STABILITY OF PLANAR WAVES IN MONO-STABLE REACTION-DIFFUSION EQUATIONS

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ABSTRACT. This paper is concerned with the asymptotic stability of planar waves in mono-stable reaction-diffusion equations in $\mathbb{R}^n$, where $n \geq 2$. Under initial perturbation that decays at space infinity, the perturbed solution converges to planar waves as $t \rightarrow \infty$. The convergence is uniform in $\mathbb{R}^n$.

1. INTRODUCTION

In this paper, we consider the stability of planar waves for the following Cauchy problem of mono-stable reaction-diffusion equations:

$$(1.1) \quad u_t = \Delta u + f(u), \quad x \in \mathbb{R}^{n-1}, \quad y \in \mathbb{R}, \quad t > 0,$$

with initial value

$$(1.2) \quad u(x, y, 0) = u_0(x, y), \quad x \in \mathbb{R}^{n-1}, \quad y \in \mathbb{R},$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_{n-1}^2 + \partial^2/\partial y^2$ and $n \geq 2$. When the reaction term $f(u)$ is of the bistable type, this problem has been studied in [7]. They obtained the stability of planar waves under any, possibly large, initial perturbations and almost periodic perturbations. They also proved the existence of a solution that oscillates permanently between two planar waves, which implies that planar waves are not asymptotically stable under more general perturbations. Concerning the stability of planar waves, see also [3, 5, 11].

It is well-known that the differential equation of (1.1) in $\mathbb{R}$ has traveling wave fronts under some assumptions on f . That is, the equation

$$(1.3) \quad u_t = u_{yy} + f(u), \quad y \in \mathbb{R}, \quad t > 0,$$

has monotone solutions with the form $u(y, t) = \phi(y - ct)$ for some constant $c \in \mathbb{R}$. Sometimes we shall call such ϕ planar waves in $\mathbb{R}^n$ with $n \geq 2$. We are interested

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in traveling wave fronts that connect the unstable state 0 and stable state 1; thus we consider

$$(1.4) \quad \begin{cases} \phi''(\xi) - c\phi'(\xi) + f(\phi(\xi)) = 0, & \xi \in \mathbb{R}, \\ \lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, \quad \lim_{\xi \rightarrow \infty} \phi(\xi) = 1. \end{cases}$$

Throughout this paper, we always assume that $u_0(x, y) \geq 0$ and

$$(F) \quad f \in C^1(\mathbb{R}), \quad f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad f(u) > 0 \text{ for } u \in (0, 1), \\ f(u) < 0 \text{ for } u \in (1, \infty).$$

A typical example is $f(u) = u(1 - u)$. If condition (F) holds, it is well-known that equation (1.4) has a monotone solution for $c \geq 2\sqrt{f'(0)}$. Fisher [2] introduced a particular case of (1.1) to model the speed of advantageous genetic traits in a population. A mathematical treatment was given in [4]. Under the condition that

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad f'(u) \leq f'(0), \quad f(u) > 0 \text{ for } u \in (0, 1),$$

they proved that if the initial function $u_0(x)$ is chosen so that

$$u_0(x) \equiv 0 \quad \text{for } x < 0, \quad u_0(x) \equiv 1 \quad \text{for } x > 0,$$

then it is indeed true in a certain sense that the solution of the initial value problem “tends” to a traveling front. Specifically, there exists a traveling front $\phi(x + ct)$ and a function $\psi(t)$ such that, as $t \rightarrow \infty$,

$$u(x, t) - \phi(x + ct + \psi(t)) \rightarrow 0 \quad \text{uniformly in } x$$

and $\psi'(t) \rightarrow 0$. Note that it is not true that $\psi(t)$ tends to a finite limit as $t \rightarrow \infty$; thus the solution u does not approach a traveling front uniformly in x . The stability of traveling wave fronts has been studied by many authors. Sattinger [9] obtained the stability of traveling fronts of (1.4) by using detailed spectral analysis, and Raugel-Kirchgässner [8] also obtained the stability of traveling wave fronts of (1.4) for the critical case. Recently, Lv-Wang [6] studied the stability of traveling wave fronts of (1.4) by using a different method than Sattinger [9].

Encouraged by [7], in this paper, we will consider equations (1.1) with (1.2) in $\mathbb{R}^n$ with $n \geq 2$. Throughout this paper, we always assume that ϕ is a monotone solution of (1.4) and satisfies the fact that $\phi(0) = 1/2$. Under the initial perturbation, we will prove that the perturbed solution converges to a planar waves as $t \rightarrow \infty$, which is different from the case that $n = 1$. Our main result is the following theorem.

Theorem 1.1. *Let $n \geq 2$ and let (F) hold. Suppose that the initial value $u_0(x, y)$ is nonnegative, smooth and bounded on $\mathbb{R}^n$ and satisfies*

$$\begin{cases} \lim_{R \rightarrow \infty} \sup_{|x| - y \geq R, y \leq 0} |u_0(x, y) - \phi(y - 1)| e^{-\lambda y} = 0, \\ \lim_{R \rightarrow \infty} \sup_{|x| + y \geq R, y \geq 0} |u_0(x, y) - \phi(y - 1)| = 0, \end{cases}$$

where $\lambda > \lambda_1(c)$ ($\lambda_1(c)$ is defined as Lemma 2.1) and ϕ is a monotone solution of (1.4) with $c \geq 2\sqrt{f'(0)}$. Then the nonnegative solution $u(x, y, t)$ of (1.1) and (1.2) satisfies

$$(1.5) \quad \lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathbb{R}^n} |u(x, y, t) - \phi(y + ct)| = 0.$$

The following theorem gives the convergence rate for (1.5) when the initial perturbation belongs to L^1 in a certain sense.

Theorem 1.2. *Let $n \geq 2$ and let (F) hold. Suppose that the initial value $u_0(x, y)$ is given by*

$$(1.6) \quad u_0(x, y) = \phi(y - v_0(x))$$

for some smooth function $v_0 \in L^1(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. Then the nonnegative solution $u(x, y, t)$ of (1.1) and (1.2) satisfies

$$(1.7) \quad \sup_{(x,y) \in \mathbb{R}^n} |u(x, y, t) - \phi(y + ct)| \leq Ct^{-\frac{n-1}{2}}, \quad t > 0,$$

where ϕ is a monotone solution of (1.4) with $c \geq 2\sqrt{f'(0)}$ and $C > 0$ is a constant depending on f , $\|v_0\|_{L^1(\mathbb{R}^{n-1})}$ and $\|v_0\|_{L^\infty(\mathbb{R}^{n-1})}$.

Similar to [7], we have the following proposition, which shows that the convergence rate (1.7) is optimal in some sense.

Proposition 1.1. *Let u_0 be as in (1.6) and assume that either $v_0 \geq 0$, $v_0 \not\equiv 0$ or $v_0 \leq 0$, $v_0 \not\equiv 0$. Then there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1(1+t)^{-\frac{n-1}{2}} \leq \sup_{(x,y) \in \mathbb{R}^n} |u(x, y, t) - \phi(y + ct)| \leq C_2t^{-\frac{n-1}{2}}, \quad t \geq 0.$$

We remark that in Theorem 1.1, $\phi(y-1)$ can be replaced by $\phi(y-\xi_0)$ with fixed $\xi_0 > 0$. In section 2, we will prove Theorem 1.1 by constructing the suitable upper and lower solutions. In the proof, we use the moving coordinate with speed c , so that the planar waves can be viewed as stationary states. Setting

$$u(x, y, t) = w(x, \xi, t), \quad \xi = y + ct,$$

the problem (1.1)-(1.2) can be rewritten as (we write $w(x, \xi, t)$ as $u(x, \xi, t)$ for simplicity)

$$(1.8) \quad \begin{cases} u_t = \Delta u - cu_\xi + f(u), & x \in \mathbb{R}^{n-1}, \quad \xi \in \mathbb{R}, \quad t > 0, \\ u(x, \xi, 0) = u_0(x, \xi), & x \in \mathbb{R}^{n-1}, \quad \xi \in \mathbb{R}. \end{cases}$$

One can also obtain the stability of planar waves in mono-stable reaction-diffusion equations under almost periodic perturbations by using the upper and lower solutions constructed in Lemma 2.6. This is similar to that of [7], and therefore we omit it here.

Theorem 1.2 and Proposition 1.1 can be proved by constructing the suitable upper and lower solutions, which are similar to those of [7]. We omit the details here.

2. PROOF OF THEOREM 1.1

In this section we will prove Theorem 1.1. Our argument is similar to the earlier work in [7]. In what follows we shall construct an upper solution and a lower solution of (1.8).

Now we begin with some auxiliary lemmas.

Lemma 2.1. *Let $\phi(\xi)$ be a monotone solution of (1.4) with $c \geq c^* := 2\sqrt{f'(0)}$. Then the following hold:*

(i) When $c > c^*$,

$$\begin{aligned} \lim_{\xi \rightarrow -\infty} \phi(\xi) e^{-\lambda_1(c)\xi} &= 1, & \lim_{\xi \rightarrow -\infty} \phi'(\xi) e^{-\lambda_1(c)\xi} &= \lambda_1(c), \\ \lim_{\xi \rightarrow -\infty} \frac{\phi'(\xi)}{\phi(\xi)} &= \lambda_1(c), & \lim_{\xi \rightarrow \infty} \frac{\phi'(\xi)}{\phi(\xi) - 1} &= \mu_1(c). \end{aligned}$$

(ii) When $c = c^*$,

$$\begin{aligned} \lim_{\xi \rightarrow -\infty} \phi(\xi) |\xi|^{-1} e^{-\lambda_1(c^*)\xi} &= 1, & \lim_{\xi \rightarrow -\infty} \phi'(\xi) |\xi|^{-1} e^{-\lambda_1(c^*)\xi} &= \lambda_1(c^*), \\ \lim_{\xi \rightarrow -\infty} \frac{\phi'(\xi)}{\phi(\xi)} &= \lambda_1(c^*), & \lim_{\xi \rightarrow \infty} \frac{\phi'(\xi)}{\phi(\xi) - 1} &= \mu_1(c^*), \end{aligned}$$

where $\lambda_1(c) > 0$ and $\mu_1(c) < 0$, satisfying

$$\lambda_1^2(c) - c\lambda_1(c) + f'(0) = 0, \quad \mu_1^2(c) - c\mu_1(c) + f'(1) = 0.$$

Proof. The proof of this lemma is based on phase plane analysis, and we only give the outline of the proof. Differentiate the differential equations of (1.4) with respect to ξ , and denote $\phi'(\xi) = \phi_1(\xi)$. Then we have

$$(2.1) \quad \phi_1'' - c\phi_1' + f'(\phi)\phi_1 = 0.$$

The limiting system of (2.1) as $\xi \rightarrow -\infty$ is

$$(2.2) \quad \phi_{1-}'' - c\phi_{1-}' + f'(0)\phi_{1-} = 0.$$

By setting $\phi_{1-}' = w$, we can write system (2.2) as a first order system of ordinary differential equation in the two components $(\phi_{1-}, w)^T$. More precisely, the system (2.2) becomes

$$(2.3) \quad W' = AW, \quad W = (\phi_{1-}, w)^T,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -f'(0) & c \end{pmatrix}.$$

One can easily obtain that the general solution of the system (2.3) has the form

$$(\phi_{1-}, w)^T = \sum_{i=1}^2 C_i h_i e^{\lambda_i \xi},$$

where $\lambda_i = \lambda_i(c)$, $i = 1, 2$,

$$\lambda_1 = \frac{c - \sqrt{c^2 - 4f'(0)}}{2}, \quad \lambda_2 = \frac{c + \sqrt{c^2 - 4f'(0)}}{2}$$

and $h_i = (1, \lambda_i)^T$ ($i = 1, 2$). Here h_i ($i = 1, 2$) are eigenvectors of the matrix A with λ_i as the corresponding eigenvalues and C_i are nonnegative constants.

When $c > c^*$, every solution of (2.3) which converges to zero as $\xi \rightarrow -\infty$ is given by

$$W(\xi) = C_1 h_1 e^{\lambda_1 \xi} + C_2 h_2 e^{\lambda_2 \xi}.$$

Applying the unstable manifold theorem, we obtain

$$(2.4) \quad \phi(\xi) = \alpha e^{\lambda_1 \xi} + h.o.t.,$$

where *h.o.t.* means the higher order term. Since $\phi(\xi)$ is a positive function, we see that the constant $\alpha > 0$. When $c = c^*$, every solution of (2.3) which converges to zero as $\xi \rightarrow -\infty$ is given by

$$W(\xi) = (C_1 h_1 \xi + C_2 h^*) e^{\lambda_1 \xi},$$

where $h^* = (1, (\sqrt{f'(0)} - 1)/2)^T$. Applying the unstable manifold theorem, we obtain

$$(2.5) \quad \phi(\xi) = \alpha |\xi| e^{\lambda_1 \xi} + \beta e^{\lambda_1 \xi} + h.o.t.,$$

where the constant $\alpha \neq 0$. It is easy to see that $\alpha > 0$ since $\phi(\xi)$ is a positive function.

Similarly, let $\xi \rightarrow \infty$. Then the limiting system of (2.1) is

$$(2.6) \quad \phi_{1+}'' - c\phi_{1+}' + f'(1)\psi_{1+} = 0.$$

By setting $\phi_{1+}' = z$, we can write system (2.6) as a first order system of ordinary differential equations

$$(2.7) \quad Z' = BZ, \quad Z = (\phi_{1+}, z)^T,$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -f'(1) & c \end{pmatrix}.$$

Solving the system (2.7), we have

$$(\phi_{1+}, z)^T = \sum_{i=1}^2 D_i p_i e^{\mu_i \xi},$$

where $\mu_i = \mu_i(c)$, $i = 1, 2$,

$$\mu_1 = \frac{c - \sqrt{c^2 - 4f'(1)}}{2} < 0, \quad \mu_2 = \frac{c + \sqrt{c^2 - 4f'(1)}}{2} > 0,$$

and $p_i = (1, \mu_i)^T$ ($i = 1, 2$). Here p_i ($i = 1, 2$) are eigenvectors of the matrix B with μ_i as the corresponding eigenvalues and D_i are arbitrary constants. Since $(\phi_{1+}, z)^T \rightarrow (0, 0)^T$ as $\xi \rightarrow \infty$, we have $D_2 = 0$.

Any solution of (2.7), which converges to zero as $\xi \rightarrow \infty$, is given by $Z(\xi) = D_1 p_1 e^{\mu_1 \xi}$. Applying the stable manifold theorem, we obtain that

$$(2.8) \quad 1 - \phi(\xi) = \alpha e^{\mu_1 \xi} + h.o.t.,$$

where the constant $\alpha > 0$ since $\phi(\xi) < 1$ for all $\xi \in \mathbb{R}$. Combining (2.4), (2.5) and (2.8), we have

$$\begin{aligned} \lim_{\xi \rightarrow -\infty} \phi(\xi) e^{-\lambda_1(c)\xi} &= 1, & \lim_{\xi \rightarrow -\infty} \phi(\xi) |\xi|^{-1} e^{-\lambda_1(c^*)\xi} &= 1, \\ \lim_{\xi \rightarrow \infty} (1 - \phi(\xi)) e^{-\mu_1(c)\xi} &= 1, & \lim_{\xi \rightarrow \infty} (1 - \phi(\xi)) e^{-\mu_1(c^*)\xi} &= 1. \end{aligned}$$

Here we have used the coordinate shift invariance of traveling wave front. The other equalities of Lemma 2.1 can be deduced by using the above equalities. $\square$

Lemma 2.1 leads to the following auxiliary lemma, which is necessary in the construction of upper and lower solutions.

Lemma 2.2. *There exists a constant $k > 0$ depending only on f such that*

$$-k\phi'(\xi) \leq \phi''(\xi) \leq k\phi'(\xi), \quad \xi \in \mathbb{R}.$$

Proof. Let $\lambda_1(c) > 0$ and $\mu_1(c) < 0$ be defined as in Lemma 2.1. Using the first equation of (1.4), we have

$$\frac{\phi''}{\phi'} = c - \frac{f(\phi)}{\phi'}.$$

Lemma 2.1 implies that

$$\lim_{\xi \rightarrow -\infty} \frac{f(\phi(\xi))}{\phi'(\xi)} = \frac{f'(0)}{\lambda_1(c)}, \quad \lim_{\xi \rightarrow \infty} \frac{f(\phi(\xi))}{\phi'(\xi)} = \frac{f'(1)}{\mu_1(c)}.$$

Thus we obtain

$$\lim_{|\xi| \rightarrow \infty} \sup \left| \frac{\phi''(\xi)}{\phi'(\xi)} \right| \leq c + \max \left\{ \frac{f'(0)}{\lambda_1(c)}, \frac{f'(1)}{\mu_1(c)} \right\}.$$

This implies that both ϕ''/ϕ' is bounded on $\mathbb{R}$. $\square$

Lemma 2.3 ([7]). *Let $k > 0$ be any constant, and let Δ_x and ∇_x denote the $(n-1)$ -dimensional Laplacian and $(n-1)$ -dimensional gradient, respectively. Let $v(x, t)$ be the solution to the Cauchy problem*

$$(2.9) \quad \begin{cases} v_t = \Delta_x v + k|\nabla_x v|^2, & x \in \mathbb{R}^{n-1}, \quad t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^{n-1}. \end{cases}$$

Suppose that $v_0(x)$ is bounded and continuous on $\mathbb{R}^{n-1}$ and satisfies $\lim_{|x| \rightarrow \infty} |v_0(x)| = 0$. Then the solution $v(x, t)$ satisfies

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} |v(x, t)| = 0.$$

Furthermore, if $v_0(x) \in L^1(\mathbb{R}^{n-1})$, then

$$\sup_{x \in \mathbb{R}^{n-1}} |v(x, t)| \leq \frac{1}{k} \|\exp(kv_0) - 1\|_{L^1(\mathbb{R}^{n-1})} t^{-\frac{n-1}{2}}, \quad t > 0.$$

Remark 2.4. The conclusion of Lemma 2.3 continues to hold for the problem

$$(2.10) \quad \begin{cases} v_t = \Delta_x v - k|\nabla_x v|^2, & x \in \mathbb{R}^{n-1}, \quad t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^{n-1}, \end{cases}$$

where $k > 0$.

Remark 2.5. Following the idea of [7], one can prove that $\phi(\xi - v^+(x, t))$ and $\phi(\xi - v^-(x, t))$ are the upper solution and lower solution of (1.8), respectively, if

$$\phi(\xi - v^-(x, 0)) \leq u(x, \xi, 0) \leq \phi(\xi - v^+(x, 0)),$$

where $v^+(x, t)$ and $v^-(x, t)$ are solutions of problem (2.10) and (2.9), respectively. Hence one can prove Theorem 1.2 and Proposition 1.1 by using Lemma 2.3.

Now, we will construct an upper solution and a lower solution of (1.1), which is similar to those in [1]. We also remark that Chen-Guo [1] considered the discrete quasilinear monostable equations.

Lemma 2.6. *Assume that (F) holds and $\phi(\xi)$ is a monotone increasing solution of (1.4) with $c \geq 2\sqrt{f'(0)}$. Then there exist $\varepsilon_0 \in (0, 1)$, $\beta > 0$, $k > 0$ and $\sigma \geq 1$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any function $v(x, t)$ satisfying*

$$v_t = \Delta_x v - k|\nabla_x v|^2, \quad x \in \mathbb{R}^{n-1}, \quad t > 0,$$

the function defined by

$$u^+(x, \xi, t) = (1 + \varepsilon e^{-\beta t})\phi(\xi - v(x, t) - \sigma \varepsilon e^{-\beta t})$$

is an upper solution. More precisely, it satisfies the inequality

$$N[u^+] := u_t^+ - \Delta u^+ + cu_\xi^+ - f(u^+) \geq 0.$$

Proof. Define a constant $k > 0$ as in Lemma 2.2. Then by using the relation $\phi'' - c\phi' + f(\phi) = 0$ and Lemma 2.2, we have

$$\begin{aligned} N[u^+] &= [(-v_t + \Delta_x v)\phi' - |\nabla_x v|^2 \phi'' + \sigma \varepsilon \beta e^{-\beta t} \phi'](1 + \varepsilon e^{-\beta t}) \\ &\quad - \varepsilon \beta \phi e^{-\beta t} - (1 + \varepsilon e^{-\beta t})(\phi'' - c\phi') - f((1 + \varepsilon e^{-\beta t})\phi) \\ &\geq [|\nabla_x v|^2(k\phi' - \phi'') + \sigma \varepsilon \beta e^{-\beta t} \phi'](1 + \varepsilon e^{-\beta t}) \\ &\quad - \varepsilon \beta \phi e^{-\beta t} - (1 + \varepsilon e^{-\beta t})f(\phi) - f((1 + \varepsilon e^{-\beta t})\phi) \\ &\geq \varepsilon e^{-\beta t}(\sigma \beta \phi' - \beta \phi + \phi \int_0^1 f'((1 + \varepsilon e^{-\beta t}\tau)\phi) d\tau). \end{aligned}$$

By assumption (F), we take constants $\beta > 0$ and $\varepsilon_0 > 0$ such that

$$(2.11) \quad -f'(s) \geq \beta > 0, \quad s \in [1 - 2\varepsilon_0, 1 + 2\varepsilon_0].$$

In addition, we take $M_1 > 0$ sufficiently large so that

$$(2.12) \quad \phi(\xi) \geq 1 - \varepsilon_0, \quad \xi \geq M_1.$$

Here we consider only the case where $c > c^*$; the other case where $c = c^*$ can be similarly proved. As $\lim_{\xi \rightarrow -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = 1$ and $\lim_{\xi \rightarrow -\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c)$, we can take $M_2 > 0$ sufficiently large such that

$$(2.13) \quad \frac{1}{2} < \phi(\xi)e^{-\lambda_1(c)\xi} < \frac{3}{2}, \quad \phi'(\xi)e^{-\lambda_1(c)\xi} > \frac{1}{2}\lambda_1(c) \quad \text{for } \xi \leq -M_2.$$

Denote

$$\varrho := \min\{\phi'(\xi), \quad -M_2 \leq \xi \leq M_1\} > 0.$$

Finally, choose $\sigma \geq 1$ sufficiently large so that

$$\sigma \geq \max\left\{1, \frac{3\beta + L_1}{\beta\lambda_1(c)}, \frac{\beta + L_1}{\beta\varrho}\right\}.$$

Let $\xi = \xi - v(x, t) - \sigma \varepsilon e^{-\beta t}$. Then for any $\xi \geq M_1$, combining (2.11) and (2.12) and using the fact that $\phi' > 0$, we have

$$N[u^+] \geq \varepsilon \phi e^{-\beta t} \left(-\beta + \int_0^1 f'((1 + \varepsilon e^{-\beta t}\tau)\phi) d\tau \right) > 0.$$

For any $\xi \leq -M_2$, by using (2.13), we obtain

$$\varepsilon^{-1} e^{\beta t} e^{-\lambda_1(c)\xi} N[u^+] \geq \frac{1}{2} \sigma \beta \lambda_1(c) - \frac{3}{2} \beta - \frac{1}{2} L_1 \geq 0.$$

For any $-M_2 \leq \xi \leq M_1$, we have

$$\varepsilon^{-1} e^{\beta t} N[u^+] \geq \sigma \beta \varrho - \beta - L_1 \geq 0.$$

In summary, we have $N[u^+] \geq 0$. □

Arguing similarly, we also obtain a lower solution as follows:

Lemma 2.7. Assume that (F) holds and $\phi(\xi)$ is a monotone increasing solution of (1.4) with $c \geq 2\sqrt{f'(0)}$. Then there exist $\varepsilon_0 \in (0, 1)$, $\beta > 0$, $k > 0$ and $\sigma \geq 1$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any function $v(x, t)$ satisfying

$$v_t = \Delta_x v + k|\nabla_x v|^2, \quad x \in \mathbb{R}^{n-1}, t > 0,$$

the function defined by

$$u^-(x, \xi, t) = (1 - \varepsilon e^{-\beta t})\phi(\xi - v(x, t) + \sigma \varepsilon e^{-\beta t})$$

is a lower solution. More precisely, it satisfies the inequality

$$N[u^-] := u_t^- - \Delta u^- + cu_\xi^- - f(u^-) \leq 0.$$

Lemma 2.8. Assume that the initial value u_0 satisfies

$$(2.14) \quad \begin{cases} \lim_{R \rightarrow \infty} \sup_{|x| - \xi \geq R, \xi \leq 0} |u_0(x, \xi) - \phi(\xi - 1)| e^{-\lambda \xi} = 0, \\ \lim_{R \rightarrow \infty} \sup_{|x| + \xi \geq R, \xi \geq 0} |u_0(x, \xi) - \phi(\xi - 1)| = 0, \end{cases}$$

where $\lambda > \lambda_1(c)$ and $\lambda_1(c)$ is defined as in Lemma 2.1. Then, for any fixed $T > 0$, the nonnegative solution $u(x, \xi, t)$ of (1.8) satisfies

$$\begin{cases} \lim_{R \rightarrow \infty} \sup_{|x| - \xi \geq R, \xi \leq 0} |u(x, \xi, T) - \phi(\xi - 1)| e^{-\lambda \xi} = 0, \\ \lim_{R \rightarrow \infty} \sup_{|x| + \xi \geq R, \xi \geq 0} |u(x, \xi, T) - \phi(\xi - 1)| = 0. \end{cases}$$

Proof. Let $w(x, \xi, t) = u(x, \xi, t) - \phi(\xi - 1)$. Then $w(x, \xi, t)$ satisfies

$$(2.15) \quad \begin{cases} w_t = \Delta w - cw_\xi + f'(\phi + \theta w)w, & x \in \mathbb{R}^{n-1}, \xi \in \mathbb{R}, t > 0, \\ w(x, \xi, 0) = u_0(x, \xi) - \phi(\xi - 1), & x \in \mathbb{R}^{n-1}, \xi \in \mathbb{R}, \end{cases}$$

where $\theta(x, \xi, t)$ is a function that satisfies $0 \leq \theta(x, \xi, t) \leq 1$. In order to prove the lemma, it suffices to consider the case where $w(x, \xi, 0) \geq 0$ and the case where $w(x, \xi, 0) \leq 0$, since the general case follows from these special cases and the comparison principle. In what follows we assume that $w(x, \xi, 0) \geq 0$, since the other case can be treated in the same way.

In order to use the comparison principle for equation (2.15), we must prove that there exists a positive constant L such that

$$|\phi(\xi) + \theta(x, \xi, t)w(x, \xi, t)| \leq L \quad \text{for } x \in \mathbb{R}^{n-1}, \xi \in \mathbb{R}, t > 0.$$

Due to $0 \leq \theta(x, \xi, t), \phi(\xi) \leq 1$, it suffices to show that $u(x, \xi, t)$ is bounded for $x \in \mathbb{R}^{n-1}, \xi \in \mathbb{R}, t > 0$. Since $u_0(x, \xi)$ is bounded, by using assumption (F), there exists a positive number K such that

$$u_0(x, t) \leq K \quad \text{for } (x, \xi) \in \mathbb{R}^n, \text{ and } f(K) < 0.$$

Hence K is an upper solution of (1.8); that is, $u(x, \xi, t) \leq K$ for $x \in \mathbb{R}^{n-1}, \xi \in \mathbb{R}, t > 0$.

Because of $w_0(x, \xi) \geq 0$, we have $w(x, \xi, t) \geq 0$ by the maximum principle. By using assumption (F) and the boundedness of $\phi + \theta w$, there exists a positive constant $M > 0$ such that

$$w_t = \Delta w - cw_\xi + f'(\phi + \theta w)w \leq \Delta w - cw_\xi + Mw.$$

Let $Z(x, \xi, \eta, \zeta, t)$ be the fundamental solution of the equation $v_t = \Delta v - cv_\xi$; that is,

$$Z(x, \xi, \eta, \zeta, t) = \frac{1}{(4\pi t)^{-\frac{n}{2}}} \exp\left(-\frac{|x - \eta|^2 + (\xi - \zeta - ct)^2}{4t}\right).$$

Then we obtain the estimate

$$w(x, \xi, t) \leq e^{Mt} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} Z(x, \xi, \eta, \zeta, t) (u_0(\eta, \zeta) - \phi(\zeta - 1)) d\eta d\zeta.$$

It follows from assumption (2.14) that, for any fixed $\varepsilon > 0$, there exists $\bar{R} > 0$ such that when $R \geq \bar{R}$, it holds that

$$(2.16) \quad \begin{aligned} & \sup_{|x| - \xi \geq R, \xi \leq 0} |u_0(x, \xi) - \phi(\xi - 1)| e^{\lambda|\xi|} \leq \frac{1}{6} e^{-T(\lambda^2 + c\lambda + M)} \varepsilon, \\ & \sup_{|x| + \xi \geq R, \xi \geq 0} |u_0(x, \xi) - \phi(\xi - 1)| \leq \frac{1}{6} \varepsilon. \end{aligned}$$

Direct calculation shows that, for $\xi \leq 0$,

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} Z(x, \xi, \eta, \zeta, T) e^{-\lambda\xi} (u_0(\eta, \zeta) - \phi(\zeta - 1)) d\eta d\zeta \\ &= \frac{A}{(4\pi T)^{-\frac{n}{2}}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x - \eta|^2 + (\xi - \zeta - cT + 2\lambda T)^2}{4T}} e^{-\lambda\zeta} (u_0(\eta, \zeta) - \phi(\zeta - 1)) d\eta d\zeta \\ &\leq \frac{A}{(4\pi T)^{-\frac{n}{2}}} \int_{|\eta| + \zeta \geq \bar{R}, \zeta \geq 0} e^{-\frac{|x - \eta|^2 + (\xi - \zeta - cT + 2\lambda T)^2}{4T}} (u_0(\eta, \zeta) - \phi(\zeta - 1)) d\eta d\zeta \\ &\quad + \frac{A}{(4\pi T)^{-\frac{n}{2}}} \int_{|\eta| - \zeta \geq R, \zeta \leq 0} e^{-\frac{|x - \eta|^2 + (\xi - \zeta - cT + 2\lambda T)^2}{4T}} (u_0(\eta, \zeta) - \phi(\zeta - 1)) e^{-\lambda\zeta} d\eta d\zeta \\ &\quad + \frac{A}{(4\pi T)^{-\frac{n}{2}}} \int_{|\eta| + |\zeta| \leq \bar{R}} e^{-\frac{|x - \eta|^2 + (\xi - \zeta - cT + 2\lambda T)^2}{4T}} (u_0(\eta, \zeta) - \phi(\zeta - 1)) e^{-\lambda\zeta} d\eta d\zeta \\ &:= I_1 + I_2 + I_3, \end{aligned}$$

where $A = e^{\lambda T(\lambda - c)}$. By using (2.16), it is easy to see that $I_i \leq \frac{1}{6} e^{-MT} \varepsilon$ ($i = 1, 2$). Due to the fact that

$$\lim_{R \rightarrow \infty} \sup_{|x| + |\xi| \geq R} \exp\left(-\frac{|x - \eta|^2 + (\xi - \zeta - cT \pm 2\lambda T)^2}{4T}\right) = 0$$

for any $|\eta|, |\zeta| < \bar{R}$, there exists a constant $\hat{R} > 0$ such that, for $|x| + |\xi| \geq \hat{R}$, $I_3 \leq \frac{1}{6} e^{-MT} \varepsilon$. Therefore, we have, for $R \geq \hat{R}$ and $\xi \leq 0$,

$$w(x, \xi, T) e^{-\lambda\xi} \leq e^{MT} (I_1 + I_2 + I_3) \leq \varepsilon;$$

that is,

$$\lim_{R \rightarrow \infty} \sup_{|x| - \xi \geq R, \xi \leq 0} |w(x, \xi, T)| e^{-\lambda\xi} = 0$$

for any fixed $T > 0$. Similarly, we can prove that $\lim_{R \rightarrow \infty} \sup_{|x| + \xi \geq R, \xi \geq 0} |w(x, \xi, T)| = 0$ $\square$

Proof of Theorem 1.1. We only show an upper estimate, since a lower estimate is obtained in a similar way. First we take constants $\sigma \geq 1$ and $k > 0$ as in Lemma 2.6.

Let a constant $\varepsilon > 0$ be arbitrarily fixed (for simplicity, we assume that $\varepsilon < 1$) and define a constant $\hat{\varepsilon} = \varepsilon/(4\|\phi'\|_{L^\infty(\mathbb{R})} + 1)$. Since $f(u) < 0$ for $u > 1$ by assumption (F), there exists a constant $T_1 \geq 0$ such that

$$u(x, \xi, T_1) \leq 1 + \frac{\hat{\varepsilon}}{\sigma}, \quad (x, \xi) \in \mathbb{R}^n.$$

Furthermore, Lemma 2.8 implies that there exists a constant $R > 0$ which satisfies

$$\begin{aligned} \sup_{|x|-\xi \geq R, \xi \leq 0} |u(x, \xi, T_1) - \phi(\xi - 1)|e^{-\lambda\xi} &\leq \frac{\hat{\varepsilon}}{\sigma}\nu, \\ \sup_{|x|+\xi \geq R, \xi \geq 0} |u(x, \xi, T_1) - \phi(\xi - 1)| &\leq \frac{\hat{\varepsilon}}{\sigma}\nu, \end{aligned}$$

where $\nu = \phi(-N_1)$ and constants $N_1 > 0$ are defined by

$$\phi(\xi) \geq e^{\lambda\xi} \quad \text{for } \xi \in (-\infty, -N_1].$$

By Lemma 2.1, it is easy to verify the existence of N_1 . Then we can take a function $v_0(x)$ which satisfies $\lim_{|x| \rightarrow \infty} v_0(x) = 0$ and

$$u(x, \xi, T_1) \leq (1 + \frac{\hat{\varepsilon}}{\sigma})\phi(\xi - v_0(x) - \hat{\varepsilon}), \quad (x, \xi) \in \mathbb{R}^n.$$

Now we consider a function $v(x, t)$ which satisfies (2.9). Lemma 2.3 implies that there exists a constant $T_2 \geq 0$ such that $v(x, t) \leq \hat{\varepsilon}$ for $t \geq T_2$. Consequently, by using the comparison principle and the upper solution constructed in Lemma 2.6, we have

$$\begin{aligned} u(x, \xi, t) &\leq (1 + \hat{\varepsilon}e^{-\beta(t-T_1)})\phi(\xi - v(x, t - T_1) - \hat{\varepsilon}e^{-\beta(t-T_1)}) \\ &\leq (1 + \hat{\varepsilon})\phi(\xi - 2\hat{\varepsilon}) \\ &\leq (1 + \hat{\varepsilon})(\phi(\xi) + 2\|\phi'(\xi)\|_{L^\infty(\mathbb{R})}\hat{\varepsilon}) \\ &\leq \phi(\xi) + (4\|\phi'(\xi)\|_{L^\infty(\mathbb{R})} + 1)\hat{\varepsilon} \\ &\leq \phi(\xi) + \varepsilon \end{aligned}$$

for $t \geq T_1 + T_2$. This completes the proof of Theorem 1.1. $\square$

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REFERENCES

1. X. F. Chen and J.-S. Guo, *Existence and asymptotic stability of traveling waves of discrete quasilinear monostable equations*, J. Differential Equations **184** (2002), 549–569. MR1929888 (2003h:35107)
2. R. A. Fisher, *The advance of advantageous genes*, Ann. Eugenics **7** (1937), 355–369.
3. T. Kapitula, *Multidimensional stability of planar traveling waves*, Trans. Amer. Math. Soc. **349** (1997), 257–269. MR1360225 (97d:35104)
4. A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, *Etude de l'equation de la diffusion avec croissance de la quantite de matiere et son application a un probleme biologique*, Bull. Univ. Moskov. Ser. Internat. Sect. **1** (1937), 1–25.
5. C. D. Levermore and J. X. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation II*, Comm. Partial Differential Equations **17** (1992), 1901–1924. MR1194744 (94c:35105)
6. G. Y. Lv and M. X. Wang, *Nonlinear stability of travelling wave fronts for delayed reaction diffusion equations*, Nonlinearity **23** (2010), 845–873. MR2602017

7. H. Matano, M. Nara and M. Taniguchi, *Stability of planar waves in the Allen-Cahn equation*, Comm. Partial Differential Equations **34** (2009), 976–1002. MR2560308 (2010j:35289)
8. G. Raugel and K. Kirchgässner, *Stability of fronts for a KPP-system, II: The critical case*, J. Differential Equations **146** (1998), 399–456. MR1631295 (2000j:35133)
9. D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Adv. Math **22** (1976), 312–355. MR0435602 (55:8561)
10. A. I. Volpert, V. A. Volpert and V. A. Volpert, *Traveling wave solutions of parabolic systems*, Translation of Mathematical Monographs, Vol. 140, Amer. Math. Soc., Providence, RI, 1994. MR1297766 (96c:35092)
11. J. X. Xin, *Multidimensional stability of traveling waves in a bistable reaction-diffusion equation I*, Comm. Partial Differential Equations **17** (1992) 1889–1899. MR1194743 (94c:35104)

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