LOGARITHMIC UP BOUNDS FOR SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we obtain a logarithmic up bound for the solution to an elliptic mixed boundary value problem. This is an improvement over the corresponding classical result in this area.

1. MAIN RESULT AND PROOF

Let Ω be a bounded domain in R^N , $N > 2$, with Lipschitz boundary $\partial\Omega$. Assume that Γ_D is a non-empty open subset of $\partial\Omega$. Consider the mixed boundary value problem

$$(1.1) \quad -\operatorname{div}(A(x)\nabla\varphi) = f \text{ in } \Omega,$$

$$(1.2) \quad \varphi = \varphi_0 \text{ on } \Gamma_D,$$

$$(1.3) \quad A(x)\nabla\varphi \cdot \nu = 0 \text{ on } \Gamma_N \equiv \partial\Omega \setminus \overline{\Gamma_D}.$$

Here $A(x)$ is an $N \times N$ matrix whose entries $a_{ij}(x)$ satisfy

$$(1.4) \quad a_{ij} \in L^\infty(\Omega), \quad \lambda|\xi|^2 \leq A(x)\xi \cdot \xi \text{ for } \xi \in R^N \text{ a.e. } x \in \Omega,$$

where $\lambda > 0$. As for other given data in the problem we have that $f \in L^{\frac{p}{2}}(\Omega)$ for some $p > N$, $\varphi_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, and ν is the unit outward normal to $\partial\Omega$. In the situation considered here, the classical theory for uniformly elliptic equations asserts that there is a unique weak solution φ in the space $W^{1,2}(\Omega)$ and that φ satisfies

$$(1.5) \quad \|\varphi\|_\infty \leq \|\varphi_0\|_\infty + c\|f\|_{\frac{p}{2}}, \quad c = c(\lambda, p, \Gamma_D, \Omega).$$

Here and in what follows we denote by $\|f\|_p$ the norm of f in $L^p(\Omega)$. The result is sharp in the sense that if $p = N$, then (1.5) fails in general. The objective of this paper is to improve the up bound for $\|\varphi\|_\infty$. To be precise, we have shown:

Theorem A. *Under the above assumptions there is a positive number $c = c(\lambda, p, \Gamma_D, \Omega)$ such that*

$$(1.6) \quad \|\varphi\|_\infty \leq \|\varphi_0\|_\infty + c\|f\|_{\frac{N}{2}} \left(\ln(\|f\|_{\frac{p}{2}} + 1) + 1 \right).$$

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A similar result for functions in $W^{s,p}(R^N)$, $sp > N$, is established in [1, 2].

To prove Theorem A, we decompose φ into $\varphi_1 + \varphi_2$, where φ_1 and φ_2 are the respective solutions of the following two problems:

$$(1.7) \quad -\operatorname{div}(A(x)\nabla\varphi_1) = f \text{ in } \Omega,$$

$$(1.8) \quad \varphi_1 = 0 \text{ on } \Gamma_D,$$

$$(1.9) \quad A(x)\nabla\varphi_1 \cdot \nu = 0 \text{ on } \Gamma_N$$

and

$$(1.10) \quad -\operatorname{div}(A(x)\nabla\varphi_2) = 0 \text{ in } \Omega,$$

$$(1.11) \quad \varphi_2 = \varphi_0 \text{ on } \Gamma_D,$$

$$(1.12) \quad A(x)\nabla\varphi_2 \cdot \nu = 0 \text{ on } \Gamma_N.$$

Now set

$$(1.13) \quad u = \frac{\varphi_1}{\|f\|_{\frac{N}{2}}},$$

$$(1.14) \quad g = \frac{f}{\|f\|_{\frac{N}{2}}}.$$

Then we have

$$(1.15) \quad -\operatorname{div}(A(x)\nabla u) = g \text{ in } \Omega,$$

$$(1.16) \quad u = 0 \text{ on } \Gamma_D,$$

$$(1.17) \quad A(x)\nabla u \cdot \nu = 0 \text{ on } \Gamma_N.$$

Lemma B. *Let c_s be the smallest positive number such that*

$$(1.18) \quad \|\phi\|_{2^*} \leq c_s \|\nabla\phi\|_2$$

for all $\phi \in W^{1,2}(\Omega)$ with $\phi = 0$ on Γ_D . It is well-known that the Sobolev constant c_s here depends only on Γ_D, Ω . To each $\alpha \in (0, \frac{4\lambda}{c_s^2})$ there corresponds a positive number c such that

$$(1.19) \quad \int_{\Omega} \exp\left(\frac{\alpha Nu}{N-2}\right) dx \leq c.$$

Proof. Let $\alpha \in (0, \frac{4\lambda}{c_s^2})$ be given. Since u is bounded, $e^{\alpha u} - 1$ is a legitimate test function for (1.15), and upon using it, we obtain

$$(1.20) \quad \lambda\alpha \int_{\Omega} e^{\alpha u} |\nabla u|^2 dx \leq \alpha \int_{\Omega} A(x)\nabla u \cdot e^{\alpha u} \nabla u dx = \int_{\Omega} g(e^{\alpha u} - 1) dx.$$

Due to the Sobolev Imbedding Theorem and the fact that $e^{\alpha u} - 1 = (e^{\frac{\alpha u}{2}} - 1)^2 + 2(e^{\frac{\alpha u}{2}} - 1)$, we estimate

$$\begin{aligned}
 \left(\int_{\Omega} (e^{\frac{\alpha u}{2}} - 1)^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} &\leq c_s^2 \int_{\Omega} |\nabla (e^{\frac{\alpha u}{2}} - 1)|^2 dx \\
 &= \frac{c_s^2 \alpha^2}{4} \int_{\Omega} e^{\alpha u} |\nabla u|^2 dx \\
 &\leq \frac{c_s^2 \alpha}{4\lambda} \int_{\Omega} g(e^{\alpha u} - 1) dx \\
 &= \frac{c_s^2 \alpha}{4\lambda} \int_{\Omega} g(e^{\frac{\alpha u}{2}} - 1)^2 dx + \frac{c_s^2 \alpha}{2\lambda} \int_{\Omega} g(e^{\frac{\alpha u}{2}} - 1) dx \\
 &\leq \frac{c_s^2 \alpha}{4\lambda} \|g\|_{\frac{N}{2}} \left(\int_{\Omega} (e^{\frac{\alpha u}{2}} - 1)^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \\
 &\quad + \frac{c_s^2 \alpha}{2\lambda} \|g\|_{\frac{2N}{N+2}} \left(\int_{\Omega} (e^{\frac{\alpha u}{2}} - 1)^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}}.
 \end{aligned}$$

Take into consideration $1 - \frac{c_s^2 \alpha}{4\lambda} > 0$ and (1.14) to deduce

$$(1.21) \quad \left(\int_{\Omega} (e^{\frac{\alpha u}{2}} - 1)^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \leq \frac{2c_s^2 \alpha \|g\|_{\frac{2N}{N+2}}}{4\lambda - c_s^2 \alpha}.$$

Note that

$$\|g\|_{\frac{2N}{N+2}} = \frac{\|f\|_{\frac{2N}{N+2}}}{\|f\|_{\frac{N}{2}}} \leq |\Omega|^{\frac{N-2}{2N}}.$$

We arrive at

$$\begin{aligned}
 \|e^{\frac{\alpha u}{2}}\|_{\frac{2N}{N-2}} &\leq \|e^{\frac{\alpha u}{2}} - 1\|_{\frac{2N}{N-2}} + |\Omega|^{\frac{N-2}{2N}} \\
 &\leq \frac{2c_s^2 \alpha |\Omega|^{\frac{N-2}{2N}}}{4\lambda - c_s^2 \alpha} + |\Omega|^{\frac{N-2}{2N}} \\
 &= \frac{4\lambda + c_s^2 \alpha}{4\lambda - c_s^2 \alpha} |\Omega|^{\frac{N-2}{2N}}.
 \end{aligned}$$

The proof of the lemma is complete.

To continue the proof of Theorem A, we assume without any loss of generality that

$$(1.22) \quad \operatorname{ess\,sup}_{\Omega} u = \|u\|_{\infty},$$

$$(1.23) \quad \|f\|_{\frac{N}{2}} \geq 1.$$

Indeed, if (1.22) is not true, we multiply through (1.15) by -1 and consider $-u$. If (1.23) is not true, we consider the problem (1.7)-(1.9) directly because we already have that (1.19) holds for $u = \varphi_1$. Let

$$(1.24) \quad v = e^u.$$

Then

$$\begin{aligned} u &= \ln v, \\ A(x)\nabla u &= \frac{1}{v}A(x)\nabla v, \\ \operatorname{div}(A(x)\nabla u) &= -\frac{1}{v^2}A(x)\nabla v \cdot \nabla v + \frac{1}{v}\operatorname{div}(A(x)\nabla v). \end{aligned}$$

Substitute these into (1.15)-(1.17) to obtain

$$(1.25) \quad -\operatorname{div}(A(x)\nabla v) + \frac{1}{v}A(x)\nabla v \cdot \nabla v = gv \text{ in } \Omega,$$

$$(1.26) \quad v = 1 \text{ on } \Gamma_D,$$

$$(1.27) \quad A(x)\nabla v \cdot \nu = 0 \text{ on } \Gamma_N.$$

The above formal calculations can easily be made vigorous. Set

$$w = (v - 1)^+ + 1.$$

Then it is easy to see that

$$(1.28) \quad w(x) = \begin{cases} v(x) & \text{if } v(x) > 1, \\ 1 & \text{if } v(x) \leq 1. \end{cases}$$

For $\beta > 0$, we use $w^\beta - 1$ as a test function in (1.25) to obtain

$$(1.29) \quad \int_{\Omega} \beta w^{\beta-1} A(x) \nabla v \cdot \nabla w dx \leq \int_{\Omega} gv(w^\beta - 1) dx.$$

This together with (1.28) implies

$$(1.30) \quad \lambda\beta \int_{\Omega} w^{\beta-1} |\nabla w|^2 dx \leq \int_{\Omega} gv(w^\beta - 1) dx \leq \int_{\Omega} |g|w^{\beta+1} dx.$$

Note that

$$\begin{aligned} w^{\beta+1} &= \left(w^{\frac{\beta+1}{2}} - 1 + 1 \right)^2 \\ &\leq 2 \left(w^{\frac{\beta+1}{2}} - 1 \right)^2 + 2. \end{aligned}$$

Keeping this in mind and with the aid of the Sobolev Imbedding Theorem and (1.30), we calculate

$$\begin{aligned} \|w^{\beta+1}\|_{\frac{N}{N-2}} &\leq 2 \left\| \left(w^{\frac{\beta+1}{2}} - 1 \right)^2 \right\|_{\frac{N}{N-2}} + 2|\Omega|^{\frac{N-2}{N}} \\ &\leq c_s^2 \frac{(\beta+1)^2}{2} \int_{\Omega} w^{\beta-1} |\nabla w|^2 dx + 2|\Omega|^{\frac{N-2}{N}} \\ &= \frac{c(\beta+1)^2}{\beta} \int_{\Omega} |g|w^{\beta+1} dx + 2|\Omega|^{\frac{N-2}{N}}, \end{aligned}$$

where $c = c(c_s, \lambda) = c(\Gamma_D, \Omega, \lambda)$. Remember $w \geq 1$ on Ω . For $p > N$, we have

$$\begin{aligned} \|w^{\beta+1}\|_{\frac{N}{N-2}} &\leq \frac{c(\beta+1)^2}{\beta} \|g\|_{\frac{p}{2}} \left(\int_{\Omega} w^{\frac{(\beta+1)p}{p-2}} dx \right)^{\frac{p-2}{p}} + 2 \left(\int_{\Omega} w^{\frac{(\beta+1)p}{p-2}} dx \right)^{\frac{p-2}{p}} |\Omega|^{\frac{2}{p} - \frac{2}{N}} \\ &= \left(\frac{c(\beta+1)^2}{\beta} \|g\|_{\frac{p}{2}} + 2|\Omega|^{\frac{2}{p} - \frac{2}{N}} \right) \|w^{\beta+1}\|_{\frac{p}{p-2}}. \end{aligned}$$

Observe that

$$(1.31) \quad \frac{(\beta+1)^2}{\beta} > 4.$$

Thus we can conclude that

$$(1.32) \quad \|w\|_{\frac{N(\beta+1)}{N-2}} \leq \left[c \frac{(\beta+1)^2}{\beta} (\|g\|_{\frac{p}{2}} + 1) \right]^{\frac{1}{\beta+1}} \|w\|_{\frac{p(\beta+1)}{p-2}}.$$

Now set

$$\chi = \frac{N/(N-2)}{p/(p-2)} > 1.$$

Fix $\beta_0 > 0$ and let

$$\beta + 1 = (1 + \beta_0)\chi^i, \quad i = 0, 1, 2, \dots$$

Subsequently,

$$\begin{aligned} \|w\|_{(1+\beta_0)\frac{p}{p-2}\chi^{i+1}} &\leq \left(\frac{c}{\beta_0} (\|g\|_{\frac{p}{2}} + 1) \right)^{\frac{1}{(1+\beta_0)\chi^i}} \left[(1 + \beta_0)^2 \chi^{2i} \right]^{\frac{1}{(1+\beta_0)\chi^i}} \|w\|_{(1+\beta_0)\frac{p}{p-2}\chi^i} \\ &\leq \left[c (\|g\|_{\frac{p}{2}} + 1) \frac{(1 + \beta_0)^2}{\beta_0} \right]^{\frac{1}{(1+\beta_0)}(\frac{1}{\chi^i} + \dots + 1)} \\ &\quad \cdot \chi^{\frac{2}{(1+\beta_0)}(\frac{i}{\chi^i} + \dots + 1)} \|w\|_{(1+\beta_0)\frac{p}{p-2}}. \end{aligned}$$

Taking $i \rightarrow \infty$ yields

$$(1.33) \quad \|w\|_{\infty} \leq c \left(\|g\|_{\frac{p}{2}} + 1 \right)^{\alpha_0} \|w\|_{(1+\beta_0)\frac{p}{p-2}},$$

where α_0 is a positive number depending on β_0, p . Let

$$q = (1 + \beta_0) \frac{p}{p-2}.$$

We can assume that

$$q < \frac{4\lambda N}{c_s^2(N-2)}.$$

If not, pick a number α so that

$$(1.34) \quad 0 < \alpha < \frac{4\lambda N}{c_s^2(N-2)} \leq q.$$

Then we have

$$\begin{aligned} \|w\|_q &= \left(\int_{\Omega} w^{q-\alpha} w^{\alpha} dx \right)^{\frac{1}{q}} \\ &\leq \|w\|_{\infty}^{\frac{q-\alpha}{q}} \|w\|_{\alpha}^{\frac{\alpha}{q}}. \end{aligned}$$

We derive from (1.33) that

$$\begin{aligned} \|w\|_{\infty} &\leq c \left(\|g\|_{\frac{p}{2}} + 1 \right)^{\alpha_0} \|w\|_{\infty}^{\frac{q-\alpha}{q}} \|w\|_{\alpha}^{\frac{\alpha}{q}} \\ &\leq \frac{1}{2} \|w\|_{\infty} + c \left(\|g\|_{\frac{p}{2}} + 1 \right)^{\alpha_0 \frac{q}{\alpha}} \|w\|_{\alpha} \end{aligned}$$

from whence it follows that

$$(1.35) \quad \|w\|_{\infty} \leq c \left(\|g\|_{\frac{p}{2}} + 1 \right)^{\alpha_0 \frac{q}{\alpha}} \|w\|_{\alpha}.$$

Recall the definition of w to obtain

$$(1.36) \quad \|v\|_\infty \leq c \left(\|g\|_{\frac{p}{2}} + 1 \right)^{\alpha_0 \frac{q}{\alpha}} (\|v\|_\alpha + |\Omega|^{\frac{1}{\alpha}}).$$

In view of (1.24), (1.34), (1.22) and (1.19), we obtain

$$(1.37) \quad e^{\|u\|_\infty} \leq c \left(\|g\|_{\frac{p}{2}} + 1 \right)^{\alpha_0 \frac{q}{\alpha}}.$$

Take the logarithm and make a note of (1.23) to get

$$(1.38) \quad \|u\|_\infty \leq c(\ln(\|f\|_{\frac{p}{2}} + 1) + 1).$$

Finally, we derive

$$\begin{aligned} \|\varphi\|_\infty &\leq \|\varphi_1\|_\infty + \|\varphi_2\|_\infty \\ &\leq c\|f\|_{\frac{N}{2}}(\ln(\|f\|_{\frac{p}{2}} + 1) + 1) + \|\varphi_0\|_\infty. \end{aligned}$$

This completes the proof.

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