ON EXPLICIT PROBABILITY DENSITIES ASSOCIATED WITH FUSS-CATALAN NUMBERS

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ABSTRACT. In this paper we give explicitly a family of probability densities, the moments of which are Fuss-Catalan numbers. The densities appear naturally in random matrices, free probability and other contexts.

In this paper we study a family of probability densities π_s , $s \in \mathbb{N}$, which are uniquely determined by the moment sequences $\{m_0, m_1, \dots, m_k, \dots\}$ [1]. Here

$$(0.1) \quad m_k = \frac{1}{sk+1} \binom{sk+k}{k}$$

are known as Fuss-Catalan numbers in free probability theory [7]. The densities π_s belong to the class of free Bessel laws [3] and are known to appear in several different contexts, for instance, random matrices [1, 3, 7], random quantum states [4], free probability and quantum groups [3, 6]. More precisely, π_s is the limit spectral distribution of random matrices in the forms such as $X_1^s X_1^{*s}$ and $X_1 \cdots X_s X_s^* \cdots X_1^*$, where $X_1, \dots, X_s$ are independent $N \times N$ random matrices (the matrix elements of different matrices are independent); in free probability we have the free convolution relation: $\pi_s = \pi_1^{\boxplus s}$ [7, 3].

More generally, T. Banica et al. [3] introduce a remarkable family of probability distributions $\pi_{s,t}$ with $(s, t) \in (0, \infty) \times (0, \infty) - (0, 1) \times (1, \infty)$, called free Bessel laws. π_s is the special case where $t = 1$, i.e., $\pi_{s,1} = \pi_s$. The moments of $\pi_{s,t}$ are called the Fuss-Catalan polynomials (Theorem 5.2, [3]):

$$(0.2) \quad m_k(t) = \sum_{j=1}^k \frac{1}{j} \binom{k-1}{j-1} \binom{sk}{j-1} t^j.$$

Indeed, the following relation holds [3]:

$$(0.3) \quad \pi_{s,1} = \pi_1^{\boxplus s}, \quad \pi_{1,t} = \pi_1^{\boxplus t}.$$

The distribution $\pi_{1,t}$, the famous Marchenko-Pastur law of parameter t , also called free Poisson law, has an explicit formula:

$$(0.4) \quad \pi_{1,t} = \max(1-t, 0) \delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x}.$$

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In the special case,

$$(0.5) \quad \pi_1 = \pi_{1,1} = \frac{1}{2\pi} \sqrt{4x^{-1} - 1}.$$

Another special case of $\pi_{s,t}$ where an explicit density formula is available is due to Penson and Solomon [8]:

$$(0.6) \quad \pi_2 = \pi_{2,1} = \frac{\sqrt[3]{2}\sqrt[3]{3}}{12\pi} \frac{\sqrt[3]{2}(27 + 3\sqrt{81 - 12x})^{2/3} - 6\sqrt[3]{x}}{x^{2/3}(27 + 3\sqrt{81 - 12x})^{1/3}} 1_{(0,27/4]}(x).$$

To the best of our knowledge, except for the special cases above there are no explicit formulae available for the other $\pi_{s,t}$. The aim of this work is to give explicit densities of $\pi_s = \pi_{s,1}$, $s \in \mathbb{N}$. The proof of the following result is based on the method to find an explicit density from a given moment sequence used in [9, 5].

Theorem 0.1. *Let π_s , $s \in \mathbb{N}$, be the unique densities determined by the Fuss-Catalan numbers in Eq. (0.1). Then we have the following formulae:*

$$(0.7) \quad \pi_s(x) = \frac{1_{(0,K]}(x)}{B(\frac{1}{2}, \frac{1}{2} + \frac{1}{s})} \int_{[0,1]^s} \frac{(\tau K - x)^{1/s-1/2}}{\sqrt{x} (\tau K)^{1/s}} F(t_1, \dots, t_s) 1_{\{\tau K \geq x\}} d^s t,$$

where $K = (s+1)^{s+1}/s^s$, $\tau = \prod_{j=1}^s t_j$ and $F(t_1) = \delta(t_1 - 1)$, while for $s > 1$

$$(0.8) \quad F(t_1, \dots, t_s) = \frac{1}{B(\frac{1}{s+1}, \frac{s-1}{2s+2}) \prod_{j=2}^s B(\frac{j}{s+1}, \frac{j}{s(s+1)})} \\ \times t_1^{\frac{1}{s+1}-1} (1-t_1)^{\frac{s-1}{2s+2}-1} \prod_{j=2}^s t_j^{\frac{j}{s+1}-1} (1-t_j)^{\frac{j}{s(s+1)}-1}.$$

Proof. First, we derive a family of symmetric distributions σ_s , the $2k$ -moments of which are m_k in Eq. (0.1).

Consider the characteristic function of σ_s as follows:

$$(0.9) \quad \int_{-\infty}^{+\infty} e^{i\xi x} \sigma_s(x) dx = \sum_{k=0}^{\infty} \frac{(-\xi^2)^k}{(2k)!} m_k = \sum_{k=0}^{\infty} \beta_k \frac{(-\xi^2)^k}{k!},$$

where

$$(0.10) \quad \beta_k = \frac{1}{sk+1} \binom{sk+k}{k} \frac{k!}{(2k)!} = \frac{1}{sk+1} \frac{(sk+k)!}{(sk)!(2k)!}.$$

A direct computation shows that the ratio

$$\begin{aligned} \frac{\beta_{k+1}}{\beta_k} &= \frac{sk+1}{s(k+1)+1} \frac{1}{(2k+1)(2k+2)} \\ &\quad \times \frac{((s+1)k+1)((s+1)k+2) \cdots ((s+1)k+s+1)}{(sk+1)(sk+2) \cdots (sk+s)} \\ &= \frac{K(k + \frac{1}{s+1})(k + \frac{2}{s+1}) \cdots (k + \frac{s}{s+1})}{4(k + \frac{1}{2})(k + \frac{2}{s}) \cdots (k + \frac{s}{s})} \frac{1}{k+1 + \frac{1}{s}} \\ &\doteq \frac{K(k+a_1)(k+a_2) \cdots (k+a_s)}{4(k+b_1)(k+b_2) \cdots (k+b_s)} \frac{1}{k+b_{s+1}}. \end{aligned}$$

Here

$$(0.11) \quad b_1 = \frac{1}{2}, \quad a_i = \frac{i}{s+1} \quad \text{and} \quad b_{i+1} = \frac{i+1}{s} \quad \text{for } i = 1, 2, \dots, s.$$

Therefore, using the generalized hypergeometric function, we rewrite

$$(0.12) \quad \int_{-\infty}^{+\infty} e^{i\xi x} \sigma_s(x) dx = {}_sF_{s+1}(a_1, \dots, a_s; b_1, \dots, b_s, b_{s+1}; -\frac{K}{4}\xi^2)$$

$$(0.13) \quad = \int_{[0,1]^s} F(t_1, \dots, t_s) {}_0F_1(b_{s+1}; -\frac{\tau K}{4}\xi^2) d^s t.$$

Note that $a_1 = b_1$ for $s = 1$ but $b_j > a_j > 0, j = 1, 2, \dots, s$, when $s > 1$. In Eq. (0.13) we have made use of Euler's integral representation of the generalized hypergeometric function [2].

Next, with the help of the integral representation of the Bessel function of the first kind [2], that is, for $\alpha > -\frac{1}{2}$,

$$(0.14) \quad \begin{aligned} J_\alpha(z) &= \frac{(z/2)^\alpha}{\Gamma(\alpha+1)} {}_0F_1(\alpha+1; -\frac{1}{4}z^2) \\ &= \frac{(z/2)^\alpha}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^1 e^{izx} (1-x^2)^{\alpha-\frac{1}{2}} dx, \end{aligned}$$

we get

$$(0.15) \quad \begin{aligned} {}_0F_1(\alpha+1; -\frac{1}{4}z^2) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_{-1}^1 e^{izx} (1-x^2)^{\alpha-\frac{1}{2}} dx \\ &= \frac{1}{B(\frac{1}{2}, \alpha+\frac{1}{2})} \int_{-1}^1 e^{izx} (1-x^2)^{\alpha-\frac{1}{2}} dx. \end{aligned}$$

Set $\alpha = 1/s$, $z = \sqrt{\tau K}\xi$. We have

$$(0.16) \quad \begin{aligned} {}_0F_1(b_{s+1}; -\frac{\tau K}{4}\xi^2) &= \frac{1}{B(\frac{1}{2}, \frac{1}{2} + \frac{1}{s})} \int_{-1}^1 e^{i\sqrt{\tau K}\xi x} (1-x^2)^{\frac{1}{s}-\frac{1}{2}} dx \\ &= \frac{1}{B(\frac{1}{2}, \frac{1}{2} + \frac{1}{s})} \int_{-\sqrt{\tau K}}^{\sqrt{\tau K}} e^{i\xi x} \frac{(\tau K - x^2)^{\frac{1}{s}-\frac{1}{2}}}{(\tau K)^{\frac{1}{s}}} dx. \end{aligned}$$

Combining (0.16) and (0.13) after interchanging the order of integration, we obtain

$$(0.17) \quad \sigma_s(x) = \frac{1}{B(\frac{1}{2}, \frac{1}{2} + \frac{1}{s})} \int_{[0,1]^s} \frac{(\tau K - x^2)^{1/s-1/2}}{(\tau K)^{1/s}} F(t_1, \dots, t_s) 1_{\{\tau K \geq x^2\}} d^s t.$$

Noting the fact that

$$(0.18) \quad \pi_s(x) = \frac{\sigma_s(\sqrt{x})}{\sqrt{x}} 1_{(0,\infty)},$$

the proof is then complete. $\square$

We remark that some properties of the density $\pi_s(x)$ follow easily from Theorem 0.1; for instance, there are no atoms, the support is $[0, K]$, and the density is analytic on $(0, K)$ (these properties were obtained earlier by implicit complex-analytic methods; see Theorem 2.1, [3]).

REFERENCES

1. Alexeev, N., Götze, F., Tikhomirov, A.: Asymptotic distribution of singular values of powers of random matrices, *Lithuanian Math. J.* **50**, no. 2 (2010), 121–132.
2. Andrews, G. E., Askey, R., Roy, R.: *Special Functions*, Cambridge University Press (1999). MR1688958 (2000g:33001)
3. Banica, T., Belinschi, S., Capitaine, M., Collins, B.: Free Bessel laws, *Canad. J. Math.* **63** (2011), 3–37.
4. Collins, B., Nechita, I., Życzkowski, K.: Random graph states, maximal flow and Fuss-Catalan distributions, *J. Phys. A: Math. Theor.* **43** (2010), 275–303. MR2658283
5. Liu, D.-Z., Wang, Z.-D., Yan, K.-H.: Density of eigenvalues and its perturbation invariance in unitary ensembles of random matrices, *Journal of Approximation Theory* **162**, no. 9 (2010), 1588–1606. MR2718887
6. Młtkowski, W.: Fuss-Catalan numbers in noncommutative probability, *Documenta Mathematica* **15** (2010), 939–955.
7. Nica, A., Speicher, R.: *Lectures on the Combinatorics of Free Probability*, Cambridge University Press (2006). MR2266879 (2008k:46198)
8. Penson, K. A., Solomon, A. I.: Coherent states from combinatorial sequences, arXiv:quant-ph/0111151v1
9. Wigner, E. P.: Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math. (2)* **62** (1955), 548–564. MR0077805 (17:1097c)

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