MULTI-SKEWED BROWNIAN MOTION AND DIFFUSION IN LAYERED MEDIA

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ABSTRACT. Multi-skewed Brownian motion $B^\alpha = \{B_t^\alpha : t \geq 0\}$ with skewness sequence $\alpha = \{\alpha_k : k \in \mathbb{Z}\}$ and interface set $S = \{x_k : k \in \mathbb{Z}\}$ is the solution to $X_t = X_0 + B_t + \int_{\mathbb{R}} L^X(t, x) d\mu(x)$ with $\mu = \sum_{k \in \mathbb{Z}} (2\alpha_k - 1) \delta_{x_k}$. We assume that $\alpha_k \in (0, 1) \setminus \{\frac{1}{2}\}$ and that S has no accumulation points. The process B^α generalizes skew Brownian motion to the case of an infinite set of interfaces. Namely, the paths of B^α behave like Brownian motion in $\mathbb{R} \setminus S$, and on $B_0^\alpha = x_k$, the probability of reaching $x_k + \delta$ before $x_k - \delta$ is α_k , for any δ small enough, and $k \in \mathbb{Z}$. In this paper, a thorough analysis of the structure of B^α is undertaken, including the characterization of its infinitesimal generator and conditions for recurrence and positive recurrence. The associated Dirichlet form is used to relate B^α to a diffusion process with piecewise constant diffusion coefficient. As an application, we compute the asymptotic behavior of a diffusion process corresponding to a parabolic partial differential equation in a two-dimensional periodic layered geometry.

1. INTRODUCTION

In [16] J.-F. Le Gall introduced a general class of stochastic processes by proving the existence and pathwise uniqueness of strong solutions to the stochastic differential equation

$$(1.1) \quad X_t = X_0 + B_t + \int_{\mathbb{R}} L^X(t, x) d\mu(x),$$

where L^X is the symmetric local time of the unknown process X , and μ is a bounded measure such that $|\mu(\{x\})| < 1$ for all $x \in \mathbb{R}$. Motivated by applications to diffusion problems in multi-layered media, this paper provides a thorough analysis of the structure of the solution to equation (1.1) in the particular case that μ is singular with respect to Lebesgue measure and is concentrated on a set $S = \{x_k : k \in \mathbb{Z}\}$ with no accumulation points.

The diffusion process B^α that solves (1.1) in the case $\mu = (2\alpha - 1)\delta_0$, $\alpha \in (0, 1)$, is known as *skew Brownian motion* and was first introduced by [13, p. 222]. The original construction consisted of independently assigning to the positive and negative excursions of reflected Brownian motion the probabilities α and $1 - \alpha$,

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respectively. The resulting process B^α behaves like standard Brownian motion when away from the “interface” at zero, but its sample paths are skewed, namely $\mathbb{P}(B_t^\alpha > 0 | B_0^\alpha = 0) = \alpha$ for all $t \geq 0$.

While skew Brownian motion has many interesting and sometimes unexpected properties, it is of particular relevance to recent physical experiments involving diffusive transport in heterogeneous media in the presence of a single membrane; e.g., see [6, 12, 21, 15, 24, 30, 5, 18, 25, 2]. The mathematical theory underlying applications to a single interface rests largely on the foundations for skew Brownian motion as developed by [13, 11, 31, 22, 18, 1].

The goal of this paper is twofold: (i) determine basic properties of multi-skewed Brownian motion as a regular diffusion according to the skewness sequence α and interface set S , and (ii) identify and exploit the role of B^α in advection-diffusion problems in layered media in one and two dimensions. The two goals are not unrelated in that many of the models that occur in the geophysical and biological sciences involve the long-time behavior of solutions to deterministic conservation laws (Fokker-Planck equations) for a concentration of dispersing substance. So it is useful to consider both the analytic as well as stochastic features of the process. The main results and goals will be summarized in this section, but to this end it will be useful to first introduce some special notation.

1.1. Notation. A diffusion process with sample space $\mathbb{R}^d$ is a four-tuple $X = (\Omega, \mathcal{F}, \{\mathbb{P}_x^X\}_{x \in \mathbb{R}^d}, \{X_t : t \geq 0\})$ where Ω is the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^d$, $\mathcal{F}$ is the σ -algebra of subsets of Ω generated by finite-dimensional cylinders, and for each $x \in \mathbb{R}^d$, $\mathbb{P}_x^X$ is a probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}_x^X(\{\omega : \omega(0) = x\}) = 1$.

For $A \in \mathcal{B}$ (the sigma algebra of Borel sets of $\mathbb{R}^d$) $\mathbb{1}_A$ denotes the indicator function of A . Also if X is a diffusion process, the following random processes are often considered: the time to hit A , $H_A^X := \inf\{t \geq 0 : X_t \in A\}$; the time to exit from A , $E_A^X := \inf\{t \geq 0 : X_t \notin A\}$; the occupation time of A up to time $t \geq 0$, $O_A^X(t) := \int_0^t \mathbb{1}_A(X_s) ds$; and the symmetric local time $L^X = \{L^X(t, x) : t \geq 0, x \in \mathbb{R}\}$, with the following convention for the occupation times formula: $2 \int_A L^X(t, x) dx = O_A^X(t)$, $t \geq 0$. Whenever any of the aforementioned random processes are evaluated on multi-skewed Brownian motion B^α , no superscript will be used; in the case of skew Brownian motion B^α (i.e., with a single interface at zero), the superscript α will be displayed.

1.2. Multi-skewed Brownian motion and its properties. As in the single interface case (see the review in [17]), several approaches can be used to construct and analyze multi-skewed Brownian motion. Here, B^α is characterized as a special case of the solution to equation (1.1). Since the excursion decomposition fails in the case of multiple interfaces, pathwise properties of B^α are explored via the representation of regular diffusions as scaled versions of Brownian motion under a random time change.

Definition 1.1. Let $S = \{x_k : k \in \mathbb{Z}\}$ and $\alpha = \{\alpha_k : k \in \mathbb{Z}\}$ be such that S has no accumulation points and $\alpha_k \in (0, 1) \setminus \{\frac{1}{2}\}$, for all $k \in \mathbb{Z}$. The diffusion process B^α that solves equation (1.1) with $\mu = \sum_{k \in \mathbb{Z}} (2\alpha_k - 1)\delta_{x_k}$ will be called *multi-skewed Brownian motion* with skewness sequence α and interface set S .

Elements of the set S will be called *interfaces*, and the intervals between interfaces will be denoted by $I_k := (x_k, x_{k+1})$, $k \in \mathbb{Z}$; α_k is referred to as the *skewness*

at the interface x_k . As stated in the next theorem, the features that give B^α its “multi-skewed” character are: (i) away from the interfaces, the paths of B^α behave as those of Brownian motion; and (ii) the process is skewed by α_k at interface x_k .

Theorem 1.2. *Let $B = \{B_t : t \geq 0\}$ be standard Brownian motion with law $\mathbb{P}_x^B$, $x \in \mathbb{R}$. Multi-skewed Brownian motion is a regular diffusion $B^\alpha = (\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}}, \{B_t^\alpha : t \geq 0\})$ satisfying*

$$(1.2) \quad \mathbb{P}_x(B_{\cdot \wedge E_{I_k}}^\alpha) = \mathbb{P}_x^B(B_{\cdot \wedge E_{I_k}}^B), \quad x \in I_k, k \in \mathbb{Z},$$

$$(1.3) \quad \mathbb{P}_{x_k}(E_{(x_k - \delta, x_k + \delta)}) = H_{x_k + \delta} = \alpha_k, \quad \delta < \min\{|I_{k-1}|, |I_k|\}.$$

It follows from Theorem 1.2 that multi-skewed Brownian motion has speed m and scale measure s that are absolutely continuous with respect to Lebesgue measure and have piecewise constant densities. A closer look at the asymptotic behavior of m and s reveals the role that the skewness α and the separation between interfaces play in the recurrence properties of B^α . One thus has the following:

Theorem 1.3. *Let $\beta_k := \frac{|I_{k-1}|\alpha_k}{|I_k|(1-\alpha_k) + |I_{k-1}|\alpha_k}$, $k \in \mathbb{Z}$. B^α is recurrent if*

$$(1.4) \quad \liminf_{k \rightarrow \infty} \beta_k < \frac{1}{2} \quad \text{and} \quad \limsup_{k \rightarrow -\infty} \beta_k > \frac{1}{2}.$$

If, in addition,

$$(1.5) \quad \frac{\alpha_k}{1 - \alpha_k} < \frac{|I_{k-1}|}{|I_k|} \quad \text{and} \quad \frac{1 - \alpha_{-k}}{\alpha_{-k}} < \frac{|I_{-k}|}{|I_{-k-1}|}$$

for all but finitely many $k \in \mathbb{N}$, then B^α is positive recurrent.

1.3. Applications to advection-diffusion in layered media. The object of Section 3 is the following initial value problem in $L^2(\mathbb{R}^2)$:

$$(1.6) \quad \partial_t u = \nabla \cdot (\tfrac{1}{2} \mathbf{D} \nabla u) + \nabla \cdot (\mathbf{U} u), \quad u(0) = u_0,$$

where

$$(1.7) \quad \mathbf{U}(\mathbf{x}) = (0, U(x)), \quad \mathbf{D}(\mathbf{x}) = \begin{bmatrix} D_x(x) & 0 \\ 0 & D_y(x) \end{bmatrix},$$

and D_x is constant on “layers” parallel to $\mathbf{U}$; that is,

$$(1.8) \quad D_x(x) = \sum_{k \in \mathbb{Z}} D_k \mathbb{1}_{J_k}(x), \quad J_k = (z_k, z_{k+1}), \quad k \in \mathbb{Z}.$$

The tensor $\mathbf{D}$ is moreover assumed elliptic on compact sets of $\mathbb{R}^2$ and bounded everywhere, with $U/\sqrt{D_y}$ bounded. A problem related to the case where $\mathbf{U}$ is perpendicular to the surface of discontinuity of D_x is treated in [1].

The stochastic process $\mathbf{X}$ “associated” with (1.6), that is, the diffusion with infinitesimal generator $\mathbf{A}^\mathbf{X} u = \nabla \cdot (\tfrac{1}{2} \mathbf{D} \nabla u) + \nabla \cdot (\mathbf{U} u)$ is characterized in the following theorem.

Theorem 1.4. *Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be the piecewise continuous linear function with*

$$(1.9) \quad \psi(0) = 0, \quad \psi'(x) = \sqrt{D_k}, \quad x \in I_k, k \in \mathbb{Z},$$

and denote its inverse by $\varphi = \psi^{-1}$. Assume $S := \{x_k = \varphi(z_k) : k \in \mathbb{Z}\}$ has no accumulation points, and consider the multi-skewed Brownian motion $B^\alpha = \{B_t^\alpha : t \geq 0\}$ with interface set S and skewness sequence $\alpha = \{\alpha_k : k \in \mathbb{Z}\}$ given by

$$(1.10) \quad \alpha_k = \frac{\sqrt{D_k}}{\sqrt{D_k} + \sqrt{D_{k-1}}}, \quad k \in \mathbb{Z}.$$

Let $W = \{W_t : t \geq 0\}$ be a standard Brownian motion independent of B^α . Then $\mathbf{X} = (X, Y) = \{(X_t, Y_t) : t \geq 0\}$, where

$$(1.11) \quad X_t = \psi(B_t^\alpha), \quad t \geq 0$$

and Y is the Itô process given by

$$(1.12) \quad dY_t = U(X_t) dt + \sqrt{D_y(X_t)} dW_t.$$

Section 4 treats a particular example of a periodic layered medium and exploits the structure of B^α to determine the asymptotic behavior of $\mathbf{X}$. Specifically, consider a periodic layered medium composed of two homogeneous materials occupying alternating layers. That is, let $J_k := (z_k, z_{k+1})$ with $z_0 = 0$ and $l_1 := |J_{2k+1}|$, $l_2 := |J_{2k}|$ for all $k \in \mathbb{Z}$. Define $M_1 := \bigcup_{k \in \mathbb{Z}} J_{2k+1}$, $M_2 := \bigcup_{k \in \mathbb{Z}} J_{2k}$ and assume that $\mathbf{D}$ and $\mathbf{U}$ take only two values, $D_2 > D_1 > 0$ and $U_2 > U_1 > 0$ with

$$(1.13) \quad D_x(\mathbf{x}) = D_y(\mathbf{x}) = D_1 \mathbb{1}_{M_1}(x) + D_2 \mathbb{1}_{M_2}(x), \quad U(\mathbf{x}) = U_1 \mathbb{1}_{M_1}(x) + U_2 \mathbb{1}_{M_2}(x),$$

for $\mathbf{x} = (x, y) \in \mathbb{R}^2$ (see Figure 1). For this particular periodic arrangement, the asymptotic behavior of $\mathbf{X}$ is given by

Theorem 1.5. *Let $\mathbf{X} = (X, Y)$ be as in (1.11) and (1.12). Then (i)*

$$(1.14) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_0 X_t = 0, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \mathbb{E}_0 X_t^2 = \frac{D_2 D_1 (l_2 + l_1)}{l_2 D_1 + l_1 D_2}.$$

(ii) Y satisfies the following central limit theorem: As $n \rightarrow \infty$, the distribution of $\{1/\sqrt{n} (Y_{[nt]} - [nt]U_{\text{eff}}) : t \geq 0\}$ converges weakly to the Wiener measure with zero mean, and “effective” diffusion coefficient $D_{\text{eff}} > 0$, where

$$(1.15) \quad U_{\text{eff}} = \frac{U_2 l_2 + U_1 l_1}{l_2 + l_1}, \quad D_{\text{eff}} = \frac{D_2 l_2 + D_1 l_1}{l_2 + l_1} + \frac{l_1^2 l_2^2 (U_2 - U_1)^2}{3(l_1 + l_2)^3} \left(\frac{l_1}{D_1} + \frac{l_2}{D_2} \right).$$

2. MULTI-SKEWED BROWNIAN MOTION AND PROPERTIES

Consider a countable set of real numbers $S := \{x_k : k \in \mathbb{Z}\}$, with $x_0 = 0$, $x_k < x_{k+1}$ for all k , and no accumulation points. Define the skewness sequence

$$(2.1) \quad \alpha = \{\alpha_k : k \in \mathbb{Z}\}, \quad \alpha_k \in (0, 1) \setminus \{1/2\}.$$

Theorem 1.2 is shown by constructing a regular diffusion $B^\alpha = (\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}}, \{B_t^\alpha : t \geq 0\})$ satisfying properties (1.2) and (1.3), and then checking that the obtained process is the solution to the stochastic differential equation (1.1).

Proof of Theorem 1.2. Denote by s the (normalized) scale measure and by m the speed measure of the desired process B^α . Assume for the moment that both m and s are absolutely continuous with respect to Lebesgue measure, $s(A) = \int_A s'(dx)$, $m(A) = \int_A m'(dx)$, $A \in \mathcal{B}$, and that s' and m' are piecewise constant functions,

$$(2.2) \quad m'(x) = m_k, \quad s'(x) = s_k, \quad x \in [x_k, x_{k+1}), \quad k \in \mathbb{Z}.$$

FIGURE 1. Advection-diffusion in a periodic layered media and the process $X = \psi(B^\alpha)$.

Equations (1.2) and (1.3), jointly with the definition of scale and speed, yield

$$(2.3) \quad m_k = \frac{2}{s_k}, \quad s_k = \frac{1 - \alpha_k}{\alpha_k} s_{k-1}, \quad k \in \mathbb{Z},$$

which by setting $s_0 = 1$ for normalization gives

$$(2.4) \quad s_k = \prod_{j=1}^k \frac{1 - \alpha_j}{\alpha_j}, \quad s_{-k} = \prod_{j=-k+1}^0 \frac{\alpha_j}{1 - \alpha_j}, \quad k \geq 1.$$

Note that since S has no accumulation points, m and s each define a Radon measure, and therefore the process B^α determined by these measures can be constructed as a random time change of Brownian motion (for details see [8, p. 371]). More specifically let $B = \{B_t : t \geq 0\}$ be Brownian motion on $\mathbb{R}$ with local times $\{L^B(t, x) : t \geq 0, x \in \mathbb{R}\}$. Let $\tilde{m}$ be the measure on $(\mathbb{R}, \mathcal{B})$ with piecewise constant density $\tilde{m}'(x) = \frac{2}{s_k^2}$, $x \in s(I_k)$, $k \in \mathbb{Z}$. Define $\phi(r) := \int_{\mathbb{R}} L^B(r, x) d\tilde{m}(x)$, $r \geq 0$, and $\tau(t) = \inf\{r \geq 0 : \phi(r) = t\}$. Let $\Phi(B) = \{s^{-1}(B_{\tau(t)}) : t \geq 0\}$ and define $\mathbb{P}_x = \mathbb{P}_{s(x)}^B \circ \Phi^{-1}$, $x \in \mathbb{R}$. The process $B^\alpha := (\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}}, \{B_t^\alpha : t \geq 0\})$ given by $B_t^\alpha = \Phi(B)(t)$, $t \geq 0$, is a canonical model for a regular diffusion satisfying properties (1.2) and (1.3). (Here, the symbol s is also used to denote the scale function, that is, $s(x) = \int_0^x s'(y) dy$, $x \in \mathbb{R}$.) To check that B^α is a multi-skewed Brownian motion in the sense of Definition (1.1), note that Tanaka's formula on $(B_t^\alpha - x_k)^+$ and $(B_t^\alpha - x_k)^-$ respectively (see [31]) gives for $k \in \mathbb{Z}$ and $t \geq 0$,

$$(2.5) \quad L(t, x_k^+) = \frac{1}{s_k} L^B(\tau(t), s(x_k)), \quad L(t, x_k^-) = \frac{1}{s_{k-1}} L^B(\tau(t), s(x_k)).$$

On the other hand, (2.5) and Tanaka's formula on $s^{-1}(B_{\tau(t)})$ give

$$B_t^\alpha = \int_0^{\tau(t)} \frac{1}{s'(B_s)} dB_s + \frac{1}{2} \sum_{k \in \mathbb{Z}} (L(t, x_k^+) - L(t, x_k^-)).$$

By definition, $L(t, x_k) = \frac{1}{2}[L(t, x_k^+) + L(t, x_k^-)]$. Combining (2.5) and (2.3) yields $(2\alpha_k - 1)L(t, x_k) = \frac{1}{2}(L(t, x_k^+) - L(t, x_k^-))$. Dubins' representation theorem gives that $\int_0^{\tau(t)} \frac{1}{s'(B_s)} dB_s$, $r \geq 0$, is in fact Brownian motion [26, p. 170]. $\square$

2.1. The semigroup of B^α . The semigroup of B^α , defined by $T_t f(x) = \mathbb{E}_x f(B_t^\alpha)$, $x \in \mathbb{R}$, $t \geq 0$, and $f \in C_b$ (the Banach space of bounded continuous functions on $\mathbb{R}$) is continuous with respect to uniform convergence on compact subsets of $\mathbb{R}$, and is strongly continuous on C_0 , the set of functions in C_b that vanish at infinity [14, Chapter 19]. The resolvent $G = \{G_\lambda : \lambda > 0\}$ of T is defined by $G_\lambda f(x) := \int_0^\infty e^{-\lambda s} T_s f(x) ds$, $x \in \mathbb{R}$, $f \in C_b$. The infinitesimal generator of T is $\mathcal{A} := \lambda - G_\lambda^{-1}$ on its domain $D_\mathcal{A} := \text{Ran}(G_\lambda)$. Furthermore, $\mathcal{A}$ can also be explicitly computed in terms of the measures m and s (see [9, Theorem 75]), which gives

$$(2.6) \quad \mathcal{A}f(x) = \frac{d}{dm} \frac{df}{ds} = \frac{1}{2} f''(x), \quad f \in D_\mathcal{A}, \quad x \in \mathbb{R} \setminus S,$$

$$(2.7) \quad D_\mathcal{A} = \{f \in C_b : f'' \text{ exists in } \mathbb{R} \setminus S, \alpha_k f'(x_k^+) = (1 - \alpha_k) f'(x_k^-), \quad k \in \mathbb{Z}\}.$$

2.2. Recurrence and invariance. A positive measure μ on $(\mathbb{R}, \mathcal{B})$ is invariant for B^α if $\int_{\mathbb{R}} T_t g d\mu = \int_{\mathbb{R}} g d\mu$ for all $t > 0$ and $\mathcal{B}$ -measurable functions g . The following lemma is well known in the case of strongly continuous semigroups on C_b . Here, T is continuous only with respect to uniform convergence on compact subsets of $\mathbb{R}$, but the properties of the resolvent G ensure that the result still holds.

Lemma 2.1. *A Radon measure is invariant for the semigroup T on C_b if and only if $\int_{\mathbb{R}} \mathcal{A}f d\mu = 0$ for all $f \in D_\mathcal{A}$. In particular, the speed measure m is invariant.*

Proof. Let μ denote an arbitrary Radon measure and let $g \in C_b$. For $\lambda > 0$, consider $f_\lambda := G_\lambda g \in D_\mathcal{A}$. Then

$$\int_{\mathbb{R}} \mathcal{A}f_\lambda(x) d\mu(x) = \int_0^\infty \lambda e^{-\lambda t} \int_{\mathbb{R}} (T_t g(x) - g(x)) d\mu(x) dt, \quad \lambda > 0.$$

The result follows by the uniqueness of the Laplace transform and the fact that $D_\mathcal{A} = \text{Ran}(G_\lambda)$. Writing $\mu = m$ and using (2.7), one also obtains

$$\int_{\mathbb{R}} \mathcal{A}f(x) dm(x) = \sum_{k \in \mathbb{Z}} \frac{1}{2} (m_{k-1} f'(x_k^-) - m_k f'(x_k^+)) = 0.$$

Therefore the speed measure m is invariant for T . $\square$

By definition, the regularity of the diffusion B^α ensures that $\mathbb{P}_x(H_y < \infty) > 0$ for all $x, y \in \mathbb{R}$. The process B^α is called (point) recurrent if $\mathbb{P}_x(H_y < \infty) = 1$ for all $x, y \in \mathbb{R}$ and is called transient otherwise. B^α would be called positive recurrent if, in addition to being recurrent, $\mathbb{P}_x(B_t^\alpha \in K)$ does not converge to zero as $t \rightarrow \infty$ for any $x \in \mathbb{R}$ and compact set K with nonempty interior [14, p. 465].

The following theorem, out of which Theorem 1.3 follows as an easy corollary, characterizes the conditions for recurrence, positive recurrence and existence of an invariant probability measure. The proof relies in part on classic results but is of interest for it exploits a birth-death Markov chain naturally associated with multi-skewed Brownian motion.

Theorem 2.2. *The process B^α is recurrent if and only if $s(\mathbb{R}^-) = s(\mathbb{R}^+) = \infty$. In the recurrent case, the following statements are equivalent: (i) B^α is positive recurrent, (ii) $m(\mathbb{R}) < \infty$, (iii) $m/m(\mathbb{R})$ is the unique invariant probability measure*

for T , (iv) as $t \rightarrow \infty$, the transition probabilities $\mathbb{P}_x(B_t^\alpha \in \cdot)$ converge weakly to $m/m(\mathbb{R})$ for all x , and (v) $\mathbb{E}_x(H_y) < \infty$ for all $x, y \in \mathbb{R}$.

Proof. Consider the discrete process given by the consecutive visits of B^α to distinct interfaces; namely, let $\eta_0 = \inf\{t \geq 0 : B_t^\alpha \in S\}$ and define recursively

$$(2.8) \quad \eta_{n+1} = \inf\{t > \eta_n : B_t^\alpha \in S \setminus \{Z_n\}\}, \quad Z_n = B_{\eta_n}^\alpha, \quad n \geq 0.$$

Let $K(n) \in \mathbb{Z}$ be such that $Z_n = x_{K(n)}$. Then

$$\eta_{n+1} = \eta_n + E_{(x_{K(n)-1}, x_{K(n)+1})}(B^\alpha \circ \theta_{\eta_n}),$$

where for $t \geq 0$, θ_t is the shift operator, $\theta_t(\omega) = \theta(t + \cdot)$, $\omega \in \Omega$. The strong Markov property of B^α gives that the process $Z = \{Z_n : n \geq 1\}$ is a Markov chain with values on S and transition probabilities and $n \geq 1$,

$$(2.9) \quad \mathbb{P}_x(Z_{n+1} = x_j | Z_n = x_k) = \begin{cases} \beta_k & j = k+1, \\ 1 - \beta_k & j = k-1, \end{cases} \quad x \in \mathbb{R}, n \geq 0,$$

where

$$(2.10) \quad \beta_k := \frac{|I_{k-1}|s_{k-1}}{|I_k|s_k + |I_{k-1}|s_{k-1}} = \frac{|I_{k-1}|\alpha_k}{|I_k|(1 - \alpha_k) + |I_{k-1}|\alpha_k}, \quad k \in \mathbb{Z}.$$

Also, B^α is recurrent if and only if Z is recurrent. The theory of discrete time birth-death process (see for example [7]) gives that Z is recurrent if and only if

$$(2.11) \quad \sum_{k=1}^{\infty} \prod_{j=1}^k \frac{1 - \beta_j}{\beta_j} = \sum_{k=1}^{\infty} \frac{s_k |I_k|}{|I_0|} = \infty, \quad \sum_{k=-\infty}^0 \prod_{j=k}^0 \frac{\beta_j}{1 - \beta_j} = \sum_{k=-\infty}^0 \frac{s_k |I_k|}{|I_0|} = \infty.$$

To prove the equivalence between statements (i) to (iii), apply Lemma 23.19 of [14] to the process $s(B^\alpha)$. Theorem IV.4.7 of [20] gives (ii) $\Leftrightarrow$ (iv). Finally, the equivalence between (i) and (v) follows from $\mathbb{E}_x H_y = \int_y^x m(z, \infty) dz$, $x, y \in \mathbb{R}$ [7, p. 422]. $\square$

To prove Theorem (1.3) stated in the Introduction, it now suffices to apply the ratio test to $\sum_{j=0}^{\pm\infty} m_k |I_k|$ and the series in (2.11).

2.3. The infinitesimal generator of B^α in L_m^2 . Problems in solute transport applications are often posed in terms of concentration profiles and/or breakthrough curves corresponding to weak solutions to conservation equations. Consequently, and as a preparation for the next section, we now compute some functional analytical properties of the semigroup $\mathbf{T}$ of B^α in $L_m^2 := L^2(\mathbb{R}, m)$, the space of square integrable functions with respect to the speed measure m .

Since m is invariant for $T = \{T_t : t \geq 0\}$, the semigroup of B^α in C_b , T can be extended to a semigroup $\mathbf{T}$ in L_m^2 by

$$(2.12) \quad \mathbf{T}_t u(x) = \mathbb{E}_x u(B_t^\alpha) = \int_{\mathbb{R}} u(y) \mathbb{P}_x(B_t^\alpha \in dy), \quad x \in \mathbb{R}, u \in L_m^2, t \geq 0.$$

In what follows, boldface will be used to denote operators corresponding to L_m^2 -extensions. As in [19, Section II.4], equation (2.12) defines a strongly continuous contraction semigroup $\mathbf{T} := \{\mathbf{T}_t : t \geq 0\}$ on L_m^2 . The infinitesimal generator $\mathbf{A}$ of $\mathbf{T}$ is a densely defined and closed operator in L_m^2 ; its domain $D_{\mathbf{A}}$ is given by

the set of functions $u \in L_m^2$ such that $\mathbf{A}u := \lim_{t \rightarrow 0} \frac{1}{t}(\mathbf{T}_t u - u)$ exists in L_m^2 . The resolvent of $\mathbf{T}$ is defined by the family of strongly continuous contractions $\mathbf{G}_\lambda := (\lambda - \mathbf{A})^{-1} : L_m^2 \rightarrow D_{\mathbf{A}}$, $\lambda > 0$. The restriction of any of the boldface operators to C_b coincides with their analogues in Section 2.1.

The infinitesimal operator in $\mathbf{A}$ is computed in Theorem 2.3. The following notation is used in the proof: For $i \geq 0$, ∂^i will denote the i -th generalized derivative, and $\mathcal{H}_m^i := \mathcal{H}^i(\mathbb{R}, m)$ the usual Sobolev space of order i ; for $k \in \mathbb{Z}$, $L_{m_k}^2 := L^2(I_k, m)$ and $\mathcal{H}_{m_k}^i := \mathcal{H}^i(I_k, m)$.

Theorem 2.3. *The infinitesimal generator of B^α in L_m^2 is given by*

$$(2.13) \quad \mathbf{A}u = \sum_{k \in \mathbb{Z}} \frac{1}{2} \partial^2(u \cdot \mathbb{1}_{I_k}), \quad u \in D_{\mathbf{A}},$$

$$(2.14) \quad D_{\mathbf{A}} = \{u \in \mathcal{H}_m^1 : \mathbf{A}u \in L_m^2 \text{ and } \alpha_k \partial u(x_k^+) = (1 - \alpha_k) \partial u(x_k^-), k \in \mathbb{Z}\}.$$

Proof. The interface conditions in (2.7) give

$$(2.15) \quad (-\mathcal{A}f, f)_{L_m^2} = \frac{1}{2}(f', f')_{L_m^2}, \quad f \in D_{\mathbf{A}}.$$

Let $u \in D_{\mathbf{A}}$. Since $\mathbf{A}$ is closed, there is a sequence $\{f_n : n \geq 0\}$ in $D_{\mathbf{A}}$ such that $f_n \xrightarrow{L_m^2} u$ and $\mathcal{A}f_n \xrightarrow{L_m^2} \mathbf{A}u$. By the continuity of $\mathcal{A}$, applying (2.15) to $f_n - f_{n+1} \in D_{\mathbf{A}}$ gives that $\{f'_n : n \geq 0\}$ is a Cauchy sequence in L_m^2 and in $L_{m_k}^2$ for any $k \in \mathbb{Z}$. Since each f_n is C^2 on $\bar{I}_k$, one then has that $u \in \mathcal{H}_{m_k}^2$, $f'_n \xrightarrow{L_{m_k}^2} \partial u$ and $\frac{1}{2}f''_n \xrightarrow{L_{m_k}^2} \frac{1}{2}\partial^2 u = \mathbf{A}u$ on I_k [27]. This establishes (2.13).

To obtain the jump condition on ∂u , fix k and choose a subsequence of $\{f_n : n \geq 0\}$ (denoted again by f_n) such that $f_n \xrightarrow{a.e.} u$ and $\mathcal{A}f_n \xrightarrow{a.e.} \mathbf{A}u$ on $\mathbb{R}$, and $f'_n \xrightarrow{a.e.} \partial u$ on I_k . Let $x \in I_k$ be such that $f'_n(x) \rightarrow \partial u(x)$. Extend ∂u continuously to $\bar{I}_k$, e.g. $\partial u(x_k^+) := \lim_{\epsilon \downarrow 0} (\mathfrak{U}(x) - \int_{x_k+\epsilon}^x \partial^2 u \, dy)$, and write

$$\partial u(x) - \partial u(x_k^+) = \int_{x_k}^x \partial^2 u(y) \, dy = \lim_{n \rightarrow \infty} \int_{x_k}^x f''_n(y) \, dy = \lim_{n \rightarrow \infty} [f'_n(x) - f'_n(x_k^+)].$$

Therefore, $f'_n(x_k^+) \rightarrow \partial u(x_k^+)$. A further subsequence $\{f_{n_j} : j \geq 0\}$ can similarly be chosen so that $f'_{n_j}(x_k^+) \rightarrow \partial u(x_k^+)$ and $f'_{n_j}(x_k^-) \rightarrow \partial u(x_k^-)$ as $j \rightarrow \infty$. This gives $m_k \partial u(x_k^+) = m_{k-1} \partial u(x_k^-)$. The function ∂u can now be defined on all of $\mathbb{R}$, and integration by parts yields

$$(2.16) \quad (\mathbf{A}u, v)_{L_m^2} = \sum_{k \in \mathbb{Z}} (\frac{1}{2} \partial^2 u, v)_{L_{m_k}^2} = -\frac{1}{2}(\partial u, \partial v)_{L_m^2}, \quad u, v \in D_{\mathbf{A}}.$$

Applying (2.16) to u and $f_n - u$ gives $u \in \mathcal{H}_m^1$.

If $\tilde{D}_{\mathbf{A}}$ denotes the set on the right-hand side of (2.14), it has been proven so far that $D_{\mathbf{A}} \subseteq \tilde{D}_{\mathbf{A}}$. For the other inclusion, extend the bilinear form defined by $\mathbf{A}$,

$$(2.17) \quad \mathcal{E}(u, v) := (-\mathbf{A}u, v)_{L_m^2} = \frac{1}{2}(\partial u, \partial v)_{L_m^2}, \quad u, v \in D_{\mathbf{A}},$$

to a coercive closed form with domain $D_{\mathcal{E}}$ equal to the completion of $D_{\mathbf{A}}$ with respect to $\|\cdot\|_{\mathcal{H}_m^1}$ and such that equation (2.17) holds for all $u \in D_{\mathbf{A}}$ and $v \in D_{\mathcal{E}}$ [19, p. 43]. Since $(\frac{1}{2} - \mathbf{A}) = \mathbf{G}_{1/2}^{-1}$ is onto L_m^2 , it follows that $D_{\mathcal{E}} = \mathcal{H}_m^1$. In terms of $\mathcal{E}$, $D_{\mathbf{A}}$ is characterized as those functions on $\mathcal{H}_m^1$ such that $v \mapsto \mathcal{E}(u, v)$ is

continuous with respect to $\|\cdot\|_{L_m^2}$. Let $u \in \tilde{D}_A$ and $v \in \mathcal{H}_m^1$. Then $Au \in L_m^2$ and

$$|\mathcal{E}(u, v)| \leq \sum_{k \in \mathbb{Z}} \|\partial^2 u\|_{L_{m_k}^2} \|v\|_{L_{m_k}^2} \leq \|Au\|_{L_m^2} \|v\|_{L_m^2}.$$

Therefore, $u \in D_A$. $\square$

3. ADVECTION-DIFFUSION IN LAYERED MEDIA

We turn our attention to the initial value problem (1.6) with D and U given by (1.7) and (1.8). On the analytic side, the existence of a diffusion process associated with parabolic operators with discontinuous coefficients has been established in very general settings and using various analytical techniques by [28, 23, 21, 10, 19, 4], and others.

Let $c(t, \mathbf{x})$ be the solution to Kolmogorov's forward equation (the adjoint of equation (1.6) in $L^2(\mathbb{R}^2)$),

$$(3.1) \quad \partial_t c = \nabla \cdot (\tfrac{1}{2} D \nabla c) - \nabla \cdot (c U).$$

Specifically, for all u in the class in $C_c^\infty(\mathbb{R}^2)$, $\partial_t(u, c(t, \cdot))_{L^2(\mathbb{R}^2)} = -\mathcal{E}(u, c(t, \cdot))$, where $\mathcal{E}$ is the bilinear form naturally associated with (1.6),

$$(3.2) \quad \mathcal{E}(u, v) = \int_{\mathbb{R}^2} \tfrac{1}{2} D \nabla u \cdot \nabla v \, d\mathbf{x} + \int_{\mathbb{R}} U \int_{\mathbb{R}} u \partial_y v \, dy \, dx, \quad u, v \in C_c^\infty(\mathbb{R}^2).$$

The scalar field $c(t, \mathbf{x})$ defined by (3.1) serves as a model for the concentration of a solute undergoing diffusion-advection in a medium with diffusion tensor $\tfrac{1}{2} D$, immersed in a fluid moving with velocity (advection) U .

In what follows we combine the theory of Dirichlet forms and martingale theory to extend the result of [25] to the case of infinite interfaces and associate equation (3.1) to a diffusion process $\mathbf{X} = \{\mathbf{X}_t : t \geq 0\}$ on $\mathbb{R}^2$ given in terms of a particular multi-skewed Brownian motion. As $\mathbf{X}$ models the microscopic motion of individual solute particles, this characterization gives insight into the effect that sharp discontinuities in D have on the essential stochastic behavior of suspended particles.

Following [19] one can establish that $\mathcal{E}(\cdot, \cdot)$ in (3.2) extends uniquely to a regular and local Dirichlet form (also denoted by $\mathcal{E}$) on $L^2(\mathbb{R}^2)$ with domain $D_{\mathcal{E}} = L^2(\mathbb{R}^2) \cap \mathcal{H}^1(\mathbb{R}^2, D)$, where $\mathcal{H}^1(\mathbb{R}^2, D)$ denotes the Sobolev space of order one with respect to the measure $dD(\mathbf{x}) := D_x D_y \, d\mathbf{x}$. By [19, Theorem V.1.11], there exists a diffusion process $\mathbf{X} = \{\mathbf{X}_t : t \geq 0\}$ on $\mathbb{R}^2$ associated with $(\mathcal{E}, D_{\mathcal{E}})$ and with infinitesimal generator given by $A^{\mathbf{X}}u = \nabla \cdot (\tfrac{1}{2} D \nabla u) + U \partial_y u$ defined on $D_{A^{\mathbf{X}}} = \{u \in D_{\mathcal{E}} : D \nabla u \in \mathcal{H}^1(\mathbb{R}^2)\}$. The process $\mathbf{X}$ is characterized in Theorem 1.4, which we now prove.

Proof of Theorem 1.4. Let $(\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}}, B^\alpha)$ be multi-skewed Brownian motion as in the statement of the theorem. Since ψ is a one-to-one continuous function, $X = \psi(B^\alpha)$ is a regular diffusion with infinitesimal generator given by

$$(3.3) \quad A^X u(x) = \sum_{k \in \mathbb{Z}} \tfrac{1}{2} \partial_x^2 ((u \circ \psi) \cdot \mathbb{1}_{I_k})(\varphi(x)) = \partial_x (\tfrac{1}{2} D_x \partial_x u)(x)$$

with domain $D_{A^X} = \{u \in L^2(\mathbb{R}) \cap \mathcal{H}^1(\mathbb{R}, D_x) : D_x \partial_x u \in \mathcal{H}^1(\mathbb{R})\}$.

Let W be standard Brownian motion independent of B^α , and let $\{\mathbb{P}_x\}_{x \in \mathbb{R}^2}$ be the family of probability measures generated by the process (B^α, W) on the two-dimensional Wiener space $(\Omega, \mathcal{F})$. The proof is finished by establishing that

$\mathbf{X} := (X, Y)$, with X constructed above and Y defined as in (1.12), solves the martingale problem for the operator $\mathbf{A}^{\mathbf{X}}$ under $\mathbb{P}_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{R}^2$. This is proven in [25, Lemma 2.3] for the case of D_x, D_y bounded away from zero. A localization argument, i.e., [29, Theorem 6.6.1], guarantees that this argument can be carried out without change to the case of $\mathbf{D}$ given by (1.7). $\square$

Remark 3.1. The set of interfaces $\{z_k : k \in \mathbb{Z}\}$ may have accumulation points, as long as $|J_k| > \delta\sqrt{D_k}$ for some $\delta > 0$, and all $k \in \mathbb{Z}$. Under this condition one may have $z_k \uparrow b$ as $k \rightarrow \infty$ and $z_k \downarrow a$ as $k \rightarrow -\infty$ for some $a < b$, and the construction $X = \psi(B_t^\alpha)$ can be carried on $(a, b) \times \mathbb{R}$.

Remark 3.2. The process X cannot have an invariant probability measure (even in the case described in Remark 3.1). To see this, note that its generator $\mathbf{A}^X$ is a selfadjoint operator and the unique solution to $\mathbf{A}^X v = 0$ is the piecewise function with $\partial_x v = 1/D_k$ on J_k , $k \in \mathbb{Z}$. On the other hand, since $|J_k| > \delta\sqrt{D_k}$ must hold for some $\delta > 0$, the invariant measure for X will be finite if and only if $\delta \sum_{k=0}^{\pm\infty} 1/\sqrt{D_k} < \sum_{k=0}^{\pm\infty} |J_k|/D_k < \infty$, which violates the boundedness of D_x . In the case of a finite state space and a finite number of layers, ergodicity is possible when reflecting boundaries are introduced, and in particular, a central limit theorem for the two-dimensional process $\mathbf{X}$ can be obtained (see [25]). This is referred to as the Taylor–Aris problem in geophysical contexts.

4. EXAMPLE: A GEOMETRY WITH ALTERNATING LAYERS

We now consider advection-diffusion in the particular two-dimensional periodic arrangement of layers with alternating widths described by equation (1.13) and the accompanying text. See also Figure 1.

If $(\Omega, \mathcal{F}, \{\mathbb{P}_{\mathbf{x}}\}_{\mathbf{x} \in \mathbb{R}^2}, \mathbf{X})$ denotes the diffusion constructed in Theorem 1.4 for these particular choices of $\mathbf{U}$ and $\mathbf{D}$, then

$$(4.1) \quad \mathbf{X}_t = (X_t, Y_t), \quad X_t = \psi(B_t^\alpha), \quad dY_t = U(X_t) dt + \sqrt{D_y(X_t)} dW_t, \quad t \geq 0.$$

Here, ψ is given in equation (1.9), W is a standard Brownian motion independent of X , and B^α is a multi-skewed Brownian motion with interfaces $S = \{x_k = \varphi(z_k) : k \in \mathbb{Z}\}$, $\varphi := \psi^{-1}$, and skewness sequence α with values

$$\alpha_{2k} = \frac{\sqrt{D_2}}{\sqrt{D_2} + \sqrt{D_1}} := \alpha, \quad \alpha_{2k-1} = \frac{\sqrt{D_1}}{\sqrt{D_2} + \sqrt{D_1}} = 1 - \alpha, \quad k \in \mathbb{Z}.$$

The speed and scale measures of B^α have densities with values

$$(4.2) \quad m'(x) = \sqrt{D_i}, \quad s'(x) = \frac{2}{\sqrt{D_i}}, \quad x \in \varphi(M_i), \quad i = 1, 2.$$

For ease of exposition, the proof of Theorem 1.5 will be given in two parts. First we consider the limits in (1.14). The central limit theorem in part (ii) of that theorem will be proved near the end of this section.

Proof of part (i) of Theorem 1.5. Consider the discrete process $Z_n = X_{\eta_n}$ tracking the visits of X to the interfaces as in (2.8) with $X_0 = Z_0 = 0$, $\eta_0 = 0$. The symmetry of the arrangement of layers makes $\{\eta_{n+1} - \eta_n : n \geq 0\}$ a sequence of i.i.d. positive random variables with the same distribution as $E_{(-l_1, l_2)}^X$ under $\mathbb{P}_0$. Let $\beta = \frac{l_1 D_2}{l_2 D_1 + l_1 D_2}$. The process Z is a Markov chain with values on $\{z_k : k \in \mathbb{Z}\}$ and i.i.d. increments $\{Z_{2(n+1)} - Z_{2n} : n \geq 0\}$ taking the value 0 with probability

$\beta^2 + (1 - \beta)^2$, and $l_1 + l_2$ or $-l_1 - l_2$ with equal probability $\beta(1 - \beta)$. Consider the renewal process $\{N(t) : t \geq 0\}$ given by $N(t) = \sup\{n \geq 0 : \eta_{2n} < t\}$. One then has the following decomposition:

$$(4.3) \quad X_t = \sum_{n=1}^{N(t)} (Z_{2n} - Z_{2(n-1)}) + (X_t - X_{\eta_{2N(t)}}), \quad t \geq 0.$$

Since $\mathbb{E}_0(Z_{2n} - Z_{2(n-1)}) = 0$ for all n , the strong law of large numbers gives

$$(4.4) \quad \lim_{t \rightarrow \infty} \frac{\mathbb{E}_0 X_t^2}{t} = \mathbb{E}_0(Z_2 - Z_0)^2 \cdot \left[\lim_{t \rightarrow \infty} \frac{\mathbb{E}_0 N(t)}{t} \right] = \frac{D_2 D_1 (l_2 + l_1)}{l_2 D_1 + l_1 D_2}. \quad \square$$

We now turn our attention to the transversal process Y and its asymptotic behavior described in (1.15). It is evident that Y is determined by the time the process X spends on M_1 and M_2 respectively. To this effect we prove first the following slightly general but elementary result.

Proposition 4.1. *Let (a, b) be an interval, $-\infty \leq a < b \leq \infty$, and assume A_1 and A_2 are disjoint sets with $(a, b) = A_1 \cup A_2$. Consider the measure $\tilde{m}$ with density $\tilde{m}' = m_1 \mathbb{1}_{A_1} + m_2 \mathbb{1}_{A_2}$, with $m_1 > m_2 > 0$ and $\tilde{m}(\{a\}) = \tilde{m}(\{b\}) = 0$. Let B be Brownian motion reflecting at the finite endpoints of (a, b) (if any). Define $\phi(r) := \int_a^b L^B(x, r) d\tilde{m}(x)$ and $\tau = \phi^{-1}$. Consider the process $\tilde{X}_t = B_{\tau(t)}$, $t \geq 0$. Then $\mathbb{P}_x$ almost surely, $x \in (a, b)$,*

$$(4.5) \quad O_{A_1}^{\tilde{X}}(t) = \frac{m_1(t - \frac{1}{2}m_2 \tau(t))}{m_1 - m_2}, \quad O_{A_2}^{\tilde{X}}(t) = \frac{m_2(\frac{1}{2}m_1 \tau(t) - t)}{m_1 - m_2}.$$

Proof. The process $\tilde{X}$ is a regular diffusion in natural scale with speed measure $\tilde{m}$ [8, p. 373]. Moreover, by the occupation times formula, $\phi(r) = \frac{m_1}{2} O_{A_1}^B(r) + \frac{m_2}{2} O_{A_2}^B(r)$. Perform a change of variables to write for $i = 1, 2$,

$$(4.6) \quad O_{A_i}^{\tilde{X}}(t) = \int_0^t \mathbb{1}_{A_i}(B_{\tau(u)}) du = \int_0^{\tau(t)} \mathbb{1}_{A_i}(B_u) \phi'(u) du = \frac{m_i}{2} O_{A_i}^B(\tau(t)).$$

Therefore, the following system of equations holds true with probability one for all $t \geq 0$:

$$(4.7) \quad \frac{2}{m_1} O_{A_1}^{\tilde{X}}(t) + \frac{2}{m_2} O_{A_2}^{\tilde{X}}(t) = \tau(t), \quad O_{A_1}^{\tilde{X}}(t) + O_{A_2}^{\tilde{X}}(t) = t.$$

Solving gives (4.5). $\square$

The asymptotic behavior of Y will follow from applying the results of [25] to an ergodic process $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{Y})$ related to $\mathbf{X}$.

Proof of part (ii) of Theorem 1.5. By Knight's theorem (see [26, p. 172]), there exist two independent Brownian motions $W^{(1)}$ and $W^{(2)}$ such that

$$(4.8) \quad Y_t = U_1 O_{M_1}^X(t) + U_2 O_{M_2}^X(t) + \sqrt{D_1} W^{(1)}(O_{M_1}^X(t)) + \sqrt{D_2} W^{(2)}(O_{M_2}^X(t)).$$

Let B be Brownian motion such that $X_t = \psi \circ s^{-1}(B_{\tau(t)})$, where τ is the time change associated with

$$(4.9) \quad \phi(r) = \frac{1}{4} D_1 O_{s(\varphi(M_1))}^B(r) + \frac{1}{4} D_2 O_{s(\varphi(M_2))}^B(r), \quad r \geq 0.$$

Let $\tilde{X}$ be skew diffusion in $\tilde{J} := (-l_1/2, l_2/2)$ with instantaneous reflection at the endpoints and diffusion coefficient given by $1/2 D_1 \mathbb{1}_{(-l_1/2, 0)} + 1/2 D_2 \mathbb{1}_{(0, l_2/2)}$ (see [25]).

Let $\bar{Y}$ be given by $d\bar{Y}_t = U(\bar{X}_t) dt + \sqrt{D_y(\bar{X}_t)} dW_t$, $t \geq 0$. The process $\bar{X}$ can be constructed via a reflected Brownian motion $\bar{B} := \{\sigma(B_t) : t \geq 0\}$ on $(-\tilde{l}_1/2, \tilde{l}_2/2)$, where $\sigma(x) := \frac{1}{2}\tilde{l}_2 - \left| \left[(x + \frac{1}{2}\tilde{l}_1) \bmod (\tilde{l}_1 + \tilde{l}_2) \right] - \frac{1}{2}(\tilde{l}_1 + \tilde{l}_2) \right|$, $x \in \mathbb{R}$, and occupation times are preserved:

$$(4.10) \quad O_{(0, \frac{1}{2}\tilde{l}_2)}^{\bar{B}}(t) = O_{M_2}^B(t), \quad O_{(-\frac{1}{2}\tilde{l}_1, 0)}^{\bar{B}}(t) = O_{M_1}^B(t), \quad t \geq 0.$$

Denote by $\bar{s}$ and $\bar{\psi}$ the restrictions of s and ψ to $\varphi(\bar{J})$. Then $\bar{X} = \bar{\psi} \circ \bar{s}^{-1}(\bar{B}_{\bar{\tau}})$, where $\bar{\phi} := \frac{1}{4}D_1O_{(-\tilde{l}_1/2, 0)}^{\bar{B}} + \frac{1}{4}D_2O_{(0, \tilde{l}_2/2)}^{\bar{B}}$ and $\bar{\tau} := \bar{\phi}^{-1}$. By (4.10), $\phi(r) = \bar{\phi}(r)$ for all r , and therefore $\tau = \bar{\tau}$. Proposition 4.1 states that the occupation times of X and $\bar{X}$ depend only on τ and $\bar{\tau}$ respectively. Then

$$(4.11) \quad O_{M_2}^X(t) = O_{(0, \tilde{l}_2/2)}^{\bar{X}}(t), \quad O_{M_1}^X(t) = O_{(-\tilde{l}_1/2, 0)}^{\bar{X}}(t), \quad t \geq 0$$

and therefore $Y \stackrel{d}{=} \bar{Y}$ under $\mathbb{P}_0$. The result follows by applying [25, Corollary 2.2] to the process $\bar{X} := (\bar{X}, \bar{Y})$. $\square$

Some further information about the preasymptotic behavior of Y (in particular, of Y^U below) can be obtained using the discrete process Z in (2.8) (see also Proposition 4.1). Decompose the stochastic integral in (4.1) as

$$(4.12) \quad Y_t = Y^U(t) + \int_0^t \sqrt{D_y(X_s)} dW_s, \quad t \geq 0,$$

where $Y^U(t) := U_1O_{M_1}^X(t) + U_2O_{M_2}^X(t)$.

Make $J := J_{-1} \cup J_0 = (-l_1, l_2)$, $\tilde{J} = s(\varphi(J))$ and let B be such that $X_t = \psi \circ s^{-1}(B_{\tau(t)})$. Proposition 4.1 gives that the $\mathbb{P}_{Z_n}$ law of $O_{M_2}^X(\eta_{n+1}) - O_{M_2}^X(\eta_n)$ is equal to the $\mathbb{P}_{z_k}$ law of $O_{M_2}^X(E_J^X) = \frac{1}{4}D_2O_{J_0}^B(E_J^B)$ for any $n \geq 0$ and $k \in \mathbb{Z}$. A similar result holds for $O_{M_1}^X(E_J^X)$. For $i = 1, 2$, let $\{T_j^{(i)} : j \geq 1\}$ be a sequence of i.i.d. random variables distributed as $O_{M_i}^X(E_J^X)$ under $\mathbb{P}_0$. Therefore we have the following equalities in the $\mathbb{P}_0$ distribution:

$$(4.13) \quad Y_{\eta_n}^U \stackrel{d}{=} \sum_{j=1}^n U_2T_j^{(2)} + U_1T_j^{(1)} \quad n \geq 1,$$

and for $j \geq 1$,

$$(4.14) \quad U_2T_j^{(2)} + U_1T_j^{(1)} \stackrel{d}{=} \int_0^{E_J^B} \frac{1}{4}U_1D_1 \mathbb{1}_{\tilde{J}_{-1}}(B_s) + \frac{1}{4}U_2D_2 \mathbb{1}_{\tilde{J}_0}(B_s) ds.$$

The moment generating function for the additive functional in (4.14) can be calculated using the Feynman–Kac formula to get

$$(4.15) \quad \mathbb{E}_0(e^{-\lambda Y_{\eta_n}^U}) = \left[\frac{S_1\sqrt{D_2U_2} + S_2\sqrt{D_1U_1}}{C_2C_1\sqrt{D_2U_2} + C_1C_2\sqrt{D_1U_1}} \right]^n, \quad \lambda \geq 0,$$

where $S_i := \sinh(\sqrt{\lambda l_i P_i})$, $C_i := \cosh(\sqrt{\lambda l_i P_i})$ and $P_i := 2l_i U_i / D_i$, $i = 1, 2$.

Remark 4.2. The example described in this section can be used as an idealization of a medium composed mainly of a matrix dissected by fissures or cracks. The dimensionless number $P_i := 2l_i U_i / D_i$ is called the Péclet number medium i . In applications, the case when P_1, P_2 are much larger than one is usually regarded as

“advection dominated transport”. In such a case, for large t , $Y^U(t) \gg \sqrt{D_i} Y^{D_i}(t)$ for $i = 1, 2$, and equation (4.15) gives an approximation to the moment generating function of the longitudinal component of the random position of particles.

From a mathematical perspective, Proposition 4.1 may be viewed as a form of a generalized “arcsine” Laplace transform for multi-skewed Brownian motion. An arcsine law for skew Brownian motion was obtained in [3] using techniques from the theory of excursions applied in the context of Brownian motion on graphs with a single vertex, of which skew Brownian motion is a special case. The following corollary exhibits this as a special case of Proposition 4.1 with an alternative straightforward proof.

Corollary 4.3. *Assume $\alpha > \frac{1}{2}$ and let $(\Omega, \mathcal{F}, \{\mathbb{P}_x\}_{x \in \mathbb{R}}, B^\alpha)$ be skew Brownian motion. Then*

$$(4.16) \quad \mathbb{P}_0(O_{(-\infty, 0)}^\alpha(t) \in du) = \frac{\alpha(1-\alpha)t}{(1-\alpha)^2(t-u) + \alpha^2u} \frac{du}{\pi\sqrt{u}\sqrt{t-u}}, \quad u \in (0, t).$$

Moreover, $\mathbb{E}_0 O_{(-\infty, 0)}^\alpha(t) = (1-\alpha)t$ and $\mathbb{E}_0 O_{[0, \infty)}^\alpha(t) = \alpha t$.

Proof. In the notation of (4.1), let $A_1 = [0, \infty)$ and $A_2 = (-\infty, 0)$. Then $\phi(r)$ has the arcsine distribution on $(m_2 r/2, m_1 r/2)$, and one can obtain the distribution of τ by writing

$$\begin{aligned} \mathbb{P}_0(\tau(t) \in dr) &= \frac{d}{dr} \mathbb{P}_0(t \leq \phi(r)) = \frac{d}{dr} \int_t^{\frac{m_1}{2}r} \frac{du}{\pi\sqrt{u - \frac{m_2}{2}r}\sqrt{\frac{m_1}{2}r - u}} \\ &= \frac{2t}{\pi\sqrt{m_1 m_2}} \frac{dr}{r\sqrt{r - \frac{2t}{m_1}}\sqrt{\frac{2t}{m_2} - r}}, \quad r \in \left(\frac{2t}{m_1}, \frac{2t}{m_2}\right). \end{aligned}$$

Finally, apply the linear transformation in (4.5) and make $m_1 = 2\alpha^2$, $m_2 = 2(1-\alpha)^2$ to get (4.16). $\square$

REFERENCES

1. T. A. Appuhamillage, V. A. Bokil, E. Thomann, Edward C. Waymire, and B. D. Wood, *Occupation and local times for skew Brownian motion with applications to dispersion across an interface*, *Annals of Applied Probability* **21** (2011), no. 1, 183–214.
2. ———, *Solute transport across an interface: A Fickian theory for skewness in breakthrough curves*, *Water Resources Research* **46** (2010), W07511.
3. M. Barlow, J. Pitman, and M. Yor, *Une extension multidimensionnelle de la loi de l’arc sinus*, *Séminaire de Probabilités, XXIII, Lecture Notes in Math.*, vol. 1372, Springer, Berlin, 1989, pp. 294–314. MR1022918 (91c:60106)
4. R. Bass and Z. Chen, *One-dimensional stochastic differential equations with singular and degenerate coefficients*, *Sankhyā: The Indian Journal of Statistics* **67** (2005), 19–45. MR2203887 (2007e:60041)
5. C. Berentsen, M. Verlaan, and C. Kruijsdijk, *Upscaling and reversibility of Taylor dispersion in heterogeneous porous media*, *Physical Review E* **71** (2005), 046308–1, 046308–16.
6. B. Berkowitz, A. Cortis, I. Dror, and H. Scher, *Laboratory experiments on dispersive transport across interfaces: The role of flow direction*, *Water Resources Research* **45** (2009).
7. R. Bhattacharya and E. Waymire, *Stochastic processes with applications*, Wiley, New York, 1990. Reprinted in *SIAM Classics in Applied Mathematics*, 2009. MR1054645 (91m:60001)
8. L. Breiman, *Probability*, *Classics in Applied Mathematics*, SIAM, New York, 1992. MR1163370 (93d:60001)
9. D. Freedman, *Brownian motion and diffusion*, Holden-Day, San Francisco, Calif., 1971. MR0297016 (45:6074)

10. M. Fukushima, *Dirichlet forms and Markov processes*, 1st ed., North-Holland Mathematical Library, North-Holland/Kodansha, Amsterdam, 1980. MR0569058 (81f:60105)
11. J.M. Harrison and L.A. Shepp, *On skew Brownian motion*, The Annals of Probability **9** (1981), no. 2, 309–313. MR606993 (82j:60144)
12. H. Hoteit, R. Mose, A. Younes, and F. Lehmann, *Three-dimensional modeling of mass transfer in porous media using the mixed hybrid finite elements and the random walk methods*, Mathematical Geology **34** (2002), no. 4, 435–456. MR1951790 (2003m:76158)
13. K. Itô and H.P. McKean, *Brownian motions on a half line*, Ill. J. Math. **7** (1963), no. 1, 181–231. MR0154338 (27:4287)
14. O. Kallenberg, *Foundations of modern probability*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002. MR1876169 (2002m:60002)
15. R. Lang, *Effective conductivity and skew Brownian motion*, Journal of Statistical Physics **80** (1995), no. 1–2, 125–146. MR1340556 (96f:82033)
16. J.-F. Le Gall, *One-dimensional stochastic differential equations involving the local times of the unknown process*, Stochastic analysis and applications (Swansea, 1983), Lecture Notes in Math., vol. 1095, Springer, Berlin, 1984, pp. 51–82. MR777514 (86g:60071)
17. A. Lejay, *On the constructions of the skew Brownian motion*, Probab. Surv. **3** (2006), 413–466 (electronic). MR2280299 (2008h:60333)
18. A. Lejay and M. Martinez, *A scheme for simulating one-dimensional diffusion processes with discontinuous coefficients*, Ann. Appl. Probab. **16** (2006), no. 1, 107–139. MR2209338 (2007b:60198)
19. Z. Ma and M. Röckner, *Introduction to the theory of (nonsymmetric) Dirichlet forms*, 1st ed., Universitext, Springer Verlag, New York, 1992. MR1214375 (94d:60119)
20. P. Mandl, *Analytical treatment of one-dimensional Markov processes*, Die Grundlehren der mathematischen Wissenschaften, Band 151, Academia Publishing House of the Czechoslovak Academy of Sciences, Prague, 1968. MR0247667 (40 #930)
21. M. Mastrangelo and T. Mouloud, *Mouvements browniens asymétriques modifiés en dimension finie et opérateurs différentiels à coefficients discontinus*, Probability and Mathematical Statistics **11** (1990), no. 1, 47–78. MR1096939 (92f:60136)
22. Y. Ouknine, *Le “skew-brownian motion” et les processus qui en dérivent*, Teor. Veroyatnost. i Primenen. **35** (1990), no. 1, 173–179. MR1050069 (91e:60242)
23. N.I. Portenko, *Generalized diffusion processes*, 1st ed., Translations of Mathematical Monographs, American Mathematical Society, 1990. MR1104660 (92b:60074)
24. N.I. Portenko, *A probabilistic representation for the solution of one problem of mathematical physics*, Ukrainian Mathematical Journal **52** (2000), no. 9, 1457–1469. MR1816940 (2002b:60135)
25. J.M. Ramirez, E.A. Thomann, E. Waymire, R. Haggerty, and B. Wood, *A generalized Taylor-Aris formula and skew diffusion*, Multiscale Model. Simul. **5** (2006), no. 3, 786–801 (electronic). MR2257235 (2008b:60180)
26. D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 1st ed., A series of comprehensive studies in mathematics, Springer-Verlag, New York, 1991. MR1083357 (92d:60053)
27. R.E. Showalter, *Hilbert space methods for partial differential equations*, Electronic Monographs in Differential Equations, San Marcos, TX, 1994, electronic reprint of the 1977 original. MR1302484 (95g:35001)
28. D.W. Stroock, *Diffusion semigroups corresponding to uniformly elliptic divergence form operators*, Lect. Notes in Math., 1321, Springer, Berlin, 1988. MR960535 (90b:35071)
29. D.W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 233, Springer-Verlag, Berlin, 1979. MR532498 (81f:60108)
30. K. Ueno, F. Kitagawa, and N. Kitamura, *Photocyanation of pyrene across an oil/water interface in a polymer microchannel chip*, Lab on a Chip: Miniaturisation for Chemistry, Biology and Bioengineering **2** (2002), 231–234.
31. J.B. Walsh, *A diffusion with a discontinuous local time*, Astérisque **52–53** (1978), 37–45.

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