POWER MEANS GENERATED BY SOME MEAN-VALUE THEOREMS

JANUSZ MATKOWSKI

(Communicated by Edward C. Waymire)

ABSTRACT. According to a new mean-value theorem, under the conditions of a function f ensuring the existence and uniqueness of Lagrange's mean, there exists a unique mean M such that

$$\frac{f(x) - f(y)}{x - y} = M(f'(x), f'(y)).$$

The main result says that, in this equality, M is a power mean if, and only if, M is either geometric, arithmetic or harmonic. A Cauchy relevant type result is also presented.

INTRODUCTION

In a recent paper [4] the following counterpart of the Lagrange mean-value theorem has been proved. *If a real function f defined on an interval $I \subset \mathbb{R}$ is differentiable, and f' is one-to-one, then there exists a unique mean function $M : f'(I) \times f'(I) \rightarrow f'(I)$, such that*

$$\frac{f(x) - f(y)}{x - y} = M(f'(x), f'(y)), \quad x, y \in I, \quad x \neq y.$$

In this connection the following problem arises. Given a mean M , determine all differentiable real functions f such that

$$(1) \quad \frac{f(x) - f(y)}{x - y} = M(f'(x), f'(y)), \quad x, y \in I, \quad x \neq y.$$

In the case when M is the geometric mean this equation has appeared in [3] and was useful in solving an open problem related to convex functions (cf. Remark 5).

In the first section we consider equation (1) in the case when $M = M^{[\varphi]}$, where

$$M^{[\varphi]}(u, v) = \varphi^{-1} \left(\frac{\varphi(u) + \varphi(v)}{2} \right), \quad u, v \in J,$$

and $\varphi : f'(I) \rightarrow \mathbb{R}$ is a continuous and strictly monotonic function; so M is a quasi-arithmetic mean of a generator φ . Assuming three times continuous differentiability of f , and twice continuous differentiability of φ , we give some necessary conditions for equality (1) (Theorem 1). Applying this result, in the next section we give a

Received by the editors August 26, 2010.

2010 *Mathematics Subject Classification.* Primary 26A24, 26E60; Secondary 39B22.

Key words and phrases. Mean-value theorem, mean, quasi-arithmetic mean, arithmetic mean, geometric mean, harmonic mean, Jensen functional equation, differential equation.

complete solution of the problem in the case when $M^{[\varphi]}$ is positively homogeneous, that is, when either $\varphi(t) = At^p + B$ or $\varphi(t) = A \log t + B$ for some real p, A, B such that $A \neq 0 \neq p$. Then $M = M^{[p]} : (0, \infty)^2 \rightarrow (0, \infty)$ is a power mean, that is,

$$M^{[p]}(u, v) := \begin{cases} \left(\frac{u^p + v^p}{2}\right)^{1/p} & \text{if } p \neq 0, \\ \sqrt{uv} & \text{if } p = 0. \end{cases}$$

The main result (Theorem 2) says that equality (1) with $M = M^{[p]}$ holds if, and only if, the mean M is either arithmetic ($M^{[1]}(u, v) = \frac{u+v}{2}$), geometric ($M^{[0]}(u, v) = \sqrt{uv}$) or harmonic ($M^{[-1]}(u, v) = \frac{2uv}{u+v}$).

Assume that the functions $f, g : I \rightarrow \mathbb{R}$ satisfy the conditions ensuring the existence and uniqueness of the classical Cauchy mean-value. Then (cf. [4]) there exists a unique mean $M : J^2 \rightarrow J$, with $J := \frac{f'}{g'}(I)$, such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = M\left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)}\right), \quad x, y \in I, \quad x \neq y.$$

Applying Theorem 2, we determine all power means M and the functions f, g satisfying this equation.

1. THE CASE WHEN M IS QUASI-ARITHMETIC

In this section we prove:

Theorem 1. *Let $I, J \subset \mathbb{R}$ be intervals. Suppose that*

$f : I \rightarrow \mathbb{R}$ is three times continuously differentiable, $f''(x) \neq 0$ for $x \in I$;

$\varphi : J \rightarrow \mathbb{R}$ is twice continuously differentiable, and $\varphi'(u) \neq 0$ for $u \in J$.

If

$$(2) \quad \frac{f(x) - f(y)}{x - y} = M^{[\varphi]}(f'(x), f'(y)), \quad x, y \in I, \quad x \neq y,$$

then there exists $C \in \mathbb{R}$, $C \neq 0$, such that

$$(3) \quad \varphi'(f'(x)) = \frac{C}{f''(x)^{2/3}}, \quad x \in I,$$

and

$$(4) \quad f''(x) \left(f'(x) - \frac{f(x) - f(y)}{x - y} \right)^3 = f''(y) \left(\frac{f(x) - f(y)}{x - y} - f'(y) \right)^3, \quad x, y \in I, \quad x \neq y.$$

Proof. Without any loss of generality we can assume that $\varphi'(x) > 0$ in J . Suppose that (2) holds true. Then, from the definition of the quasi-arithmetic mean,

$$2\varphi\left(\frac{f(x) - f(y)}{x - y}\right) = \varphi(f'(x)) + \varphi(f'(y)), \quad x, y \in I, \quad x \neq y.$$

Differentiating both sides, first with respect to x and then with respect to y , we get

$$(5) \quad 2\varphi'\left(\frac{f(x) - f(y)}{x - y}\right) \frac{f'(x)(x - y) - f(x) + f(y)}{(x - y)^2} = \varphi'(f'(x)) f''(x),$$

$$(6) \quad 2\varphi'\left(\frac{f(x) - f(y)}{x - y}\right) \frac{f'(y)(y - x) - f(y) + f(x)}{(x - y)^2} = \varphi'(f'(y)) f''(y)$$

for all $x, y \in I$, $x \neq y$. Subtracting the respective sides of these equalities and dividing the obtained differences by $x - y$ we have

$$(7) \quad 2\varphi' \left(\frac{f(x) - f(y)}{x - y} \right) \frac{[f'(x) + f'(y)](x - y) - 2[f(x) - f(y)]}{(x - y)^3} \\ = \frac{\varphi'(f'(x)) f''(x) - \varphi'(f'(y)) f''(y)}{x - y}$$

for all $x, y \in I$, $x \neq y$. Applying L'Hospital's rule (or Taylor's theorem) we easily get

$$\lim_{y \rightarrow x} \frac{[f'(x) + f'(y)](x - y) - 2[f(x) - f(y)]}{(x - y)^3} = \frac{f'''(x)}{6}, \quad x \in I.$$

Hence, letting $y \rightarrow x$ in (7), we obtain

$$2\varphi'(f'(x)) \frac{f'''(x)}{6} = \varphi''(f'(x)) [f''(x)]^2 + \varphi'(f'(x)) f'''(x),$$

whence

$$3\varphi''(f'(x)) [f''(x)]^2 + 2\varphi'(f'(x)) f'''(x) = 0, \quad x \in I.$$

Assume first that $f''(x) \neq 0$ for all $x \in I$. Then, by the Darboux property of a derivative, f'' is of a constant sign in I . Dividing both sides by $f''(x)\varphi'(f'(x))$ we hence get

$$3 \frac{\varphi''(f'(x))}{\varphi'(f'(x))} f''(x) + 2 \frac{f'''(x)}{f''(x)} = 0, \quad x \in I,$$

or, equivalently,

$$(3 \log \varphi'(f'(x)) + 2 \log f''(x))' = 0, \quad x \in I,$$

whence, after simple calculation,

$$\varphi'(f'(x)) = \frac{C}{f''(x)^{2/3}}, \quad x \in I,$$

for some $C \in \mathbb{R}$, $C \neq 0$.

Hence, dividing the respective sides of (5) and (6) (by the assumption it can be done), we obtain

$$\frac{f'(x)(x - y) - f(x) + f(y)}{f'(y)(y - x) - f(y) + f(x)} = \left(\frac{f''(y)}{f''(x)} \right)^{1/3}, \quad x, y \in I, \quad x \neq y,$$

which implies (4). This completes the proof. $\square$

Remark 1. The function $f(x) = \alpha x + \beta$, $x \in I$, satisfies equation (1) with an arbitrary mean $M : J^2 \rightarrow J$. Therefore in Theorem 1 we assume that f' is not a constant function.

Remark 2. From equation (1) we have

$$f(x) - f(y) = M(f'(x), f'(y))(x - y), \quad x, y \in I.$$

Since $f(x) - f(y) = [f(x) - f(z)] + [f(z) - f(y)]$, for all $x, y, z \in I$, we get

$$M(f'(x), f'(y))(x - y) = M(f'(x), f'(z))(x - z) + M(f'(z), f'(y))(z - y).$$

Assuming that f' is one-to-one and putting $h := (f')^{-1}$, we hence obtain

$$M(u, v)[h(u) - h(v)] = M(u, w)[h(u) - h(w)] + M(w, v)[h(w) - h(v)]$$

for all $u, v, w \in J := f'(I)$.

If $f(x) = \frac{a}{2}x^2 + bx + c$ for some $a, b, c \in \mathbb{R}$, then $h(u) = \frac{u-b}{a}$. It is easy to verify that h and $M(u, v) = \frac{u+v}{2}$ satisfy this equality.

2. THE CASE WHEN M IS A POWER MEAN

In this part we assume that M in equation (1) is a power mean. The main result reads as follows.

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $f : I \rightarrow \mathbb{R}$ is differentiable, $f'(x) > 0$ for $x \in I$, and f' is not constant in I . Then*

$$(8) \quad \frac{f(x) - f(y)}{x - y} = M^{[p]}(f'(x), f'(y)), \quad x, y \in I, \quad x \neq y,$$

for some $p \in \mathbb{R}$ if, and only if, one of the following cases occurs:

(i): $p = 0$ (that is, $M^{[p]}$ is the geometric mean) and for some $a, b, c \in \mathbb{R}$, $ac - b \neq 0$,

$$f(x) = \frac{ax + b}{x + c}, \quad x \in I;$$

(ii): $p = 1$ (that is, $M^{[p]}$ is the arithmetic mean) and for some $a, b, c \in \mathbb{R}$, $a \neq 0$,

$$f(x) = \frac{a}{2}x^2 + bx + c, \quad x \in I;$$

(iii): $p = -1$ (that is, $M^{[p]}$ is the harmonic mean) and for some $a, b, c, k \in \mathbb{R}$, $a \neq 0 \neq k$,

$$f(x) = k\sqrt{ax + b} + c, \quad x \in I.$$

Proof. First consider the case when $p = 0$. If f satisfies equation (8), then

$$\frac{f(x) - f(y)}{x - y} = \sqrt{f'(x)f'(y)}, \quad x, y \in I, \quad x \neq y,$$

whence

$$f(x) - f(y) = \sqrt{f'(x)f'(y)}(x - y), \quad x, y \in I.$$

Since $f(x) - f(y) = [f(x) - f(z)] + [f(z) - f(y)]$ for all $x, y, z \in I$, we hence get

$$\sqrt{f'(x)f'(y)}(x - y) = \sqrt{f'(x)f'(z)}(x - z) + \sqrt{f'(z)f'(y)}(z - y), \quad x, y, z \in I.$$

Setting here $z = \frac{x+y}{2}$, $x \neq y$, and then dividing both sides by $x - y$, we get

$$2\sqrt{f'(x)f'(y)} = \sqrt{f'(x)f'\left(\frac{x+y}{2}\right)} + \sqrt{f'\left(\frac{x+y}{2}\right)f'(y)},$$

which, obviously, also remains true for all $x, y \in I$. Dividing both sides of this equality by $2\sqrt{f'(x)f'\left(\frac{x+y}{2}\right)f'(y)}$ we hence get

$$\frac{1}{\sqrt{f'\left(\frac{x+y}{2}\right)}} = \frac{\frac{1}{\sqrt{f'(x)}} + \frac{1}{\sqrt{f'(y)}}}{2}, \quad x, y \in I;$$

that is, $\frac{1}{\sqrt{f'(\frac{x+y}{2})}}$ is the arithmetic mean of $\frac{1}{\sqrt{f'(x)}}$ and $\frac{1}{\sqrt{f'(y)}}$ (and $\sqrt{f'(\frac{x+y}{2})}$ is the harmonic mean of $\sqrt{f'(x)}$ and $\sqrt{f'(y)}$). It follows that the function $\gamma := 1/\sqrt{f'}$ satisfies the Jensen functional equation

$$\gamma\left(\frac{x+y}{2}\right) = \frac{\gamma(x) + \gamma(y)}{2}, \quad x, y \in I.$$

Since γ is Lebesgue measurable, there are $k, m \in \mathbb{R}$ such that $\gamma(x) = kx + m$, for all $x \in I$ (cf. M. Kuczma [2, Chapter XIII, Section 2]). Hence

$$f'(x) = \frac{1}{(kx + m)^2}, \quad x \in I,$$

where $k \neq 0$, as f' is not constant, whence, for some real a ,

$$f(x) = a - \frac{1}{k(kx + m)} = \frac{ax + (\frac{am}{k} - \frac{1}{k^2})}{x + \frac{m}{k}}, \quad x \in I.$$

Setting $b := \frac{am}{k} - \frac{1}{k^2}$, $c := \frac{m}{k}$ we get

$$f(x) = \frac{ax + b}{x + c}, \quad x \in I,$$

and

$$ac - b = a \frac{m}{k} - \left(\frac{am}{k} - \frac{1}{k^2}\right) = \frac{1}{k^2} \neq 0.$$

It is easy to verify that f satisfies equation (8).

Now assume that $p \neq 0$. In this case,

$$\varphi(t) = At^p + B, \quad t > 0,$$

for some $A, B \in \mathbb{R}$, $A \neq 0$, is a generator of the mean $M^{[p]}$, and equation (8) can be written in the form

$$\frac{f(x) - f(y)}{x - y} = \left(\frac{[f'(x)]^p + [f'(y)]^p}{2} \right)^{1/p}, \quad x, y \in I, \quad x \neq y.$$

Assume that $f : I \rightarrow \mathbb{R}$ satisfies this equation. Then, obviously, f is of the class C^∞ in I .

Let $I_0 \subset I$ be a maximal open and non-empty interval such that $f''(x) \neq 0$ for all $x \in I_0$. By the Darboux property of a derivative, f'' is of a constant sign in I_0 . Since, by assumption, f'' does not vanish everywhere in I , such an interval exists. In view of Theorem 1, there is $C \neq 0$ such that

$$pA[f'(x)]^{p-1} = \frac{C}{|f''(x)|^{2/3}}, \quad x \in I_0,$$

whence

$$(9) \quad [f'(x)]^{p-1} |f''(x)|^{2/3} = \frac{C}{pA}, \quad x \in I_0.$$

If $p = 1$ we hence get

$$f''(x) = a, \quad x \in I_0,$$

for some $a \in \mathbb{R}$, $a \neq 0$. Consequently

$$f(x) = \frac{a}{2}x^2 + bx + c, \quad x \in I_0,$$

for some $b, c \in \mathbb{R}$ and, obviously, $I = I_0$. Since

$$\frac{f(x) - f(y)}{x - y} = a \frac{x + y}{2} + b = \frac{(ax + b) + (ay + b)}{2} = M^{[1]}(f'(x), f'(y))$$

for all $x, y \in I$, $x \neq y$, equation (8) is satisfied.

In the sequel we assume that $p \neq 1$. Since f'' is of a constant sign in I_0 , from (9) we get

$$(10) \quad [f'(x)]^q f''(x) = a, \quad x \in I_0,$$

for some $a \neq 0$ and

$$(11) \quad q := \frac{3}{2}(p - 1).$$

If $q = -1$, then (10) implies that

$$\log f'(x) = ax + b, \quad x \in I_0,$$

for some $b \in \mathbb{R}$, whence

$$f(x) = \frac{1}{a} e^{ax+b} + \delta, \quad x \in I_0,$$

and, of course, $I = I_0$. From (11) we get

$$p = \frac{1}{3}.$$

Since

$$\frac{f(x) - f(y)}{x - y} = \frac{e^{ax+b} - e^{ay+b}}{a(x - y)}, \quad x, y \in I, \quad x \neq y,$$

and

$$M^{[1/3]}(f'(x), f'(y)) = \left(\frac{[e^{ax+b}]^{1/3} + [e^{ay+b}]^{1/3}}{2} \right)^3, \quad x, y \in I, \quad x \neq y,$$

equation (8) is not fulfilled.

If $q \neq -1$, then, from (10),

$$\frac{1}{q+1} [f'(x)]^{q+1} = ax + b, \quad x \in I_0;$$

that is,

$$(12) \quad f'(x) = [(q+1)(ax+b)]^{1/(q+1)}, \quad x \in I_0.$$

For $q = -2$, we hence get

$$f'(x) = -\frac{1}{ax+b}, \quad x \in I_0,$$

whence, for some $m \in \mathbb{R}$,

$$(13) \quad f(x) = -\log |ax+b| + m, \quad x \in I_0,$$

and, of course, $I = I_0$. In this case, (11) implies that

$$p = -\frac{1}{3}.$$

From (13) we have

$$\frac{f(x) - f(y)}{x - y} = \frac{1}{x - y} \log \left| \frac{ay + b}{ax + b} \right|, \quad x, y \in I, \quad x \neq y,$$

$$M^{[-1/3]}(f'(x), f'(y)) = \frac{8}{(\sqrt[3]{ax + b} + \sqrt[3]{ay + b})^3}, \quad x, y \in I,$$

and, obviously, equation (8) is not satisfied.

Assume that $q \notin \{-1, -2\}$. From (12) we get

$$f(x) = \frac{1}{(q+2)a} [(q+1)(ax+b)]^{(q+2)/(q+1)} + m, \quad x \in I_0,$$

for some $k, m, r \in \mathbb{R}$, $k \neq 0$. Thus

$$(14) \quad f(x) = k(ax+b)^r + m, \quad x \in I_0,$$

where

$$(15) \quad r := \frac{q+2}{q+1} \neq 0, \quad k := \frac{(q+1)^{(q+2)/(q+1)}}{(q+2)a} \neq 0.$$

Making use of (14) and setting $u := ax + b$, $v := ay + b$ for $x, y \in I_0$, $x \neq y$, we get

$$\frac{f(x) - f(y)}{x - y} = k \frac{(ax+b)^r - (ay+b)^r}{x - y} = k \frac{u^r - v^r}{\frac{u-b}{a} - \frac{v-b}{a}} = ka \frac{u^r - v^r}{u - v},$$

and

$$f'(x) = kar(ax+b)^{r-1}, \quad f'(y) = kar(ay+b)^{r-1}.$$

Thus

$$M^{[p]}(f'(x), f'(y)) = kar \left(\frac{(ax+b)^{p(r-1)} + (ay+b)^{p(r-1)}}{2} \right)^{1/p}$$

$$= kar \left(\frac{u^{p(r-1)} + v^{p(r-1)}}{2} \right)^{1/p},$$

whence, by (8),

$$\frac{u^r - v^r}{u - v} = r \left(\frac{u^{p(r-1)} + v^{p(r-1)}}{2} \right)^{1/p}$$

for all $u, v \in J_0 := aI_0 + b$, $u \neq v$. The right side,

$$R(u, v) := r \left(\frac{u^{p(r-1)} + v^{p(r-1)}}{2} \right)^{1/p},$$

is an analytic function (in the real sense) in $(0, \infty)^2$. This equality implies that the function on the left side extended to the diagonal by the formula

$$L(u, v) := \begin{cases} \frac{u^r - v^r}{u - v} & \text{for } u \neq v, \\ ru^{r-1} & \text{for } u = v \end{cases}$$

is also analytic in $(0, \infty)^2$ (cf. also Remark 3 below) and we have

$$R(u, v) - L(u, v) = 0, \quad u, v > 0.$$

Setting $v = 1$ we get

$$g(u) := R(u, 1) - L(u, 1) = 0, \quad u > 0,$$

and, of course,

$$g^{(k)}(1) = 0 \quad \text{for all } k \in \{0, 1, \dots\}.$$

After some calculations we get

$$g''(1) = \frac{1}{12}r(1-r)[3p(r-1) - (r+1)]$$

and

$$g^{(4)}(1) = \frac{1}{80}r(r-1)[10p^3(r-1)^3 + 15p^2(1-r)^3 + 10p(1-r)(3r^2 - 24r + 43) + 11r^3 - 69r^2 + 61r + 141].$$

Since $p \neq 0$ and $p \neq 1$, from the equality $g''(1) = 0$ we get

$$(16) \quad p = \frac{r+1}{3(r-1)}.$$

Setting this value into the equality $g^{(4)}(1) = 0$ we get

$$g^{(4)}(1) = \frac{r(2-r)(r-1)(r+1)(2r-1)}{540} = 0.$$

By (15) and (16), respectively, we can omit the cases when $r = 0$ or $r = 1$. If $r = -1$, then, by (16), we get $p = 0$, the case already considered. Therefore, applying (16), we conclude that either $r = 2$ and $p = 1$ or $r = \frac{1}{2}$ and $p = -1$.

If $r = 2$ and $p = 1$, then, by (14), we get

$$f(x) = k(ax+b)^2 \quad \text{for } x \in I_0 \quad \text{and} \quad M^{[1]}(u, v) = \frac{u+v}{2}, \quad u, v > 0,$$

and, if $r = \frac{1}{2}$ and $p = -1$, we get

$$f(x) = k\sqrt{ax+b} \quad \text{for } x \in I_0 \quad \text{and} \quad M^{[-1]}(u, v) = \frac{2uv}{u+v}, \quad u, v > 0.$$

It is easy to verify that in both cases equation (8) is fulfilled. Moreover, in each of these cases, the regularity of the solutions implies that $I = I_0$. This completes the proof. $\square$

Remark 3. The analyticity of the function L can also be obtained as follows. Treating L as a function of two complex variables, u and v , it is easy to see that, at any point of the diagonal points $(u, u) \neq (0, 0)$, the function L is separately analytic (holomorphic) at (u, u) with respect to each variable. Therefore, by the famous theorem of Hartogs [1], L is analytic at (u, u) with respect to both variables.

Remark 4. The necessity of the positivity of f' in Theorem 2 follows from the definition of the power means. Defining $M^{[p]} : (-\infty, 0)^2 \rightarrow (-\infty, 0)$ by the formula

$$M^{[p]}(u, v) := \begin{cases} -\left(\frac{(-u)^p + (-v)^p}{2}\right)^{1/p} & \text{if } p \neq 0, \\ -\sqrt{uv} & \text{if } p = 0, \end{cases}$$

we can formulate the counterpart of Theorem 2 for f such that $f' < 0$.

Remark 5. Equation (8) for $p = 0$ has appeared to be useful in solving a problem of Zs. Páles concerning existence of discontinuous Jensen affine (convex and concave) functions in the sense of Beckenbach with respect to the two-parameter family of curves generated by a Tchebycheff system [3].

Applying Theorem 2 we can prove the following:

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval. Suppose that $f, g : I \rightarrow \mathbb{R}$ are differentiable, $f'(x)g'(x) > 0$ for $x \in I$ and f'/g' is not constant in I . If M is a power mean, then*

$$(17) \quad \frac{f(x) - f(y)}{g(x) - g(y)} = M \left(\frac{f'(x)}{g'(x)}, \frac{f'(y)}{g'(y)} \right), \quad x, y \in I, \quad x \neq y,$$

if, and only if, one of the following cases occurs:

(i): M is the geometric mean, g is arbitrary, and for some $a, b, c \in \mathbb{R}$, $ac - b \neq 0$,

$$f(x) = \frac{ag(x) + b}{g(x) + c}, \quad x \in I;$$

(ii): M is the arithmetic mean, g is arbitrary, and for some $a, b, c \in \mathbb{R}$, $a \neq 0$,

$$f(x) = \frac{a}{2}g(x)^2 + bg(x) + c, \quad x \in I;$$

(iii): M is the harmonic mean, g is arbitrary, and for some $a, b, c, k \in \mathbb{R}$, $a \neq 0 \neq k$,

$$f(x) = k\sqrt{ag(x) + b} + c, \quad x \in I.$$

Proof. Suppose that the functions f and g satisfy the assumptions and equation (17) where $M = M^{[p]}$ for some $p \in \mathbb{R}$. Obviously $g(I)$ is an interval and the function $f \circ g^{-1} : g(I) \rightarrow \mathbb{R}$ satisfies all assumptions of Theorem 2. Take arbitrary $u, v \in g(I)$, $u \neq v$. Setting $x := g^{-1}(u)$, $y := g^{-1}(v)$ in (17) we get

$$\begin{aligned} \frac{f \circ g^{-1}(u) - f \circ g^{-1}(v)}{u - v} &= M^{[p]} \left(\frac{f' \circ g^{-1}(u)}{g' \circ g^{-1}(u)}, \frac{f' \circ g^{-1}(v)}{g' \circ g^{-1}(v)} \right) \\ &= M^{[p]} \left((f \circ g^{-1})'(u), (f \circ g^{-1})'(v) \right), \end{aligned}$$

which means that the function $f \circ g^{-1}$ satisfies equation (8). Theorem 2 implies that either $p = 0$ and for some $a, b, c \in \mathbb{R}$, $ac - b \neq 0$,

$$f \circ g^{-1}(u) = \frac{au + b}{u + c}, \quad u \in g(I),$$

or $p = 1$ and for some $a, b, c \in \mathbb{R}$, $a \neq 0$,

$$f \circ g^{-1}(u) = \frac{a}{2}u^2 + bu + c, \quad u \in g(I),$$

or $p = -1$ and for some $a, b, c, k \in \mathbb{R}$, $a \neq 0 \neq k$,

$$f \circ g^{-1}(u) = k\sqrt{au + b} + c, \quad u \in g(I).$$

Setting in these formulas $u = g(x)$ for $x \in I$, we obtain the result. $\square$

3. FINAL REMARK

Replacing f' by h of the equation for $p = 0$, $p = 1$, and $p = -1$ (the cases itemized in Theorem 2) we get three functional equations:

$$\begin{aligned}\frac{f(x) - f(y)}{x - y} &= \sqrt{h(x)h(y)}, & x, y \in I, x \neq y, \\ \frac{f(x) - f(y)}{x - y} &= \frac{h(x) + h(y)}{2}, & x, y \in I, x \neq y, \\ \frac{f(x) - f(y)}{x - y} &= \frac{2h(x)h(y)}{h(x) + h(y)}, & x, y \in I, x \neq y.\end{aligned}$$

It is not difficult to show that, without any regularity assumptions on f and h , each of these equations characterizes the respective function f and its derivative h .

It is interesting that this fact remains true if we “pexiderize” these equations by replacing $f(y)$ by $\phi(y)$ and $h(y)$ by $\gamma(y)$.

REFERENCES

1. F. Hartogs, *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen*, Math. Ann. **62** (1906), 1–88. MR1511365
2. M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Uniwersytet Śląski - PWN, Warszawa - Kraków - Katowice, 1985 (Second edition, edited with a preface of Attila Gilányi, Birkhäuser Verlag, Basel, 2009). MR788497 (86i:39008)
3. J. Matkowski, *Generalized convex functions and a solution of a problem of Zs. Páles*, Publ. Math. Debrecen, **73**/3-4 (2008), 421–460. MR2466385 (2009i:26048)
4. J. Matkowski, *A mean-value theorem and its applications*, J. Math. Anal. Appl. **373** (2011), 227–234. MR2684472

FACULTY OF MATHEMATICS, COMPUTER SCIENCE AND ECONOMETRICS, UNIVERSITY OF ZIELONA GÓRA, PODGÓRNA 50, PL-65246 ZIELONA GÓRA, POLAND – AND – INSTITUTE OF MATHEMATICS, SILESIAN UNIVERSITY, BANKOWA 14, PL-42007 KATOWICE, POLAND

E-mail address: J.Matkowski@wmie.uz.zgora.pl