A DIFFERENTIAL GEOMETRIC CHARACTERIZATION OF THE CAYLEY HYPERSURFACE

ZEJUN HU, CECE LI, AND DONG ZHANG

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ABSTRACT. The so-called Cayley hypersurface, constructed by Eastwood and Ezhov, is a higher-dimensional extension of the classical Cayley surface. In this paper, we establish a differential geometric characterization of the Cayley hypersurface, which is an answer to Eastwood and Ezhov's question.

1. INTRODUCTION

Let $\mathbb{R}^{n+1}$ be the $(n+1)$ -dimensional real unimodular-affine space equipped with its canonical flat connection D and a parallel volume form θ . For a non-degenerate equiaffine hypersurface M of $\mathbb{R}^{n+1}$, the affine normal ξ induces an affine connection ∇ and a semi-Riemannian *affine metric* h . Let $C = \nabla h$ be the cubic form. Then we have the Pick-Berwald theorem which states that $C = 0$ implies that M is locally a non-degenerate quadric. Besides non-degenerate quadrics, the next simplest one might be the Cayley surface in $\mathbb{R}^3$, given by

$$x_3 = x_1x_2 - \frac{1}{3}x_1^3,$$

which is non-degenerate and affine homogeneous ([14], p. 119). In [13], Nomizu and Pinkall showed that the Cayley surface can be characterized, up to an affine congruence, as the unique affine surface whose cubic form is non-zero and is parallel relative to the induced affine connection, i.e., $C \neq 0$ and $\nabla C = 0$.

For general dimensions, Eastwood and Ezhov [7] exhibited a homogeneous hypersurface in $\mathbb{R}^{n+1}$, named as *Cayley hypersurface*, defined by

$$(1) \quad \Phi(x_1, \dots, x_{n+1}) \equiv \sum_{k=1}^{n+1} (-1)^k \frac{1}{k} \sum_{i_1 + \dots + i_k = n+1} x_{i_1} x_{i_2} \cdots x_{i_k} = 0.$$

This is the Cayley surface when $n = 2$. Obviously, the *generalized* Cayley hypersurface is the graph of $x_{n+1} = \Psi(x_1, \dots, x_n)$, where Ψ is a polynomial function of

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degree $(n + 1)$ in $\mathbb{R}^n$ defined by

$$\Psi(x_1, \dots, x_n) \equiv \sum_{k=2}^{n+1} (-1)^k \frac{1}{k} \sum_{i_1 + \dots + i_k = n+1} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Eastwood and Ezhov have shown that the Cayley hypersurface has remarkable properties:

Proposition ([7]). *For an n -dimensional Cayley hypersurface, the following hold:*

- (a) *It is affine homogeneous and has vanishing Pick invariant.*
- (b) *It is uniquely ruled by $\lfloor \frac{n}{2} \rfloor$ planes.*
- (c) *Its affine normals are everywhere parallel.*
- (d) *Its affine automorphism group contains a transitive abelian subgroup.*
- (e) *The isotropy subgroup of its affine automorphism group is 1-dimensional.*

Eastwood and Ezhov [7] further conjectured that the last three properties of the Proposition should characterize the Cayley hypersurface. This conjecture was proved by Choi and Kim [2] under the additional condition that the domain in $\mathbb{R}^{n+1}$, being bounded by the hypersurface which is expressed as a graph of a function defined on $\mathbb{R}^n$, is affine homogeneous. On the other hand, Eastwood and Ezhov [7] raised an interesting question as follows:

How do we extend Nomizu and Pinkall's differential geometric characterization of the Cayley surface to the higher-dimensional Cayley hypersurface?

In this paper, we will show that, as an improper affine hypersphere with vanishing Pick invariant, the Cayley hypersurface has a flat affine metric and satisfies $\hat{\nabla}C = 0$ and $K^{n-1} \neq 0$, where $\hat{\nabla}$ is the Levi-Civita connection of the affine metric and K is the difference tensor defined by $K(X, Y) = K_X Y = \nabla_X Y - \hat{\nabla}_X Y$. Most importantly, we will show that the converse holds. This establishes a differential geometric characterization for the Cayley hypersurface and gives an affirmative answer to Eastwood and Ezhov's question. Precisely, we will prove the following.

Main Theorem. *Let M be a non-degenerate affine hypersurface of $\mathbb{R}^{n+1}$ with vanishing Pick invariant. If $\hat{\nabla}C = 0$, $K^{n-1} \neq 0$ and the affine metric h is of constant sectional curvature, then M is locally affine equivalent to the Cayley hypersurface.*

Remark 1.1. To make a comparison with Nomizu and Pinkall's characterization of the Cayley surface, we point out that it was shown in [4] and [15] that a hypersurface satisfying $C \neq 0$ and $\nabla C = 0$ is a graph immersion of a polynomial function of degree at most 3. Hence we cannot use the condition $\nabla C = 0$ to characterize the higher-dimensional Cayley hypersurface. Related to the Main Theorem, we mention an interesting result of F. Dillen and L. Vrancken [5] which states that the conditions $\nabla K = 0$ and $K^{n-1} \neq 0$ indeed characterize a hypersurface of $\mathbb{R}^{n+1}$, which is also a flat improper affine hypersphere and is given by $x_{n+1} = G(x_1, \dots, x_n)$, where

$$(2) \quad G(x_1, \dots, x_n) = \sum_{k=2}^{n+1} \frac{(-2)^{k-2}}{k!} \sum_{i_1 + \dots + i_k = n+1} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Remark 1.2. The Cayley hypersurface is an improper affine hypersphere which has constant affine sectional curvature and vanishing Pick invariant. Closely related to this fact, we would point out that L. Vrancken [16] has classified affine hyperspheres with constant affine sectional curvature and non-vanishing Pick invariant.

Remark 1.3. Many researches have been carried out with the condition $\hat{\nabla}C = 0$. For locally strongly convex (i.e. h is definite) affine hypersurfaces with $\hat{\nabla}C = 0$, after continuous efforts by F. Dillen, L. Vrancken et al. and obtaining the classifications for low dimensions ([12], [3], [6], [9], [10]), Z.J. Hu, H. Li and L. Vrancken [11] eventually obtained a complete classification. For general non-degenerate affine hypersurfaces with $\hat{\nabla}C = 0$ and the fact that h is not definite, the classification problem remains open. Besides [12] for $n = 2$, the first two authors recently solved the problem for the 3-dimensional case [8].

2. DIFFERENTIAL GEOMETRIC PROPERTIES OF CAYLEY HYPERSURFACES

In this section, we will show that the Cayley hypersurface has the following remarkable differential geometric properties:

Proposition 2.1. *The Cayley hypersurface in $\mathbb{R}^{n+1}$ is a flat improper affine hypersphere which has the properties that $\hat{\nabla}C = 0$ and $K^{n-1} \neq 0$.*

First of all, we will briefly recall some basic formulas in affine differential geometry. For more details, we refer to [14]. For a non-degenerate affine hypersurface $F : M \hookrightarrow \mathbb{R}^{n+1}$, let S be the *affine shape operator* (or *Weingarten operator*). Then, for any tangent vector fields X and Y , we have the Gauss and Weingarten formulas

$$D_X Y = \nabla_X Y + h(X, Y)\xi, \quad D_X \xi = -SX.$$

M is called an *affine hypersphere* if $S = \lambda id$, and one easily proves that $\lambda = \text{const}$ for $n \geq 2$. F is called a *proper affine hypersphere* if $\lambda \neq 0$ and an *improper affine hypersphere* if $\lambda = 0$. For an improper affine hypersphere the affine normal ξ is constant.

The curvature tensor $\hat{R}$ of $\hat{\nabla}$ is related to S and K by the Gauss equation

$$\hat{R}(X, Y)Z = \frac{1}{2}[h(Y, Z)SX - h(X, Z)SY + h(SY, Z)X - h(SX, Z)Y] - [K_X, K_Y]Z.$$

In particular, for affine hyperspheres we have $S = \lambda id$, and thus

$$(3) \quad \hat{R}(X, Y)Z = \lambda(h(Y, Z)X - h(X, Z)Y) - [K_X, K_Y]Z.$$

Then we have the relation

$$(4) \quad \chi = \lambda + J,$$

where $J = \frac{1}{n(n-1)}h(K, K)$ is the so-called Pick invariant and χ is the normalized scalar curvature of the affine metric h . From the well-known relation $C(X, Y, Z) = -2h(K(X, Y), Z)$ we see that $\hat{\nabla}C = 0$ is equivalent to $\hat{\nabla}K = 0$. Moreover, K satisfies the *apolarity condition*, i.e., $\text{tr } K_X = 0$ for all X , and it has the property that $h(K(X, Y), Z)$ is totally symmetric in X, Y and Z .

To prove Proposition 2.1, we notice that from the affine homogeneity of the Cayley hypersurface, it is sufficient to show that the conclusion holds around some point $p = (x_1^0, x_2^0, \dots, x_{n+1}^0) \in \mathbb{R}^{n+1}$, where x_i^0 ($i = 1, 2, \dots, n+1$) are all sufficient small positive numbers.

We will need the following.

Lemma 2.1. *The Cayley hypersurface (1) can be parameterized, around $p \in \mathbb{R}^{n+1}$, by*

$$(5) \quad \begin{aligned} x_1 &= v_1, \quad x_2 = v_2 + \frac{1}{2!}v_1^2, \quad \dots, \quad x_n = \sum_{i_1+2i_2+\dots+ni_n=n} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n}, \\ x_{n+1} &= \sum_{i_1+2i_2+\dots+ni_n=n+1} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n}, \end{aligned}$$

where v_i ($i = 1, 2, \dots, n$) are all sufficient small positive numbers.

Proof. It is sufficient to show that the $(n+1)$ functions $\{x_i\}_1^{n+1}$ in (5) satisfy the relation

$$(6) \quad x_{n+1} = \sum_{k=2}^{n+1} \frac{(-1)^k}{k} \sum_{i_1+\dots+i_k=n+1} x_{i_1} x_{i_2} \dots x_{i_k}.$$

For $k = 1, 2, \dots, n$, we set $v_k = u_k^k$. Then (5) can be rewritten as

$$\begin{aligned} x_k &= \sum_{i_1+2i_2+\dots+ki_k=k} \frac{1}{i_1!i_2!\dots i_k!} u_1^{i_1} u_2^{2i_2} \dots u_k^{ki_k}, \quad k = 1, 2, \dots, n, \\ x_{n+1} &= \sum_{i_1+2i_2+\dots+ni_n=n+1} \frac{1}{i_1!i_2!\dots i_n!} u_1^{i_1} u_2^{2i_2} \dots u_n^{ni_n}. \end{aligned}$$

Using Taylor expansion we obtain

$$\begin{aligned} e^{u_1+u_2^2+\dots+u_n^n} &= e^{u_1} e^{u_2^2} \dots e^{u_n^n} \\ &= \left(1 + u_1 + \frac{u_1^2}{2!} + \dots\right) \left(1 + u_2^2 + \frac{u_2^{2 \times 2}}{2!} + \dots\right) \dots \left(1 + u_n^n + \frac{u_n^{2 \times n}}{2!} + \dots\right) \\ &= 1 + u_1 + \left(\frac{u_1^2}{2!} + u_2^2\right) + \left(\frac{u_1^3}{3!} + u_1 u_2^2 + u_3^3\right) + \dots \\ (7) \quad &= 1 + \sum_{k=1}^n \sum_{i_1+2i_2+\dots+ki_k=k} \frac{1}{i_1!i_2!\dots i_k!} u_1^{i_1} u_2^{2i_2} \dots u_k^{ki_k} \\ &\quad + \sum_{k=n+1}^{\infty} \sum_{i_1+2i_2+\dots+ni_n=k} \frac{1}{i_1!i_2!\dots i_n!} u_1^{i_1} u_2^{2i_2} \dots u_n^{ni_n} \\ &:= 1 + \sum_{k=1}^{\infty} f_k, \end{aligned}$$

where f_k is a homogeneous polynomial of degree k . Obviously, for $j = 1, \dots, n+1$, we have $f_j = x_j$, and thus relation (6) is equivalent to

$$f_{n+1} = \sum_{k=2}^{n+1} \frac{(-1)^k}{k} \sum_{i_1+\dots+i_k=n+1} f_{i_1} f_{i_2} \dots f_{i_k}.$$

On the other hand, from (7) and the fact that $\sum_{k=1}^{\infty} f_k$ is positive and small, we get

$$\begin{aligned} \sum_{k=1}^n u_k^k &= \ln \left(1 + \sum_{k=1}^{\infty} f_k\right) \\ &= (f_1 + f_2 + \dots) - \frac{(f_1 + f_2 + \dots)^2}{2} + \dots + (-1)^{k-1} \frac{(f_1 + f_2 + \dots)^k}{k} + \dots. \end{aligned}$$

From this and by comparing terms of degree $n+1$ in $\{u_i\}_1^n$ on both sides, we obtain

$$\begin{aligned} 0 &= f_{n+1} - \frac{1}{2} \sum_{i_1+i_2=n+1} f_{i_1} f_{i_2} + \cdots + \frac{(-1)^n}{n+1} \sum_{i_1+\cdots+i_{n+1}=n+1} f_{i_1} f_{i_2} \cdots f_{i_{n+1}} \\ &= f_{n+1} - \sum_{k=2}^{n+1} \frac{(-1)^k}{k} \sum_{i_1+\cdots+i_k=n+1} f_{i_1} f_{i_2} \cdots f_{i_k}, \end{aligned}$$

and the conclusion follows. $\square$

Proof of Proposition 2.1. Lemma 2.1 shows that around $p \in \mathbb{R}^{n+1}$, the Cayley hypersurface can be seen as

$$F: \mathbb{R}^n \supset U \longrightarrow \mathbb{R}^{n+1},$$

where

$$\begin{aligned} F(v_1, \dots, v_n) &= (v_1, v_2 + \frac{1}{2!}v_1^2, \dots, \sum_{i_1+2i_2+\cdots+ni_n=n} \frac{1}{i_1!i_2!\cdots i_n!} v_1^{i_1} v_2^{i_2} \cdots v_n^{i_n}, \\ &\quad \sum_{i_1+2i_2+\cdots+ni_n=n+1} \frac{1}{i_1!i_2!\cdots i_n!} v_1^{i_1} v_2^{i_2} \cdots v_n^{i_n}). \end{aligned}$$

From (5), direct computations give that

$$(8) \quad \frac{\partial x_k}{\partial v_j} = \begin{cases} x_{k-j}, & j < k, \\ 1, & j = k, \\ 0, & j > k, \end{cases} \quad \frac{\partial x_{n+1}}{\partial v_j} = x_{n-j+1}, \quad k, j = 1, \dots, n.$$

Denote by $\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}$ the coordinate vector fields and $F_*(\frac{\partial}{\partial v_j}) = F_{v_j}$. Then

$$(9) \quad F_{v_j} = (\overbrace{0, \dots, 0}^{j-1}, 1, x_1, \dots, x_{n-j+1}), \quad j = 1, \dots, n.$$

Let us take $\xi = (0, \dots, 0, 1)$ as a tentative transversal vector field. For the corresponding volume element θ we have $\theta(\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}) = \det[F_{v_1}, \dots, F_{v_n}, \xi] = 1$, so that $\{\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}\}$ is a unimodular basis. From (8) and (9), we have

$$D_{\frac{\partial}{\partial v_i}} F_{v_j} = \begin{cases} F_{v_{i+j}}, & i+j < n+1, \\ \xi, & i+j = n+1, \\ 0, & i+j > n+1. \end{cases}$$

Hence, we obtain that

$$(10) \quad \nabla_{\frac{\partial}{\partial v_i}} \frac{\partial}{\partial v_j} = \begin{cases} \frac{\partial}{\partial v_{i+j}}, & i+j \leq n, \\ 0, & \text{otherwise,} \end{cases} \quad h_{ij} = h(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}) = \begin{cases} 1, & i+j = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\det_\theta h = \det[h_{ij}] = (-1)^{n(n-1)/2}$, so ξ is exactly the affine normal and the Cayley hypersurface is an improper affine hypersphere. From (10) and $C = \nabla h$ we have

$$C(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}, \frac{\partial}{\partial v_l}) = \begin{cases} -2, & i+j+l = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

According to $C(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}, \frac{\partial}{\partial v_l}) = -2h(K\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}, \frac{\partial}{\partial v_l})$ and $(h^{ij}) = (h_{ij})$, we easily get

$$(11) \quad K\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j} = \begin{cases} \frac{\partial}{\partial v_{i+j}}, & i+j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

From $\hat{\nabla}_X Y = \nabla_X Y - K_X Y$ we then obtain $\hat{\nabla}\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j} = 0$ for all i, j .

It follows that the Cayley hypersurface is a flat improper affine hypersphere and satisfies $\hat{\nabla}C = 0$. From (11), we get $(K\frac{\partial}{\partial v_1})^{n-1}\frac{\partial}{\partial v_1} = \frac{\partial}{\partial v_n}$. Hence $K^{n-1} \neq 0$. $\square$

3. A DIFFERENTIAL GEOMETRIC CHARACTERIZATION OF CAYLEY HYPERSURFACES

In this section we want to show the converse, thus completing the proof of the Main Theorem.

Proposition 3.1. *Let M be a non-degenerate affine hypersurface of $\mathbb{R}^{n+1}$ with constant affine sectional curvature a and $\hat{\nabla}C = 0$. Then either M is a quadric (i.e. $C = 0$) or M is a flat affine hypersphere.*

Proof. From the assumption, we have

$$(12) \quad \hat{R}(X, Y)Z = a(h(Y, Z)X - h(X, Z)Y),$$

whereas the condition $\hat{\nabla}C = 0$, or equivalently $\hat{\nabla}K = 0$, implies that M is an affine hypersphere [1] and that for all X, Y, Z and W , we have

$$(13) \quad \hat{R}(X, Y)K(Z, W) = K(\hat{R}(X, Y)Z, W) + K(Z, \hat{R}(X, Y)W).$$

Let us choose a local orthonormal basis $\{e_1, \dots, e_n\}$ such that $h(e_i, e_j) = \epsilon_i \delta_{ij}$ and $\epsilon_i = \pm 1$. Then from (13), we get

$$h(e_i, \hat{R}(e_i, e_j)K(e_i, e_i)) = 2h(e_i, K(\hat{R}(e_i, e_j)e_i, e_i)),$$

and (12) then implies that for $i \neq j$,

$$(14) \quad 0 = h(e_i, \hat{R}(e_i, e_j)K_{e_i}e_i - 2K_{e_i}\hat{R}(e_i, e_j)e_i) = 3a\epsilon_i h(K_{e_i}e_i, e_j).$$

This shows that we have either $a = 0$ or $a \neq 0$ and $K_{e_i}e_i = \mu_i e_i$ for all i . In the latter case, the apolarity condition implies that $\mu_j = \text{tr } K_{e_j} = 0$ for each j . Therefore, we get $K = 0$ and thus $C = 0$. $\square$

Let us now consider a non-degenerate affine hypersurface M of $\mathbb{R}^{n+1}$ which has constant affine sectional curvature and satisfies the conditions $\hat{\nabla}C = 0$, $J = 0$ and $K^{n-1} \neq 0$. From (3), (4) and Proposition 3.1, we see that M is a flat improper affine hypersphere which satisfies $K_X K_Y = K_Y K_X$ for all X and Y . We will need the following.

Lemma 3.1 (Lemma 3.3 of [5]). *If $K_X K_Y = K_Y K_X$, then K_X is nilpotent. In particular, $(K_X)^n = 0$.*

Having that $K \neq 0$ and $K_X K_Y = K_Y K_X$ for all X and Y , we can define, for every natural number m , a totally symmetric tensor K^m by

$$K^m(X_1, \dots, X_m, X_{m+1}) = K_{X_1} K_{X_2} \cdots K_{X_m} X_{m+1}.$$

Then K^m vanishes identically if and only if $(K_v)^m v = 0$ for any vector v . By assumption, $K^{n-1} \neq 0$ at a point p . Then (see Remark 3.3 in [5]), for any tangent vector v at p , we have $(K_v)^n v = 0$ and there exists a tangent vector u at p such that $h((K_u)^{n-1} u, u) \neq 0$.

Define

$$(15) \quad y_1 = u, \quad y_i = (K_u)^{i-1} u, \quad i = 2, \dots, n.$$

Then $y_n \neq 0$. The facts $K_u y_i = y_{i+1}$ and $K_u y_n = 0$ imply that $y_1, y_2, \dots, y_n$ are n linearly independent vectors. Since K_u is a symmetric operator with respect to h , we also get that

$$(16) \quad h(y_i, y_j) = \begin{cases} h(y_{i+j-1}, y_1), & i+j \leq n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Put $\alpha_k = h(y_k, y_1)$, $1 \leq k \leq n$. As $\alpha_n = h((K_u)^{n-1} u, u) \neq 0$, by re-scaling u , and if necessary changing the sign of ξ , we may assume that $\alpha_n = 1$.

Lemma 3.2. *Under conditions as stated in the Main Theorem, M is a flat improper affine hypersphere. Moreover, if $K^{n-1} \neq 0$ at $p \in M$, then there exists a basis $\{x_1, \dots, x_n\}$ of $T_p M$ such that*

$$(17) \quad K_{x_i} x_j = \begin{cases} x_{i+j}, & i+j \leq n, \\ 0, & \text{otherwise,} \end{cases} \quad h(x_i, x_j) = \begin{cases} 1, & i+j = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The first assertion is a consequence of Proposition 3.1. To prove the second assertion, we choose $u \in T_p M$ such that $h((K_u)^{n-1} u, u) = 1$. Let $y_1, \dots, y_n$ be the basis of $T_p M$ as constructed in (15). Given an arbitrary real number β , set $x_1 = y_1 + \beta y_2$ and $x_j = K_{x_1} x_{j-1}$ for all $j = 2, \dots, n$. Then

$$\begin{aligned} x_2 &= y_2 + 2\beta y_3 + \dots, \quad x_3 = y_3 + 3\beta y_4 + \dots, \quad \dots, \\ x_{n-1} &= y_{n-1} + (n-1)\beta y_n, \quad x_n = y_n. \end{aligned}$$

It follows that $h(x_1, x_n) = h(y_1, y_n) = 1$ and $h(x_1, x_{n-1}) = h(y_1, y_{n-1}) + n\beta$. Then we can choose β such that $h(x_1, x_{n-1}) = 0$. Hence, we may assume that $\alpha_{n-1} = 0$.

Next we assume that we have a basis $\{y_1, \dots, y_n\}$ for which $\alpha_n = 1$ and $\alpha_{n-1} = \dots = \alpha_{n-i} = 0$, for some $i \geq 1$. Then by putting $x_1 = y_1 + \beta y_{2+i}$ and $x_j = K_{x_1} x_{j-1}$ for all $j = 2, \dots, n$, we get

$$\begin{aligned} x_2 &= y_2 + 2\beta y_{3+i} + \dots, \\ x_3 &= y_3 + 3\beta y_{4+i} + \dots, \quad \dots, \\ x_{n-j} &= y_{n-j} + (n-j)\beta y_{n-j+i+1} + \dots, \quad \dots, \\ x_{n-i-1} &= y_{n-i-1} + (n-i-1)\beta y_n, \\ x_{n-i} &= y_{n-i}, \quad \dots, \quad x_n = y_n. \end{aligned}$$

It is easily seen that

$$\begin{aligned} h(x_1, x_n) &= h(y_1, y_n) = \alpha_n = 1, \\ h(x_1, x_{n-j}) &= h(y_1, y_{n-j}) = \alpha_{n-j} = 0, \quad 1 \leq j \leq i, \\ h(x_1, x_{n-i-1}) &= h(y_1, y_{n-i-1}) + (n-i)\beta. \end{aligned}$$

Hence, by choosing β appropriately, we may also assume that $\alpha_{n-i-1} = 0$.

Continuing this process, using $K_{y_i} y_j = (K_{y_1})^{i+j-1} y_1$ and (16), we complete the proof. $\square$

Remark 3.1. The second assertion of Lemma 3.2 is similar to Dillen and Vrancken's Lemma 6.1 in [5], where the conclusion is obtained under the conditions $\nabla K = 0$ and $K^{n-1} \neq 0$.

Proof of the Main Theorem. Lemma 3.2 shows that M is an h -flat improper affine hypersphere. As M is flat, we can extend the basis chosen in Lemma 3.2 to local parallel flat coordinate vector fields $\frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_n}$, with corresponding coordinates $(v_1, \dots, v_n)$, the point p corresponding to 0. Then we have

$$h\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = \begin{cases} 1, & i + j = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$K\left(\frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_j}\right) = \sum_{\ell=1}^n A_{ij}^{\ell} \frac{\partial}{\partial v_{\ell}}.$$

Since $\hat{\nabla} K = 0$ and $\hat{\nabla} \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} = 0$, the initial conditions (17) at p shows that A_{ij}^{ℓ} are all constant and satisfy

$$A_{ij}^{\ell} = \begin{cases} 1, & \ell = i + j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\frac{\partial^k F}{\partial v_{i_1} \dots \partial v_{i_k}} = F_{v_{i_1} \dots v_{i_k}}$. Due to the fact that the affine normal ξ of improper affine hypersphere is constant and that $D_X Y = K_X Y + \hat{\nabla}_X Y + h(X, Y)\xi$, the immersion F is determined by the following system of differential equations:

$$F_{v_i v_j} = \begin{cases} F_{v_{i+j}}, & i + j < n + 1, \\ \xi, & i + j = n + 1, \\ 0, & i + j > n + 1. \end{cases}$$

This is equivalent to

$$(18) \quad F_{v_{i_1} \dots v_{i_k}} = \begin{cases} F_{v_{i_1 + \dots + i_k}}, & i_1 + \dots + i_k < n + 1, \\ \xi, & i_1 + \dots + i_k = n + 1, \\ 0, & i_1 + \dots + i_k > n + 1. \end{cases}$$

Set $A_{n+1} = \xi$ and $A_i = F_{v_i}(0) = y_i$ for all $i = 1, \dots, n$. Then

$$F_{v_{i_1} \dots v_{i_k}}(0) = A_{i_1 + \dots + i_k}, \quad \text{if } i_1 + \dots + i_k \leq n + 1.$$

By using Taylor expansion and (18), we obtain, up to an affine translation,

$$(19) \quad \begin{aligned} F(v_1, \dots, v_n) = & \left(\sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \right) F|_0 + \frac{1}{2!} \left(\sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \right)^2 F|_0 + \dots \\ & + \frac{1}{n!} \left(\sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \right)^n F|_0 + \frac{1}{(n+1)!} \left(\sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \right)^{n+1} F|_0. \end{aligned}$$

By using (18), we easily see the following:

$$\begin{aligned}
 \left(\sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \right) F|_0 &= \sum_{i=1}^n v_i A_i, \\
 \frac{1}{\ell!} \left(\sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \right)^\ell F|_0 &= \frac{1}{\ell!} \sum_{k=\ell}^{n+1} \left(\sum_{j_1+j_2+\dots+j_\ell=k} v_{j_1} v_{j_2} \dots v_{j_\ell} \right) A_k \\
 &= \sum_{k=\ell}^{n+1} \left(\sum_{\substack{i_1+\dots+i_n=\ell \\ i_1+2i_2+\dots+ni_n=k}} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n} \right) A_k, \quad 2 \leq \ell \leq n, \\
 \left(\sum_{i=1}^n v_i \frac{\partial}{\partial v_i} \right)^{n+1} F|_0 &= v_1^{n+1} A_{n+1}.
 \end{aligned}$$

From the above calculations, the right hand side of (19) becomes a combination of A_{n+1} and A_k ($k = 1, \dots, n$) with coefficients as follows:

$$\sum_{\ell=2}^{n+1} \sum_{\substack{i_1+\dots+i_n=\ell \\ i_1+2i_2+\dots+ni_n=n+1}} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n} = \sum_{i_1+2i_2+\dots+ni_n=n+1} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n}$$

and

$$\begin{aligned}
 \sum_{\ell=1}^k \sum_{\substack{i_1+\dots+i_n=\ell \\ i_1+2i_2+\dots+ni_n=k}} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n} &= \sum_{\ell=1}^k \sum_{\substack{i_1+\dots+i_k=\ell \\ i_1+2i_2+\dots+ki_k=k}} \frac{1}{i_1!i_2!\dots i_k!} v_1^{i_1} v_2^{i_2} \dots v_k^{i_k} \\
 &= \sum_{i_1+2i_2+\dots+ki_k=k} \frac{1}{i_1!i_2!\dots i_k!} v_1^{i_1} v_2^{i_2} \dots v_k^{i_k},
 \end{aligned}$$

where we use the fact that all indices $i_1, \dots, i_n$ are non-negative integers. Hence, we can finally rewrite (19) as follows:

$$\begin{aligned}
 F(v_1, \dots, v_n) &= v_1 A_1 + (v_2 + \frac{1}{2!} v_1^2) A_2 + (v_3 + v_2 v_1 + \frac{1}{3!} v_1^3) A_3 + \dots \\
 &+ \left(\sum_{i_1+2i_2+\dots+ki_k=k} \frac{1}{i_1!i_2!\dots i_k!} v_1^{i_1} v_2^{i_2} \dots v_k^{i_k} \right) A_k + \dots \\
 &+ \left(\sum_{i_1+2i_2+\dots+ni_n=n} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n} \right) A_n \\
 &+ \left(\sum_{i_1+2i_2+\dots+ni_n=n+1} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n} \right) A_{n+1}.
 \end{aligned}$$

Since $A_1, \dots, A_{n+1}$ are constant linearly independent vectors, up to an affine transformation, we can write

$$\begin{aligned}
 F(v_1, \dots, v_n) &= \left(v_1, v_2 + \frac{1}{2!} v_1^2, \dots, \sum_{i_1+2i_2+\dots+ni_n=n} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n}, \right. \\
 &\quad \left. \sum_{i_1+2i_2+\dots+ni_n=n+1} \frac{1}{i_1!i_2!\dots i_n!} v_1^{i_1} v_2^{i_2} \dots v_n^{i_n} \right).
 \end{aligned}$$

According to Lemma 2.1, M is locally affine equivalent to the Cayley hypersurface. $\square$

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DEPARTMENT OF MATHEMATICS, ZHENGZHOU UNIVERSITY, ZHENGZHOU 450052, PEOPLE'S REPUBLIC OF CHINA

E-mail address: huzj@zzu.edu.cn

DEPARTMENT OF MATHEMATICS, ZHENGZHOU UNIVERSITY, ZHENGZHOU 450052, PEOPLE'S REPUBLIC OF CHINA

E-mail address: ceceli@sina.com

DEPARTMENT OF MATHEMATICS, ZHENGZHOU UNIVERSITY, ZHENGZHOU 450052, PEOPLE'S REPUBLIC OF CHINA

E-mail address: zd20082100333@163.com