SUPER-ADDITIVE SEQUENCES AND ALGEBRAS OF POLYNOMIALS

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ABSTRACT. If K is a field with discrete valuation ν and $D = \{a \in K : \nu(a) \geq 0\}$, then an algebra $D[x] \subseteq A \subseteq K[x]$ has associated to it a sequence of fractional ideals $\{\mathcal{I}_n : n = 0, 1, 2, \dots\}$ with $\mathcal{I}_n$ consisting of 0 and the leading coefficients of elements of A of degree no more than n and a sequence of integers $\{a(n) : n = 0, 1, 2, \dots\}$ with $a(n) = -\nu(\mathcal{I}_n)$. Combinatorial properties of this integer sequence reflect algebraic properties of A , and these are used to identify the degrees of generators of A and to characterize finitely generated algebras A by a periodicity property of this sequence.

1. INTRODUCTION

Let K be a field with discrete valuation $\nu : K^* \rightarrow \mathbb{Z}$ and $D = \{a \in K : \nu(a) \geq 0\}$ the corresponding valuation domain. Let $\mathcal{M} = \{a \in D : \nu(a) > 0\}$ be the unique maximal ideal of D . If $D[x] \subseteq A \subseteq K[x]$ is a subalgebra, then there is associated to A a nested sequence of fractional ideals $\{\mathcal{I}_n : n = 0, 1, 2, \dots\}$ called the characteristic ideals of A with $\mathcal{I}_n$ consisting of 0 and the leading coefficients of elements of A of degree no more than n . The characteristic sequence of A , given by $\{a(n) : n = 0, 1, 2, \dots\}$ with $a(n) = -\nu(\mathcal{I}_n)$, encapsulates some of the algebraic structure of A . Because A is an algebra, the $a(n)$'s satisfy the inequality $a(n) \geq a(i) + a(j)$ for all $i, j \geq 0$ with $i + j = n$ and so are examples of super-additive sequences. The plan of this paper is to describe some combinatorial properties of such sequences and point out how these results reflect the structure of the algebra A .

A particular class of algebras of the sort described above are those defined by integrality conditions. If E is a subset of D , then $\text{Int}(E, D) = \{f \in K[x] : f(E) \subseteq D\}$ is called the algebra of integer valued polynomials on E . The study of such algebras has a long history going back at least to the work of Polya and Ostrowski ([12], [11]) in 1919 (see [6] for a systematic treatment and a comprehensive bibliography up to 1997). For algebras of this and related sorts the results of Bhargava ([1], [2]) describe the characteristic sequences of these algebras in terms of specific orderings of the set E , and this leads to a recursive method for computing their characteristic sequences in some cases ([8], [4]). The method involves decomposing the sequence as a shuffle or a merge of two or more subsequences. We will investigate

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this construction for super-additive sequences generally and show that it gives a lower bound for the characteristic sequence of the intersection of two such algebras.

One question about algebras which is always of interest is that of determining a generating set. In the case of the algebras $\text{Int}(E, D)$ for sets E which are homogeneous, i.e. a finite union of cosets of some power of the ideal $\mathcal{M}$, it was shown in [5] that a finite generating set exists if and only if $|D/\mathcal{M}| = \infty$. The original motivation for the present paper was to find a method for computing a generating set in this case. Since then it has been shown ([4]) that a variant of $\text{Int}(E, D)$, called the algebra of polynomials integer valued on E of finite order k , $\text{Int}_k(E, D)$, is finitely generated for all E , and our method applies in this case also.

In more detail the paper is organized as follows. In section 2 we develop the basic properties of super-additive sequences, define the shuffle product and show that the shuffle product of super-additive sequences is super-additive. This is accomplished by relating super-additive sequences both to sub-additive sequences and to the classes of super and sub-additive functions. The latter, which has many applications, has a substantial literature, and we establish our results by appealing to known results in this area. (A curious feature of this development is that there is a product for functions analogous to the shuffle product of sequences which also preserves super-additivity but which seems not to have been previously studied.) In section 3 we relate generators for A to points at which the super-additive sequence is strictly super-additive, i.e. n for which $a(n) > a(i) + a(j)$ for all $i, j > 0$ with $i + j = n$. The proof in [4] that $\text{Int}_h(D, E)$ is finitely generated depends on showing that the corresponding characteristic sequence has the periodicity property that there exist h, q such that $a(n + q) = a(n) + h$ for all $n \geq 0$ and that this property is preserved by the shuffle product. In general, it is true that a characteristic sequence having this property implies that the corresponding algebra is finitely generated; however the converse does not hold. Instead we show in section 4 that characteristic sequences of finitely generated algebras have the asymptotic periodicity property that beyond some point they agree with a super-additive sequence satisfying the condition above and that this characterizes them.

2. SUPER-ADDITIVE SEQUENCES AND THE SHUFFLE PRODUCT

Definition 2.1. (i) A nondecreasing sequence $\{a(n) : n = 0, 1, 2, \dots\}$ of nonnegative integers with $a(0) = 0$ is super-additive if $a(i + j) \geq a(i) + a(j)$ for all $i, j \geq 0$ and sub-additive if $a(i + j) \leq a(i) + a(j)$ for all $i, j \geq 0$.

(ii) A nondecreasing function $f : \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}^{\geq 0}$ with $f(0) = 0$ is super-additive if $f(x + y) \geq f(x) + f(y)$ for all $x, y \geq 0$ and sub-additive if $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$.

Throughout this paper the term sequence will mean a sequence of nonnegative integers and function will mean a real valued function on $\mathbb{R}^{\geq 0}$. Examples of the sequences defined above are $a(n) = \lfloor cn \rfloor$, for any positive real number c , which is super-additive, and $a(n) = \lceil cn \rceil$, which is sub-additive. $f(x) = x^2$ and $f(x) = \sqrt{x}$ are examples of a super- and a sub-additive function respectively.

As noted in the introduction characteristic sequences of subalgebras $D[x] \subseteq A \subseteq K[x]$ give examples of super-additive sequences. In fact any super-additive sequence arises in this way. If $\{a(n) : n = 0, 1, 2, \dots\}$ is such a sequence and $p \in D$ is a uniformizing element, i.e. $\nu(p) = 1$, then the D module spanned by the

set $\{x^n/p^{a(n)} : n = 0, 1, 2, \dots\}$ is an algebra because of super-additivity and has $\{a(n) : n = 0, 1, 2, \dots\}$ as a characteristic sequence.

Lemma 2.2. (i) If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing super-additive sequence, then $a(\lfloor x \rfloor)$ is a super-additive function.

(ii) If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing sub-additive sequence, then $a(\lceil x \rceil)$ is a sub-additive function

(iii) If $f(x)$ is a nondecreasing super-additive function, then $\lfloor f(n) \rfloor$ is a super-additive sequence.

(iv) If $f(x)$ is a nondecreasing sub-additive function, then $\lceil f(n) \rceil$ is a sub-additive sequence.

Proof. If $a(n)$ is super-additive, then

$$a(\lfloor x + y \rfloor) \geq a(\lfloor x \rfloor + \lfloor y \rfloor) \geq a(\lfloor x \rfloor) + a(\lfloor y \rfloor);$$

while if it is sub-additive,

$$a(\lceil x + y \rceil) \leq a(\lceil x \rceil + \lceil y \rceil) \leq a(\lceil x \rceil) + a(\lceil y \rceil).$$

If $f(x)$ is super-additive, then

$$\lfloor f(i + j) \rfloor \geq \lfloor f(i) + f(j) \rfloor \geq \lfloor f(i) \rfloor + \lfloor f(j) \rfloor;$$

while if it is sub-additive,

$$\lceil f(i + j) \rceil \leq \lceil f(i) + f(j) \rceil \leq \lceil f(i) \rceil + \lceil f(j) \rceil. \quad \square$$

Lemma 2.3. If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing sequence, then $a(\lfloor x \rfloor)$ is an upper semi-continuous function and $a(\lceil x \rceil)$ is a lower semi-continuous function.

Proof. Since if $\lfloor x' \rfloor \leq x' \leq x < \lfloor x' \rfloor + 1$, then $a(\lfloor x \rfloor) = a(\lfloor x' \rfloor)$, it follows that $\lim_{x \rightarrow x_0^+} a(\lfloor x \rfloor) = a(\lfloor x_0 \rfloor)$ and similarly for $\lim_{x \rightarrow x_0^-} a(\lceil x \rceil)$. $\square$

In this paper we are primarily concerned with super-additive sequences. There is a substantial literature concerning super and sub-additive functions, and the previous lemma will allow us to translate those results to the situation in which we are interested.

The next definition follows that in section 2 of [10]:

Definition 2.4. (i) If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing sequence, let $\overrightarrow{a}(m) = \max\{n : a(n) \leq m\}$ and $\overleftarrow{a}(m) = \min\{n : m \leq a(n)\}$

(ii) If $f(x)$ is a nondecreasing function on $\mathbb{R}^{\geq 0}$, let $\overrightarrow{f}(x) = \sup\{y : f(y) \leq x\}$ and $\overleftarrow{f}(x) = \inf\{y : x \leq f(y)\}$.

Proposition 2.5 ([10], Lemma 2.1). (i) If f is lower semi-continuous, then $\overrightarrow{f}$ is upper semi-continuous.

(ii) If f is upper semi-continuous, then $\overleftarrow{f}$ is lower semi-continuous.

Proposition 2.6 ([10], Lemma 2.2). (i) If f is lower semi-continuous, then $f = \overleftarrow{\overrightarrow{f}}$.

(ii) If f is upper semi-continuous, then $f = \overrightarrow{\overleftarrow{f}}$.

Proposition 2.7 ([10], Lemma 2.3). (i) If f is lower semi-continuous, then f is sub-additive iff $\overrightarrow{f}$ is super-additive.

(ii) If f is upper semi-continuous, then f is super-additive iff $\overleftarrow{f}$ is sub-additive.

Corollary 2.8. (i) If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing super-additive sequence, then $\{\overleftarrow{a}(n) : n = 0, 1, 2, \dots\}$ is a sub-additive sequence.

(ii) If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing nondecreasing sub-additive sequence, then $\{\overrightarrow{a}(n) : n = 0, 1, 2, \dots\}$ is a super-additive sequence.

(iii) If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing sequence, then $a(n) = \overleftrightarrow{a}(n)$.

Proof. (i) $a(\lfloor x \rfloor)$ is a super-additive upper semi-continuous function; hence $\overleftarrow{a}(\lfloor x \rfloor)$ is a sub-additive function, and so $\lfloor \overleftarrow{a}(\lfloor m \rfloor) \rfloor$ is a sub-additive sequence. But

$$\begin{aligned} \lfloor \overleftarrow{a}(\lfloor m \rfloor) \rfloor &= \lfloor \inf\{y : m \leq a(\lfloor y \rfloor)\} \rfloor \\ &= \min\{n : m \leq a(n)\} \\ &= \overleftarrow{a}(m). \end{aligned}$$

A similar argument proves (ii).

(iii) Clearly

$$\min\{\ell : a(\ell) \geq a(m)\} \leq m,$$

which implies

$$\max\{n : \min\{\ell : a(\ell) \geq n\} \leq m\} \geq a(m);$$

but if $n > a(m)$ and $a(\ell) \geq n$, then $a(\ell) > a(m)$, and so $\ell > m$, which implies we have equality. $\square$

Definition 2.9 ([8]). If $\{a_i(n) : n = 0, 1, 2, \dots\}$ for $i = 1$ or 2 are nondecreasing sequences, then the shuffle of a_1 and a_2 , denoted $a_1 \wedge a_2$, is the unique nondecreasing sequence having both a_1 and a_2 as disjoint subsequences with every element of $a_1 \wedge a_2$ being in one or the other of these subsequences. Thus there are strictly increasing maps

$$\phi_1, \phi_2 : \mathbb{Z}^{\geq 0} \rightarrow \mathbb{Z}^{\geq 0}$$

such that $\phi_1(\mathbb{Z}^{\geq 0}) \cap \phi_2(\mathbb{Z}^{\geq 0}) = \emptyset$,

$$\phi_1(\mathbb{Z}^{\geq 0}) \cup \phi_2(\mathbb{Z}^{\geq 0}) = \mathbb{Z}^{\geq 0},$$

and

$$a_1 \wedge a_2(n) = \begin{cases} a_1(m) & \text{if } n = \phi_1(m), \\ a_2(m) & \text{if } n = \phi_2(m), \end{cases}$$

and such that $\{a_1 \wedge a_2(n) : n = 0, 1, 2, \dots\}$ is nondecreasing.

The significance of this product is illustrated by the special case of the algebras $\text{Int}(E, \mathbb{Z}_{(2)})$ with $E \subseteq \mathbb{Z}_{(2)}$. If E_0 and E_1 denote the sets $E \cap 2\mathbb{Z}_{(2)}$ and $E \cap (1+2\mathbb{Z}_{(2)})$ respectively, then the characteristic sequence of $\text{Int}(E, \mathbb{Z}_{(2)})$ is the shuffle product of those of $\text{Int}(E_0, \mathbb{Z}_{(2)})$ and $\text{Int}(E_1, \mathbb{Z}_{(2)})$. A similar result holds generally for the intersections of a subset E of D with the various cosets of $\mathcal{M}$ and gives a method for recursively computing the characteristic sequence of E if E is finite, [8], or homogeneous, [9].

Proposition 2.10. If $\{a_i(n) : n = 0, 1, 2, \dots\}$ for $i = 1, 2$ are nondecreasing sequences with $a_1(0) = a_2(0) = 0$, then $\{\overleftarrow{a_1 \wedge a_2}(n) : n = 0, 1, 2, \dots\} = \{\overleftarrow{a_1}(n) + \overleftarrow{a_2}(n) : n = 0, 1, 2, \dots\}$.

Proof. We proceed by induction on m to show that $\overleftarrow{a_1 \wedge a_2}(m) = \overleftarrow{a_1}(m) + \overleftarrow{a_2}(m)$. For $m = 0$ both sides are 0. For $m > 0$ for any nondecreasing sequence $a(n)$, we have

$$\overleftarrow{a}(m) = |\{n : a(n) = m - 1\}| + \overleftarrow{a}(m - 1),$$

and also, for any nondecreasing sequences $a_1(n)$ and $a_2(n)$ and any $m > 0$,

$$|\{n : a_1 \wedge a_2(n) = m - 1\}| = |\{n : a_1(n) = m - 1\}| + |\{n : a_2(n) = m - 1\}|.$$

Thus

$$\begin{aligned} \overleftarrow{a_1 \wedge a_2}(m) &= |\{n : a_1 \wedge a_2(n) = m - 1\}| + \overleftarrow{a_1 \wedge a_2}(m - 1) \\ &= |\{n : a_1(n) = m - 1\}| + |\{n : a_2(n) = m - 1\}| \\ &\quad + \overleftarrow{a_1}(m - 1) + \overleftarrow{a_2}(m - 1) \\ &= \overleftarrow{a_1}(m) + \overleftarrow{a_2}(m). \end{aligned} \quad \square$$

Corollary 2.11. *If $\{a(n) : n = 0, 1, 2, \dots\}$ for $i = 1$ or 2 are super-additive sequences, then $\{a_1 \wedge a_2(n) : n = 0, 1, 2, \dots\}$ is super-additive.*

Proof.

$$a_1 \wedge a_2 = \overleftarrow{\overrightarrow{a_1 \wedge a_2}} = \overleftarrow{\overrightarrow{a_1}} + \overleftarrow{\overrightarrow{a_2}},$$

and the sum of sub-additive sequences is sub-additive. $\square$

This description of the shuffle product of sequences translates directly to the case of upper semi-continuous functions:

Definition 2.12. If $f(x)$ and $g(x)$ are upper semi-continuous functions, let $(f \wedge g)(x) = \overleftarrow{\overrightarrow{f(x)}} + \overleftarrow{\overrightarrow{g(x)}}$.

Corollary 2.13. *If $f(x)$ and $g(x)$ are upper semi-continuous super-additive functions, then the function $(f \wedge g)(x)$ is super-additive.*

Lemma 2.14. *If $\{a_i(n) : n = 0, 1, 2, \dots\}$ for $i = 1$ and 2 are nondecreasing sequences with $a_1(0) = a_2(0) = 0$ and $c(n) = a_1 \wedge a_2(n)$, then*

$$c(n) = \max\{\min(a_1(i), a_2(j)) : i + j = n\}.$$

Proof. We proceed by induction, noting first that

$$\min(a_1(0), a_2(0)) = 0 = c(0)$$

and

$$\max(\min(a_1(1), a_2(0)), \min(a_1(0), a_2(1))) = 0 = c(1).$$

If the result holds for $c(n)$, then $\{c(i) : i = 0, \dots, n\}$ equals the union of $\{a_1(i) : i = 0, \dots, k\}$ and $\{a_2(i) : i = 0, \dots, \ell\}$ in nondecreasing order for some k and ℓ with $(k + 1) + (\ell + 1) = n + 1$ and $a_1(k) \leq a_2(\ell + 1)$ and $a_2(\ell) \leq a_1(k + 1)$. If $i + j = n + 1 = k + \ell + 2$ with $i \leq k$, then $\min(a_1(i), a_2(j)) = a_1(i)$ since $a_1(i) \leq a_1(k) \leq a_2(\ell + 1) \leq a_2(j)$; and similarly if $i + j = n + 1$ with $j \leq \ell$, then $\min(a_1(i), a_2(j)) = a_2(j)$. Thus the set $\{\min(a_1(i), a_2(j)) : i + j = n + 1\}$ contains $\{a_1(i) : i = 0, \dots, k\}$ and $\{a_2(i) : i = 0, \dots, \ell\}$ and $\min(a_1(k + 1), a_2(\ell + 1))$, and the maximum of this set is this last element which equals $c(n + 1)$. $\square$

If $D[x] \subseteq A, B \subseteq K[x]$ are algebras, let $p \in \mathcal{M}$ be such that $\nu(p) = 1$ and let $\{f_i(x)/p^{a(i)} : i = 0, 1, 2, \dots\}$ and $\{g_i(x)/p^{b(i)} : i = 0, 1, 2, \dots\}$ be regular bases (i.e. bases with $f_i, g_i \in D[x]$ monic of degree i) for A and B respectively. These exist since the characteristic ideals of A and B are principally generated by powers of p . For any i, j we then have $f_i(x)g_j(x)/p^{\min(a(i), b(j))} \in A \cap B$. Thus in particular if for each n the pair i, j is chosen so that $i + j = n$ and $\min(a(i), b(j))$ is largest and this value is denoted $c(n)$ and $h_n(x) = f_i(x)g_j(x)$, then $h_n(x)/p^{c(n)} \in A \cap B$,

and by the previous lemma $c(n) = a \wedge b(n)$. Thus the shuffle of the characteristic sequences of A and B gives a lower bound for the characteristic sequence of $A \cap B$.

3. S -PERIODICITY

Definition 3.1. A nondecreasing sequence $\{a(n) : n = 0, 1, 2, \dots\}$ with $a(0) = 0$ will be called S -periodic if there are positive integers q and h such that

$$a(n + q) = a(n) + h$$

for all $n \geq 0$. The integer q will be referred to as the period of the sequence and h as the height. It is not assumed that the period, q , is the smallest positive integer for which this condition holds. That integer will be referred to as the minimal period.

Example 3.2. The sequence $\{h \cdot \lfloor n/q \rfloor : n = 0, 1, 2, \dots\}$ is S -periodic of period q and height h . More generally if $q > r_1 \geq r_2 \geq \dots \geq r_h \geq 0$ and $q > s_h \geq s_{h-1} \geq$

$\dots \geq s_1 \geq 0$ are integers, then the sequences $\{\sum_{i=1}^h \lfloor \frac{n+r_i}{q} \rfloor : n = 0, 1, 2, \dots\}$ and $\{\sum_{i=1}^h \lceil \frac{n-s_i}{q} \rceil : n = 0, 1, 2, \dots\}$ are S -periodic of period q and height h .

The name S -periodic (step periodic) is chosen to reflect that fact that such a sequence can be expressed as $h \cdot \lfloor n/q \rfloor + a(n \bmod q)$, i.e., the sum of a step sequence and a periodic one.

Lemma 3.3. If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing S -periodic sequence with period q and height h , then the sequences $\overleftarrow{a}(n)$ and $\overrightarrow{a}(n)$ are S -periodic of period h and height q .

Proof.

$$\begin{aligned} \overleftarrow{a}(m) &= \min\{n : m \leq a(n)\} \\ &= \min\{n : m \leq a(n+q) - h\} \\ &= \min\{n : h + m \leq a(n+q)\} \\ &= \min\{n' - q : h + m \leq a(n')\} \\ &= \min\{n' : h + m \leq a(n')\} - q \\ &= \overleftarrow{a}(m+h) - q. \end{aligned}$$

The calculation for $\overrightarrow{a}(n)$ is similar. □

Proposition 3.4. If $\{a(n) : n = 0, 1, 2, \dots\}$ is a nondecreasing S -periodic sequence of period q and height h and $a(0) = 0$, then

$$a(n) = \sum_{i=1}^h \left\lfloor \frac{n+r_i}{q} \right\rfloor = \sum_{i=1}^h \left\lceil \frac{n-s_i}{q} \right\rceil$$

with $r_i = q - \overleftarrow{a}(i)$ and $s_i = \overrightarrow{a}(i)$.

Proof. The sequence $\{\sum_{i=1}^h \lfloor \frac{n+r_i}{q} \rfloor : n = 0, 1, 2, \dots\}$ is S -periodic of period q and height h ; hence it suffices to check that it agrees with $\{a(n) : n = 0, 1, 2, \dots\}$ for $0 \leq n \leq q$. If the r_i 's are as in the statement of the proposition so that

$q > r_1 \geq r_2 \geq \cdots \geq r_h \geq 0$, then for each $0 < j \leq h$ we have $a(n) = j$ if $q - r_j \leq n < q - r_{j+1}$ and $a(n) = 0$ if $n < q - r_1$. If $q - r_j \leq n < q - r_{j+1}$, then

$$\left\lfloor \frac{n + r_i}{q} \right\rfloor = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{if } i > j, \end{cases}$$

and so $\sum_{i=1}^h \left\lfloor \frac{n + r_i}{q} \right\rfloor = j = a(n)$ as required. The proof of the second equality is similar. $\square$

Note that since $\lfloor \frac{n}{q} \rfloor = \sum_{i=0}^{k-1} \lfloor \frac{n + iq}{kq} \rfloor$, this representation does not require q to be the minimal period and that by replacing q by a multiple it may be assumed that the period of $a(n)$ is as large as desired.

Corollary 3.5. (i) *The S -periodic sequence*

$$\left\{ \sum_1^h \left\lfloor \frac{n + r_i}{q} \right\rfloor : n = 0, 1, 2, \dots \right\}$$

is super-additive if and only if the r_i 's satisfy the inequalities

$$(q - r_i) + (q - r_j) \geq (q - r_{i+j}) \quad \text{if } i + j < h$$

and

$$(q - r_i) + (q - r_j) \geq (2q - r_{i+j-h}) \quad \text{if } h \leq i + j.$$

(ii) *The S -periodic sequence*

$$\left\{ \sum_1^h \left\lceil \frac{n - s_i}{q} \right\rceil : n = 0, 1, 2, \dots \right\}$$

is sub-additive if and only if the s_i 's satisfy the inequalities

$$s_i + s_j \leq (s_{i+j}) \quad \text{if } i + j < h$$

and

$$s_i + s_j \leq (q + s_{i+j-h}) \quad \text{if } h \leq i + j.$$

Proof. $\{a(n) : n = 0, 1, 2, \dots\}$ is super-additive if and only if $\{\overleftarrow{a}(n) : n = 0, 1, 2, \dots\}$ is sub-additive and is sub-additive if and only if $\{\overrightarrow{a}(n) : n = 0, 1, 2, \dots\}$ is super-additive. $\square$

Proposition 3.6. *If $\{a_i(n) : n = 0, 1, 2, \dots\}$ for $i = 1$ or 2 are S -periodic sequences of periods q_1 and q_2 and heights h_1 and h_2 , then $\{(a_1 + a_2)(n) : n = 0, 1, 2, \dots\}$ is S -periodic of period $\text{lcm}(q_1, q_2)$ and height $(h_1q_2 + h_2q_1)/\text{gcd}(q_1, q_2)$.*

Proof. If $q_1 = q_2 = q$, then clearly $a_1 + a_2(n)$ is S -periodic of period q and height $h_1 + h_2$. Since $a_1(n)$ is periodic of period q_1 , it is also periodic of period k_1q_1 and height k_1h_1 for any k_1 . Similarly for $a_2(n)$. Thus taking $k_1 = q_2/\text{gcd}(q_1, q_2)$ and $k_2 = q_1/\text{gcd}(q_1, q_2)$, we obtain the result. $\square$

Proposition 3.7 ([4], Proof of Theorem 3.1). *If $\{a_i(n) : n = 0, 1, 2, \dots\}$ for $i = 1$ or 2 are S -periodic sequences of periods q_1 and q_2 and heights h_1 and h_2 , then $\{a_1 \wedge a_2(n) : n = 0, 1, 2, \dots\}$ is S -periodic of period $(q_1h_2 + q_2h_1)/\text{gcd}(h_1, h_2)$ and height $\text{lcm}(h_1, h_2)$.*

Proof. Since

$$\overleftarrow{a_1 \wedge a_2} = \overleftarrow{a_1} + \overleftarrow{a_2},$$

the result follows from the previous proposition. $\square$

The finite sums $\sum_{i=1}^{\ell} \lfloor \frac{x}{q_i} \rfloor$ also give examples of S -periodic super-additive sequences. In fact:

Proposition 3.8. *If ℓ is finite and $q_1 \leq q_2 \leq \dots \leq q_\ell$ is a nondecreasing list and $a(n) = \sum_{i=1}^{\ell} \lfloor \frac{n}{q_i} \rfloor$, then $\{a(n) : n = 0, 1, 2, \dots\}$ is an S -periodic super-additive sequence of period $Q = \text{lcm}(\{q_i\}_{i=1}^{\ell})$ and height $\sum_{j=1}^{\ell} Q/q_j$.*

Proof. Let $Q_j = Q/q_j$. We, then have

$$\left\lfloor \frac{n}{q_i} \right\rfloor = \sum_{j=0}^{Q_j-1} \left\lfloor \frac{n + jq_i}{Q} \right\rfloor;$$

hence

$$a(n) = \sum_{i=1}^{\ell} \sum_{j=0}^{Q_j-1} \left\lfloor \frac{n + jq_i}{Q} \right\rfloor. \quad \square$$

It is not the case that any S -periodic super-additive sequence can be represented as a finite sum $\sum_{i=1}^{\ell} \lfloor \frac{n}{q_i} \rfloor$ however.

Example 3.9. The sequence $\{a(n) : n = 0, 1, 2, \dots\}$ with $a(n) = \lfloor \frac{n+3}{5} \rfloor + \lfloor \frac{n+2}{5} \rfloor + \lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{5} \rfloor$ is super-additive since the values $q = 5$, $r_1 = 3$, $r_2 = 2$, $r_3 = 0$, $r_4 = 0$ satisfy the inequalities of Corollary 3.5. If this sequence were equal to $\sum_{i=1}^{\ell} \lfloor \frac{n}{q_i} \rfloor$, then each q_i would be a divisor of 5 and so, since $a(1) = 0$, would equal 5. This would imply $a(n) = 0$ for $n < 5$; however $a(2) = 1$.

4. GENERATORS

Definition 4.1. If $\{a(n) : n = 0, 1, 2, \dots\}$ is a super-additive sequence, then a positive integer g will be called a generator of a if $a(i) + a(j) < a(g)$ for all i, j positive integers with $i + j = g$ and 1 will be called a generator if $a(1) > 0$.

Example 4.2. If $a(n) = \nu_p(n!)$ is the p -adic norm with respect to a fixed prime, p , of $n!$, then $a(n)$ is super-additive since if $i + j = n$, then $i!j!$ divides $n!$. The generators for this sequence are p^k for $k = 1, 2, \dots$ since p divides $p^k!/i!j!$ if $i + j = p^k$ with $i, j > 0$, and if $p^k > r > 0$, then p does not divide $(p^k + r)!/p^k!r!$.

Proposition 4.3. *If $\{a(n) : n = 0, 1, 2, \dots\}$ is the characteristic sequence of an algebra $D[x] \subseteq A \subseteq K[x]$ and G is the set of generators of this sequence and if $H = \{h_n(x)/p^{a(n)} : n = 0, 1, 2, \dots\}$ is a regular basis for A , then the set $\{h_g(x)/p^{a(g)} : g \in G\}$ is a multiplicative generating set for the algebra A .*

Proof. It suffices to show that each $h_n(x)/p^{a(n)}$ is in the algebra generated by the set $\{h_g(x)/p^{a(g)} : g \in G\}$. We do this by induction on n . If $n \in G$ there is nothing to prove. If $n \notin G$, then $a(n) = a(i) + a(j)$ for some $i, j > 0$, and so the leading coefficient of $h_i(x)h_j(x)/p^{a(i)+a(j)}$ is a D unit multiple of that of $h_n(x)$; i.e. there exists a unit $u \in D$ such that $h_n(x)/p^{a(n)} - uh_i(x)h_j(x)/p^{a(i)+a(j)}$ is of degree less than n . By induction this implies that $h_n(x)/p^n$ is in the algebra generated by $\{h_g(x)/p^{a(n)} : g \in G\}$ as required. $\square$

Example 4.4. The algebra $\text{Int}(\mathbb{Z}, \mathbb{Z}_{(p)})$ has as a regular $\mathbb{Z}_{(p)}$ basis the binomial polynomials $\binom{x}{n}$. Since the n -th of these is the quotient of a monic polynomial by $n!$, the characteristic sequence is that given in the previous example and the generators are as given there. From this it follows that the Fermat polynomials defined recursively by $f_0(x) = x$, $f_{i+1}(x) = (f_i^p(x) - f_i(x))/p$ also form a generating set over $\mathbb{Z}_{(p)}$ for $\text{Int}(\mathbb{Z}, \mathbb{Z}_{(p)})$. Since this algebra occurs in algebraic topology as the K -homology of $\mathbb{C}P^\infty$ ([7]), this fact has been of use in homological algebra calculations concerning that space.

While the previous proposition justifies the choice of the term generator for a super-additive sequence, it is still somewhat misleading since it implies that a sequence $a(n)$ can be recovered from its list of generators or that an arbitrary increasing list determines a unique sequence. For these to hold we must also retain the multiplicities of the generators:

Definition 4.5. If $\{a(n) : n = 0, 1, 2, \dots\}$ is super-additive sequence and g a generator of a , then the multiplicity of g is the integer e given by

$$e = a(g) - \max\{a(i) + a(j) : i + j = g, i, j > 0\}.$$

If $g = 1$ is a generator, then its multiplicity is $a(1)$.

Proposition 4.6. If $\{(g_i, e_i) : i = 1, 2, \dots\}$ is a set of pairs of positive integers with $g_{i+1} > g_i$ for all i , then there is a unique super-additive sequence $a(n)$ with generators $\{g_i\}$ of multiplicities $\{e_i\}$.

Proof. Define $a(n)$ recursively by $a(n) = 0$ for $n < g_1$ and

$$a(n) = \begin{cases} \max\{a(i) + a(j) : i + j = n, i, j > 0\} & \text{if } n \neq g_i, \\ \max\{a(i) + a(j) : i + j = n, i, j > 0\} + e_i & \text{if } n = g_i. \end{cases} \quad \square$$

If $\{a(n) : n = 0, 1, 2, \dots\}$ is a super-additive sequence, then the relation between the generators of $a(n)$ and the values of the sub-additive sequence $\overleftarrow{a}(n)$ is given in the following proposition:

Proposition 4.7. If $\{a(n) : n = 0, 1, 2, \dots\}$ is a super-additive sequence, then the generators are those values of the sequence $\{\overleftarrow{a}(n) : n = 0, 1, 2, \dots\}$ with the property that

$$\overleftarrow{a}(n) < \overleftarrow{a}(k) + \overleftarrow{a}(\ell)$$

if $k + \ell = n$ and $k, \ell > 0$.

Proof. First note that if $a(n) = a(n-1)$, then n cannot be a generator of a . Since $a(n) \leq a(n-1) + a(1)$, then $a(n) = a(n-1) + a(1)$ since a is super-additive. Next note that if $a(n-1) < a(n)$, then n is a value of $\overleftarrow{a}$, since in this case $\overleftarrow{a}(a(n)) = n$. If n is not a generator, then $a(n) = a(i) + a(j)$ for some $i, j > 0$ with $i + j = n$. By

decreasing i, j if necessary we may assume that $a(n) = a(i) + a(j)$, that $i + j \leq n$ and that $a(i-1) < a(i)$ and $a(j-1) < a(j)$. Thus

$$i = \overleftarrow{a}(a(i))$$

and

$$j = \overleftarrow{a}(a(j)),$$

and so

$$n = \overleftarrow{a}(a(n)) \geq \overleftarrow{a}(a(i)) + \overleftarrow{a}(a(j)) = i + j.$$

Since a is super-additive, $\overleftarrow{a}$ is sub-additive; hence

$$n = \overleftarrow{a}(a(i)) + \overleftarrow{a}(a(j)).$$

Thus the generators of a are among the values of $\overleftarrow{a}$ satisfying the condition in the statement of the proposition.

Conversely, suppose that $m = \overleftarrow{a}(n)$ is a value of $\overleftarrow{a}$ and that $\overleftarrow{a}(n) = m = \overleftarrow{a}(k) + \overleftarrow{a}(\ell)$ for some $k, \ell > 0$ with $k + \ell = n$. It need not be the case that either k or ℓ is a value of a . However, there exist $k' \geq k$ and $\ell' \geq \ell$ such that $\overleftarrow{a}(k') = \overleftarrow{a}(k)$ and $\overleftarrow{a}(\ell') = \overleftarrow{a}(\ell)$ and $k' = a(i)$ and $\ell' = a(j)$. Since $k', \ell' > 0$ it follows that $i, j > 0$ also. We thus have

$$m = \overleftarrow{a}(a(m)) = \overleftarrow{a}(n) = \overleftarrow{a}(a(i)) + \overleftarrow{a}(a(j)) = i + j$$

and

$$a(m) = n \leq k' + \ell' = a(i) + a(j).$$

Since a is superadditive, equality must hold in the last equation, and so m is not a generator of a . $\square$

Corollary 4.8. *If $\{a_i(n) : n = 0, 1, 2, \dots\}$ for $i = 1$ or 2 are super-additive sequences, then the generators of $a_1 \wedge a_2$ are the values of $\overleftarrow{a}_1 + \overleftarrow{a}_2$ satisfying the condition*

$$(\overleftarrow{a}_1 + \overleftarrow{a}_2)(n) < (\overleftarrow{a}_1 + \overleftarrow{a}_2)(k) + (\overleftarrow{a}_1 + \overleftarrow{a}_2)(\ell) \text{ if } k + \ell = n, \quad k, \ell > 0.$$

Definition 4.9. A sequence is finitely generated if its list of generators is finite.

Proposition 4.10. *If $\{a(n) : n = 0, 1, 2, \dots\}$ is S -periodic, then it is finitely generated.*

Proof. Since $a(n+q) = a(n) + h = a(n) + a(q)$, a cannot have a generator of size $> q$. $\square$

It is not the case that all finitely generated sequences are S -periodic.

Example 4.11. The sequence $a(0) = a(1) = 0$, $a(2) = a(3) = 2$, $a(4) = 4$, $a(n) = n$ for $n \geq 5$ is generated by $\{(2, 2), (5, 1)\}$ but is clearly not S -periodic.

Proposition 4.12. *If $\{a(n) : n = 0, 1, 2, \dots\}$ is a finitely generated super-additive sequence with generators $g_1 \dots g_\ell$ and $a_i = a(g_i)$ for $i = 1, \dots, \ell$, then there exists an integer N and an S -periodic sequence $b(n)$ such that $b(n) = a(n)$ for $n > N$. The minimal period of $b(n)$ divides g_i for each index i for which a_i/g_i is maximal.*

Proof. Let g_k be one of the generators g_i for which a_i/g_i is maximal. For any n larger than the largest of the g_i 's we have

$$\begin{aligned} a(n) &= \max\{a(i) + a(j) : i + j = n, i, j > 0\} \\ &= \max\left\{\sum_{i=1}^{\ell} c_i a_i : \sum_{i=1}^{\ell} c_i g_i \leq n, c_i \geq 0\right\}. \end{aligned}$$

If $j \neq k$ and $c_j \geq g_k$, then

$$c_j g_j + c_k g_k = (c_j - g_k) g_j + (c_k + g_j) g_k$$

and

$$c_j a_j + c_k a_k \leq (c_j - g_k) a_j + (c_k + g_j) g_k$$

since $g_j a_k \geq g_k a_j$. Thus

$$a(n) = \max\left\{\sum_{i=1}^{\ell} c_i a_i : \sum_{i=1}^{\ell} c_i g_i \leq n, 0 \leq c_i < g_k \text{ if } i \neq k\right\},$$

and, if n is large enough ($n > N = g_k \sum_{i=0}^{\ell} g_i$), then we also may assume $c_k > 0$:

$$a(n) = \max\left\{\sum_{i=1}^{\ell} c_i a_i : \sum_{i=1}^{\ell} c_i g_i \leq n, c_i < g_k \text{ if } i \neq k \text{ and } c_k > 0\right\}.$$

The set of coefficients over which this maximum is taken,

$$\mathcal{C}_1 = \left\{(c_1 \dots c_{\ell}) : \sum_{i=1}^{\ell} c_i g_i \leq n, c_i < g_k \text{ if } i \neq k \text{ and } c_k \geq 0\right\},$$

is in 1 – 1 correspondence with the set

$$\mathcal{C}_2 = \{(c_1 \dots c_{\ell}) : \sum_{i=1}^{\ell} c_i g_i = n + g_k, 0 \leq c_i < g_k \text{ if } i \neq k \text{ and } c_k > 0\}$$

via $(c_1 \dots c_{\ell})$ corresponding with $(c_1 \dots c_k + 1, \dots, c_{\ell})$.

Since $a(n) = \max\{\sum_{i=1}^{\ell} c_i a_i : (c_1 \dots c_{\ell}) \in \mathcal{C}_1\}$ we have

$$\begin{aligned} a(n + g_k) &= \max\left\{\sum_{i=1}^{\ell} c_i a_i : (c_1 \dots c_{\ell}) \in \mathcal{C}_2\right\} \\ &= a_k + \max\left\{\sum_{i=1}^{\ell} c_i a_i : (c_1 \dots c_{\ell}) \in \mathcal{C}_1\right\} \\ &= a_k + a(n). \end{aligned}$$

Given $n \geq 0$, choose r such that $n + r g_k > N$ and define $b(n) = a(n + r g_k) - r a_k$. The previous calculation shows that this definition is independent of the choice of r and $b(n) = a(n)$ if $n > N$. Also if $i + j = n$ and r, s are such that $r i > N$ and $s j > N$, then $b(i) + b(j) = a(i + r g_k) - r a_k + a(j + s g_k) - s a_k \leq a(i + r g_k + j + s g_k) - (r + s) a_k = b(n)$. Thus $\{b(n) : n = 0, 1, 2, \dots\}$ is super-additive, S -periodic with period dividing g_k , and agrees with $\{a(n) : n = 0, 1, 2, \dots\}$ for n sufficiently large. $\square$

Corollary 4.13. *If $\{a(n) : n = 0, 1, 2, \dots\}$ is a finitely generated super-additive sequence with generators $g_1 < g_2 < \dots < g_{\ell}$ and $a(n)$ is S -periodic, then $a(g_{\ell})/g_{\ell} > a(g_i)/g_i$ for all $i < \ell$.*

Proof. Since g_ℓ is a generator, $a(g_\ell)/g_\ell \neq a(g_i)/g_i$ for any g_i dividing g_ℓ . Thus if $a(g_\ell)/g_\ell = a(g_i)/g_i$ for some $i < \ell$, then $a(n)$ has a minimal period dividing both g_ℓ and g_i , hence smaller than either. But no S -periodic sequence has a generator larger than its minimal period. $\square$

Remark 4.14. This corollary gives a necessary condition for a finitely generated sequence to be S -periodic which is not sufficient. If $a(n)$ is the sequence with generators 2 and 5 of multiplicities 3 and 2, then $a(2) = 3$, $a(5) = 8$ and so $a(2)/2 = 3/2 < 8/5 = a(5)/5$. The sequence $a(n)$ begins with 0, 0, 3, 3, 6, 8, 9; hence it is not S -periodic since $9 = a(6) \neq 8 + 0 = a(5) + a(1)$.

Proposition 4.15. *If $a(n)$ is a super-additive sequence which is asymptotically S -periodic, i.e. there exists an S -periodic sequence $b(n)$ such that $a(n) = b(n)$ for $n \geq N$, then $a(n)$ is finitely generated.*

Proof. Since $b(n)$ is S -periodic it has a representation of the form

$$b(n) = \sum_{i=0}^{h-1} \left\lfloor \frac{n + r_i}{q} \right\rfloor,$$

and we may assume, by replacing q by a multiple if necessary, that $q > N$. Note that since $b(n)$ is an S -periodic function it has the property $b(tq + s) = th + b(s) = b(tq) + b(s)$. Also note that if the result holds in the special case $a(n) = 0$ for $n < q$, then it will hold in general. We claim that in this case $a(n)$ is generated by a generator q of multiplicity h together with generators $\{q + r_i : r_i > 0\}$, each of multiplicity 1. Let $c(n)$ be the sequence with these generators. We proceed by induction on n . It is clear that $c(n) = 0$ for $n < q$ and $c(n) = b(n) = a(n)$ for $n \leq 2q$. Suppose that $c(n) = b(n)$ for $n < mq + s$ with $0 < s \leq q$ and consider $c(mq + s)$. We have $b(mq + s) = b((m-1)q + (m+s)) = b((m-1)q) + b(m+s) = c((m-1)q) + c(m+s)$. Thus $c(mq + s) \geq b(mq + s)$. On the other hand, if $i + j = mq + s$ with $i, j \geq q$, then $c(i) + c(j) = b(i) + b(j) \leq b(mq + s)$ since $b(n)$ is super-additive; while if $i + j = mq + s$ with $0 < i < q$, then $c(i) + c(j) = c(j) = b(j) \leq b(mq + s)$ since $b(n)$ is nondecreasing. Thus $c(mq + s) \leq b(mq + s)$. $\square$

Remark 4.16. It should be noted that the hypothesis in this proposition is not the same as requiring that for some positive integers q and h we have $a(n+q) = a(n) + h$ for all $n > N$. This weaker condition is satisfied by the super-additive sequence $a(0) = 0$, $a(n) = n - 1$ for $n > 0$, which is not finitely generated since $a(i) + a(j) = i + j - 2 = a(i + j) - 1$ for all $i, j > 0$.

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