term on the right-hand side of inequality (3.2) is negative when z is substituted by M_1 , i.e. $z' < 0$ at $z = M_1$. Hence for any time t , we have $b_{\min}\ell(t) \leq z(t) \leq M_1$. This yields

$$(3.3) \quad \ell(t) \leq \frac{M_1}{b_{\min}} \equiv \ell_{\max} < \infty.$$

Similarly take another positive constant $m_1 \leq z(0)$. By decreasing m_1 if necessary, we have $z' > 0$ at $z = m_1$. Hence for any time t it is immediate that $m_1 \leq z(t) \leq b_{\max}\ell(t)$, which gives

$$(3.4) \quad \ell_{\min} \equiv \frac{m_1}{b_{\max}} \leq \ell(t).$$

The proof of this theorem is now complete. $\square$

The above theorem implies that the front boundary $x = f(t)$ and the rear boundary $x = r(t)$ always keep a minimum positive distance from each other, and they will never get too far apart.

We will need the following comparison lemmas:

Lemma 3.2. *Let the maximal time of existence of a smooth solution $T < \infty$ and $r, f, m \in C[0, T)$ with $r(t) < m(t) < f(t)$ for $0 \leq t < T$. Take any $T_1 \in (0, T)$ and define the domain $D_{T_1} = \{(x, t) \mid r(t) \leq x \leq m(t), 0 \leq t < T_1\}$. Assume that $\sigma_1, \sigma_2 \in C^{2,1}(\overline{D_{T_1}})$, $a, b \in C(\overline{D_{T_1}})$, and there exists a $\delta > 0$, which may depend on T , such that $a(x, t) \geq \delta$ in D_T . Suppose*

$$(3.5) \quad \begin{aligned} & a(x, t)\sigma_{1xx} - \eta\sigma_{1x}^2 + \gamma b(x, t) - \sigma_{1t} \geq a(x, t)\sigma_{2xx} - \eta\sigma_{2x}^2 + \gamma b(x, t) - \sigma_{2t} \text{ in } \overline{D_{T_1}}, \\ & \sigma_1(m(t), t) \leq \sigma_2(m(t), t), \sigma_1(r(t), t) \leq \sigma_2(r(t), t) \text{ for } 0 \leq t < T, \text{ and } \sigma_1(x, 0) \leq \sigma_2(x, 0) \text{ for } 0 \leq x \leq m(0). \end{aligned}$$

Then $\sigma_1 \leq \sigma_2$ in $\overline{D_{T_1}}$. Moreover $\sigma_1 < \sigma_2$ in the interior of D_{T_1} unless $\sigma_1 \equiv \sigma_2$ in the whole domain. A similar result can be shown to be true if D_{T_1} is replaced by the domain $\{(x, t) \mid m(t) < x < f(t), 0 < t < T_1\}$.

Proof. A direct consequence of the maximum principle. $\square$

Lemma 3.3. *Let the assumptions of Lemma 3.2 hold. Suppose that $\sigma_1 \not\equiv \sigma_2$, $\sigma_1(r(t), t) = \sigma_2(r(t), t) = 0$ for $0 \leq t < T$, and the left boundary $x = r(t)$ satisfies the interior disc condition in the (x, t) plane for $t \in [0, T)$. Then for any $t \in (0, T)$, $(\sigma_1)_x < (\sigma_2)_x$ at the boundary point $(r(t), t)$. Similarly, if the right boundary satisfies the interior disc condition, then $(\sigma_1)_x > (\sigma_2)_x$ at the boundary point $(f(t), t)$ for $t \in (0, T)$.*

Proof. This proof is identical to Lemma 2.5 in [1] and is a consequence of Hopf's lemma [10]. Note that for $t < T$, smooth solutions exist so that the functions $f(t)$ and $r(t)$ are C^2 . Then the interior disc conditions will be satisfied. $\square$

The next lemma proves the necessary properties for the solution to an ODE which will be used in the construction of barrier functions for comparison.

Lemma 3.4. *Consider the second-order differential equation*

$$(3.6) \quad h'' + C_1 h'(M - h') + C_2 + C_3 h' = 0$$

with given $C_1 > 0$, $C_2 > 0$, and $C_3 \geq 0$, $h(0) = 0$ and $h'(0) = M > 0$. Then the following hold:

- (i) $h(t)$ exists on $[0, \infty)$,
- (ii) $h(t)$ has a unique global maximum at t_c , is non-decreasing on $(0, t_c)$ and non-increasing on (t_c, ∞) ,
- (iii) $h''(t) < 0$ for all t ,
- (iv) $\lim_{M \rightarrow \infty} t_c = 0$,
- (v) $\lim_{M \rightarrow \infty} h(t_c) = \infty$.

Proof. Local existence and uniqueness of h is clear for an initial value problem in ODE and the solution can be continued as long as h and h' stay bounded in finite time. Solving (3.6) for h'' we get

$$(3.7) \quad h'' = C_1(h')^2 - (C_1M + C_3)h' - C_2.$$

Whenever $h'(t) = 0$, then $h'' \leq -C_2 < 0$ so that any critical point of h has to be a strict local maximum. Thus h has no local minimum and because $h'(0) > 0$, there is at most one local maximum.

We can see that $h''(0) < 0$ for $0 < h'(0) = M < \infty$. Substituting $v = h'$, (3.7) is then equivalent to

$$(3.8) \quad v' = C_1(v - \mu_-)(v - \mu_+),$$

where $\mu_{\pm}$, the roots of a quadratic equation, are given by

$$(3.9) \quad \mu_{\pm} = \frac{(C_1M + C_3) \pm \sqrt{(C_1M + C_3)^2 + 4C_1C_2}}{2C_1}.$$

It is noted that $\mu_+ > M > 0 > \mu_-$. A plot of the phase line of this first-order autonomous equation shows that when $v(0) = M$, the solution v exists for all time, and $v = h'$ decreases monotonically from M to 0 in finite time t_c and then approaches μ_- as $t \rightarrow \infty$. Hence the solution of h exists for all time and it has a unique local maximum point at $t = t_c$. Moreover $h'' = v' < 0$ for all t . Integrating (3.8) we get

$$(3.10) \quad \int \frac{dv}{C_1(v - \mu_-)(v - \mu_+)} = \int dt,$$

which yields

$$(3.11) \quad \frac{1}{C_1} \frac{1}{\mu_+ - \mu_-} \log \left| \frac{v - \mu_+}{v - \mu_-} \right| = t + C,$$

where C is an unknown constant due to indefinite integration. Since we know $\mu_- < v < \mu_+$, the above equation reduces to

$$(3.12) \quad h'(t) = v(t) = \frac{\mu_- ke^{\hat{c}t} + \mu_+}{1 + ke^{\hat{c}t}}$$

for some positive constant k (which contains the unknown constant C), while $\hat{c}$ is defined by

$$\hat{c} \equiv C_1(\mu_+ - \mu_-) = \sqrt{(C_1M + C_3)^2 + 4C_1C_2}.$$

With $v(0) = M$, it is immediate that

$$(3.13) \quad k = \frac{\mu_+ - M}{M - \mu_-}.$$

We can now prove assertion (iv) of the lemma. Since $v(t_c) = 0$ we can solve (3.12) to obtain (recall $\mu_- < 0$)

$$(3.14) \quad t_c = \frac{1}{\hat{c}} \log \left(\frac{\mu_+}{-k\mu_-} \right).$$

To evaluate $\lim_{M \rightarrow \infty} t_c$, we must study how $\hat{c}$, μ_+ , μ_- and k behave as $M \rightarrow \infty$. Easy calculations yield the asymptotic expansions

$$\begin{cases} \hat{c} &= C_1 M + C_3 + \frac{2C_2}{M} + O\left(\frac{1}{M^2}\right), \\ \mu_+ &= M + \frac{C_3}{C_1} + \frac{C_2}{C_1 M} + O\left(\frac{1}{M^2}\right), \\ \mu_- &= -\frac{C_2}{C_1 M} + O\left(\frac{1}{M^2}\right), \\ k &= \frac{C_3}{C_1 M} + \frac{C_2}{C_1 M^2} + O\left(\frac{1}{M^3}\right), \end{cases} \quad \text{as } M \rightarrow \infty.$$

Putting all the asymptotic information into (3.14), we get

$$\begin{aligned} t_c &= \frac{1}{C_1 M + O(1)} \log \left(\frac{M + O(1)}{-\left(\frac{C_3}{C_1 M} + O\left(\frac{1}{M^2}\right)\right) \left(-\frac{C_2}{C_1 M} + O\left(\frac{1}{M^2}\right)\right)} \right) \\ (3.15) \quad &= \frac{3 \log M}{C_1 M} + O\left(\frac{1}{M}\right), \end{aligned}$$

$$\begin{aligned} t_c &= \frac{1}{C_1 M + O(1)} \log \left(\frac{M + O\left(\frac{1}{M}\right)}{-\left(\frac{C_2}{C_1 M^2} + O\left(\frac{1}{M^3}\right)\right) \left(-\frac{C_2}{C_1 M} + O\left(\frac{1}{M^2}\right)\right)} \right) \\ (3.16) \quad &= \frac{4 \log M}{C_1 M} + O\left(\frac{1}{M}\right), \end{aligned}$$

so that $t_c \rightarrow 0$ as $M \rightarrow \infty$.

Having proved assertion (iv), we turn our attention to assertion (v). Now we will use the fact that $\frac{dv}{dt} = \frac{dv}{dh} \frac{dh}{dt} = g(v) \equiv C_1(v - \mu_-)(v - \mu_+)$. Thus $\frac{dv}{dh} v = g(v)$ so that $\int_M^0 \frac{v}{g(v)} dv = \int_0^{h(t_c)} dh$, which simplifies to

$$(3.17) \quad h(t_c) = \frac{1}{\hat{c}} \left[\mu_+ \log \left(\frac{\mu_+}{\mu_+ - M} \right) + \mu_- \log \left(\frac{M - \mu_-}{-\mu_-} \right) \right].$$

As $M \rightarrow \infty$, the first term on the right of the above equation dominates and it can be checked that

$$h(t_c) = \frac{2 - \operatorname{sgn}(C_3)}{C_1} \log M + O(1).$$

This establishes assertion (v) and completes the proof of this lemma. $\square$

4. ESTIMATES ON BOUNDARY SPEEDS

We now establish bounds on the speeds of the moving boundaries. These in turn lead to high norm a priori bounds on the solutions.

Lemma 4.1. *Let $T < \infty$ be the maximal time of existence. Then there exists a positive constant M_2 , which may depend on T and the initial conditions σ_0 and w_0 , such that $\sigma_x(r(t), t) \leq M_2$ for all $0 \leq t < T$.*

Proof. Recall that σ satisfies (1.1b), which can be recast as

$$(4.1) \quad \sigma_t = b(x, t) (\eta \sigma_x)_x - \eta \sigma_x^2 + \gamma b(x, t),$$

where $b = w - \sigma$. We will regard this exact solution b as given in Q_T and construct an upper solution to the scalar parabolic equation (4.1) in σ . This upper comparison function $\bar{\sigma}$ will satisfy $\bar{\sigma}(r(t), t) = 0$, which is the same as σ , at the moving boundary $x = r(t)$.

First because of Lemmas 2.1 and 2.2, we have $-M_0 \leq \sigma = w - b \leq \max_{0 \leq x \leq \ell_0} w_0(x) \equiv w_{max}$ in Q_T . Now let h be the solution to (3.6) in Lemma 3.4 with $C_1 = \frac{\eta}{b_{min}}$, $C_2 = \frac{\gamma b_{max}}{b_{min}}$, $C_3 = \frac{V_d}{b_{min}}$ and $M = h'(0) > 0$. To avoid confusion, we think of h as a function of y and label the t_c in Lemma 3.4 as y_c . We now choose M sufficiently large such that $y_c \leq \ell_{min}$, $h(x) \geq \sigma_0(x)$ for $x \in [0, y_c]$, $h'(0) > \sigma'_0(0)$, and $h(y_c) \geq w_{max}$. These are always possible because of the properties of h .

Let $\bar{\sigma}(x, t) = h(x - r(t))$ in the domain $Q_1 = \{(x, t) \mid r(t) \leq x \leq m(t), 0 \leq t < T\}$, where $m(t) = r(t) + y_c$. Denote $y = x - r(t)$. We shall show that $\bar{\sigma}$ is an upper comparison function for (4.1). It is clear that on the left boundary, $\bar{\sigma}(r(t), t) = h(0) = 0 = \sigma(r(t), t)$. On the right boundary, $\bar{\sigma}(m(t), t) = h(y_c) \geq w_{max} \geq \sigma(m(t), t)$. Also, $\bar{\sigma}(x, 0) = h(x) \geq \sigma_0(x)$ for $0 \leq x \leq y_c$. Therefore $\bar{\sigma} \geq \sigma$ on the parabolic boundary of Q_1 .

Next $\bar{\sigma}_x(r(t), t) = h'(0) > \sigma'_0(0) = \sigma_x(0, 0)$. We claim that $\bar{\sigma}_x(r(t), t) > \sigma_x(r(t), t)$ for all $t \in [0, T]$. If it is not, then by continuity, there exists a $\tau_1 < T$ such that $h'(0) = \bar{\sigma}_x(r(t), t) > \sigma_x(r(t), t)$ for $t < \tau_1$ and $\bar{\sigma}_x(r(\tau_1), \tau_1) = \sigma_x(r(\tau_1), \tau_1)$. With $h'(y) \geq 0$ and $h''(y) \leq 0$ when $y \in [0, y_c]$, we have for $t \in [0, \tau_1]$,

$$\begin{aligned} & b(x, t) \bar{\sigma}_{xx} - \eta \bar{\sigma}_x^2 + \gamma b(x, t) - \bar{\sigma}_t \\ & \leq b(x, t) h'' - \eta h'^2 + \gamma b(x, t) + h' \{V_d + \eta \sigma_x(r(t), t)\} \\ & \leq b_{min} h'' - \eta h'^2 + \gamma b_{max} + h' \{V_d + \eta h'(0)\} = 0. \end{aligned} \quad \square$$

Lemma 4.2. *Let $T < \infty$ be the maximal time of existence. Then there exists a positive constant M_3 , which may depend on T and the initial conditions σ_0 and w_0 , such that $\sigma_x(f(t), t) \geq -M_3$ for all $0 \leq t < T$.*

Proof. As in the proof of Lemma 4.1, we have $-M_0 \leq \sigma \leq w_{max}$. Let h be a solution to (3.6) in Lemma 3.4 with $C_1 = \frac{\eta}{b_{min}}$, $C_2 = \frac{\gamma b_{max}}{b_{min}}$, $C_3 = 0$ and $M = h'(0) > 0$. We again choose M sufficiently large such that $y_c \leq \ell_{min}$, $h(\ell_0 - x) \geq \sigma_0(x)$ for $x \in [\ell_0 - y_c, \ell_0]$, $h'(0) > -\sigma'_0(\ell_0)$, and $h(y_c) \geq w_{max}$.

Let $\bar{\sigma}(x, t) = h(f(t) - x)$ in the domain $Q_2 = \{(x, t) \mid m(t) \leq x \leq f(t), 0 \leq t < T\}$, where $m(t) = f(t) - y_c$. Denote $y = f(t) - x$. We shall show that $\bar{\sigma}$ is an upper comparison function for (4.1) in Q_2 . It is clear that on the right boundary,

$\bar{\sigma}(f(t), t) = h(0) = 0 = \sigma(f(t), t)$. On the left boundary, $\bar{\sigma}(m(t), t) = h(y_c) \geq w_{max} \geq \sigma(m(t), t)$. Also, $\bar{\sigma}(x, 0) = h(\ell_0 - x) \geq \sigma_0(x)$ for $\ell_0 - y_c \leq x \leq \ell_0$. Therefore $\bar{\sigma} \geq \sigma$ on the parabolic boundary of Q_2 .

Now $\bar{\sigma}_x(f(t), t) = -h'(0) < \sigma'_0(\ell_0) = \sigma_x(\ell_0, 0)$. We claim that $\bar{\sigma}_x(f(t), t) < \sigma_x(f(t), t)$ for all $t \in [0, T)$. If it is not, then by continuity, there exists a $\tau_1 < T$ such that $-h'(0) = \bar{\sigma}_x(f(t), t) < \sigma_x(f(t), t)$ for $t < \tau_1$ and $\bar{\sigma}_x(f(\tau_1), \tau_1) = \sigma_x(f(\tau_1), \tau_1)$. Then since $h' \geq 0$ and $h'' \leq 0$ on $[0, y_c]$, we have in Q_2 ,

$$\begin{aligned} & b(x, t)\bar{\sigma}_{xx} - \eta\bar{\sigma}_x^2 + \gamma b(x, t) - \bar{\sigma}_t \\ & \leq b(x, t)h'' - \eta h'^2 + \gamma b_{max} - h'\left\{\frac{1}{\ell} + \eta\sigma_x(f(t), t)\right\} \\ & \leq b_{min}h'' - \eta h'^2 + \gamma b_{max} - h'\left\{\frac{1}{\ell_{max}} - \eta h'(0)\right\} \\ & \leq b_{min}h'' - \eta h'^2 + \eta h'(0)h' + \gamma b_{max} = 0. \end{aligned} \quad \square$$

Up to now we have not invoked the assumption on the initial condition that $\sigma_0 \geq 0$. From Remark 2 immediately after Lemma 2.1, we know that $\sigma \geq 0$ in Q_T . Hence we have

Lemma 4.3. *For $0 \leq t < T$, we have $\sigma_x(r(t), t) \geq 0$ and $\sigma_x(f(t), t) \leq 0$.*

Proof. This follows from $\sigma \geq 0$ and the boundary conditions $\sigma = 0$ at both $x = r(t)$ and $x = f(t)$. $\square$

From (1.3), Lemmas 4.1, 4.2 and 4.3, we can infer that $|r'(t)| \leq M_2\eta + V_d$ and $|f'(t)| \leq M_3\eta + \frac{1}{\ell_{min}}$ for all $0 \leq t < T$. In other words, there exists a constant $M_4 > 0$, which may depend on T , such that $\max\{|f'(t)|, |r'(t)|, |\ell'(t)|\} \leq M_4$ for all $0 \leq t < T$, where $\ell(t) = f(t) - r(t)$. Next from (2.9) and Lemma 2.2, $\|w_x\|_{L^\infty}$ is bounded. Since $|\sigma_x|$ is bounded at both moving boundaries and $b = w - \sigma$, by increasing M_4 if necessary, we can assume that $\max\{|b_x(r(t), t)|, |b_x(f(t), t)|\} \leq M_4$ for $0 \leq t < T$. Such controls on the boundary speeds and the domain size will allow us to perform further bootstrap to obtain high norm bounds on the solutions so that they remain smooth up to and including $t = T < \infty$. It will then enable the solution to extend beyond $t = T$. Such a contradiction to the definition of T leads to the global existence of a smooth solution.

Let $x = r(t) + \bar{x}\ell(t)$, $\hat{b}(\bar{x}, t) = b(r(t) + \bar{x}\ell(t), t)$, $\hat{f}_3(\bar{x}, t) = f_3(r(t) + \bar{x}\ell(t), t)$, and $\hat{f}_4(\bar{x}, t) = f_4(r(t) + \bar{x}\ell(t), t)$. Then (2.12) becomes

$$(4.2) \quad \hat{b}_t = \frac{1}{\ell^2} \left(\eta \hat{b} \hat{b}_{\bar{x}} \right)_{\bar{x}} + \left(\frac{\hat{f}_3 + r' + \ell' \bar{x}}{\ell} \right) \hat{b}_{\bar{x}} + (\hat{f}_4 - \gamma) \hat{b},$$

and the transformed $\bar{x}$ -domain $[0, 1]$ is now fixed with $\hat{b}(1, t) = b_0$. We will treat (4.2) as a linear uniformly parabolic equation in $\hat{b}$ in divergence form with the coefficients of $\hat{b}_{\bar{x}}$ and $\hat{b}$ being L^∞ functions. The diffusion coefficient $\hat{b}$ is known to satisfy $b_{min} \leq \hat{b} \leq b_{max}$.

We also need to control the boundary condition of $\hat{b}$ at $\bar{x} = 0$. Recall that $\|w_x\|_{L^\infty}$ is bounded. Thus

$$(4.3) \quad \frac{d}{dt} w(r(t), t) = (w_x r' + w_t) \Big|_{x=r(t)} = (w_x r' - \eta \sigma_x w_x) \Big|_{x=r(t)} = V_d w_x(r(t), t)$$

is bounded. With $b = w - \sigma$ and $\sigma(r(t), t) = 0$, we have

$$\frac{\partial \hat{b}}{\partial t} \Big|_{\bar{x}=0} = \frac{d}{dt} b(r(t), t) = \frac{d}{dt} w(r(t), t)$$

being bounded for $0 \leq t < T$. Hence the boundary condition $\hat{b}(0, t)$ is Lipschitz continuous in t as well. We can now apply a global Nash-DeGiori-Moser type estimate (theorem 10.1, p. 204, [3]) to (4.2) to conclude that there exist a $0 < \alpha < 1$ and a constant $M_5 > 0$ such that the Hölder norm $\|\hat{b}\|_{C^{\alpha, \alpha/2}([0, 1] \times [0, T])} \leq M_5$.

We next define $B = \hat{b}^2$. Then (4.2) becomes

$$B_t = \frac{\eta \hat{b}}{\ell^2} B_{\bar{x}\bar{x}} + \left(\frac{\hat{f}_3 + r' + \ell' \bar{x}}{\ell} \right) B_{\bar{x}} + 2(\hat{f}_4 - \gamma) B,$$

which has Lipschitz continuous boundary conditions in t at both $\bar{x} = 0$ and $\bar{x} = 1$.

In addition the coefficient $\frac{\eta \hat{b}}{\ell^2}$ is Hölder continuous in $(\bar{x}, t)$. Using theorem 9.1, p. 341, [3], we obtain $B \in W_q^{2,1}([0, 1] \times [0, T])$ for any $q > 1$. This implies that B , and hence $\hat{b}$, are in $C^{1+\alpha_1, (1+\alpha_1)/2}([0, 1] \times [0, T])$ for any $\alpha_1 < 1$. Since $b_x(x, t) = \frac{1}{\ell(t)} \hat{b}_x(\frac{x-r(t)}{\ell(t)}, t)$, it is clear that $b_x \in C^{\alpha_1, \alpha_1/2}(\overline{Q_T})$. From (2.9) we can infer that $w_x = g(x_0) e^{\gamma t} b$ for some smooth function g . Since $x = \tilde{x}(x_0, t)$ and $\partial \tilde{x} / \partial x_0 \neq 0$, an application of the implicit function theorem leads to x_0 being a C^1 function of (x, t) . Hence both w_x and $\sigma_x = w_x - b_x$ are in $C^{\alpha_1, \alpha_1/2}(\overline{Q_T})$. This allows us to treat σ_x^2 in (1.1b) as a Hölder continuous source term. Moreover the other coefficient $b = w - \sigma$ is also Hölder continuous.

Employing (1.3), both $r(t)$ and $f(t)$ will have $C^{1+\alpha_1/2}$ norm bounds. The boundary is now smooth enough to allow us to employ a Schauder type estimate to obtain a $C^{2+\alpha_1, (2+\alpha_1)/2}$ norm bound on σ . Further bootstrap gives a $C^{2+\beta, (2+\beta)/2}(\overline{Q_T})$ norm bound, where β is the specified smoothness of the initial conditions in (1.4). The proof of Theorem 1.1 is now complete.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06269-3009

E-mail address: `choi@math.uconn.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAVEN, 300 BOSTON POST ROAD, WEST HAVEN, CONNECTICUT 06516

E-mail address: `cmiller@newhaven.edu`