LIOUVILLE THEOREMS FOR THE ANCIENT SOLUTION OF HEAT FLOWS

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ABSTRACT. Let M be a complete Riemannian manifold with Ricci curvature bounded from below: $Ric(M) \geq -\kappa$. Let N be a simply connected complete Riemannian manifold with nonpositive sectional curvature. Using a gradient estimate, we prove Liouville's theorem for the ancient solution of heat flows.

1. INTRODUCTION

In 1964, Eells and Sampson introduced the heat flow method to study the problem of existence of harmonic maps between compact Riemannian manifolds. Let (M, g) be a compact Riemannian manifold and let N be a closed submanifold of the Euclidean space R^k . They proved that, if the sectional curvature K_N of N is nonpositive, then any map $\phi \in C^1(M, N)$ can be deformed to a harmonic map. The heat equation of harmonic maps is the following parabolic system:

$$(1.1) \quad \frac{\partial u}{\partial t} = \tau(u) = \Delta u + A(u)(\nabla u, \nabla u),$$

where $\tau(u)$ is called the tension field of the mapping u , and $A(y) : T_y(N) \times T_y(N) \rightarrow (T_y(N))^\perp$ is the second fundamental form of N in R^k .

In [5], Li and Tam considered this equation in the case that M is noncompact with Ricci curvature bounded from below and N is a complete Riemannian manifold with nonpositive sectional curvature [5, Theorem 4.1].

If $N = R$, then u is a function, and the equation (1.1) becomes the following heat equation:

$$(1.2) \quad \frac{\partial u}{\partial t} = \Delta u.$$

As to this equation, Li-Yau [3] obtained the following parabolic gradient estimate.

Theorem A. *Let M be a complete manifold of dimension $m \geq 2$ with $Ricci(M) \geq -\kappa$ for some $\kappa \geq 0$. Suppose that u is any positive solution to the heat equation in $B(x_0, R) \times [t_0 - T, t_0]$. Then for $a > 1$,*

$$(1.3) \quad \frac{|\nabla u|^2}{u^2} - a \frac{u_t}{u} \leq \frac{c_n}{R^2} + \frac{c_n}{T} + c_n \kappa,$$

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in $B(x_0, R/2) \times [t_0 - T, t_0]$. Here c_n depends only on the dimension n and a .

For the time-dependent solution of the heat equation, it is well known that the elliptic Cheng-Yau type gradient estimate [1] cannot be true in general. The typical example is the function $u(x, t) = e^{-|x|^2/4t}/(4\pi t)^{n/2}$. However, in the case of compact manifolds, Hamilton's estimate shows that one can get a certain elliptic gradient estimate. In [6], the author obtained

Theorem B. *Let M be a compact manifold without boundary and with $\text{Ricci}(M) \geq -\kappa$ for some $\kappa \geq 0$. Let u be a smooth positive solution of the heat equation with $u \leq M$ for all $(x, t) \in M \times (0, \infty)$. Then*

$$(1.4) \quad \frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2\kappa \right) \ln \frac{M}{u}.$$

It would be highly desirable to have a noncompact version of Hamilton's estimate. In [4], they discovered that the elliptic Cheng-Yau estimate actually holds for the heat equation, after inserting a necessary logarithmic correction term. Their main result is the following:

Theorem C. *Let M be a Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(M) \geq -\kappa$ for some $\kappa \geq 0$. Suppose that u is any positive solution to the heat equation in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$. Suppose also that $u \leq M$ in $Q_{R,T}$. Then there exists a dimensional constant c such that*

$$\frac{|\nabla u(x, t)|}{u(x, t)} \leq c \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{\kappa} \right) \left(1 + \ln \frac{M}{u(x, t)} \right)$$

in $Q_{R/2, T/2}$.

Moreover, if M has nonnegative Ricci curvature and u is any positive solution of the heat equation on $M \times (0, \infty)$, then there exist dimensional constants c_1, c_2 such that

$$(1.5) \quad \frac{|\nabla u(x, t)|}{u(x, t)} \leq c_1 \frac{1}{t^{1/2}} \left(c_2 + \ln \frac{u(x, 2t)}{u(x, t)} \right)$$

for all $x \in M$ and $t > 0$.

Remark 1.1. Without seeing the paper [4], we obtained the Liouville theorem for the positive quasi-harmonic function by a different method in one of our papers which is to appear. It is just the time-independent version of (1.5) for $M = R^m$.

The main purpose of this paper is to consider the elliptic gradient estimate for the ancient solution of the equation (1.1) when the target manifold N is a general complete manifold. We can obtain that

Theorem 1.2. *Let M be a complete Riemannian manifold with Ricci curvature bounded from below: $\text{Ric}(M) \geq -\kappa$. Let N be a simply connected complete Riemannian manifold with nonpositive sectional curvature. Let u be a solution of the equation (1.1). Assume that $y_0 \notin u(B(x_0, R) \times [t_0 - T, t_0])$. Let $\rho(y)$ be the distance between y and y_0 in N . Then, if $b > 2 \sup\{\rho(u(x, t)) | (x, t) \in B(x_0, R) \times [t_0 - T, t_0]\}$, we have*

$$(1.6) \quad \sup_{Q_{R/2, T/2}} \frac{|\nabla u|(x, t)}{b^2 - \rho^2(u(x, t))} \leq \frac{C}{b} \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{\kappa} \right),$$

where $C > 0$ depends only on m and N .

As a corollary of the above theorem, we can get the following Liouville theorem for the ancient solution of the equation (1.1) with certain growth conditions.

Fixing $p \in M$, let $d(x)$ be the distance between x and p .

Theorem 1.3. *Let M be a complete, noncompact manifold with nonnegative curvature. Let N be a simply connected Riemannian manifold with nonpositive sectional curvature. Let u be an ancient solution to the equation (1.1) such that $\rho(u(x, t)) = o(d(x) + \sqrt{|t|})$ near infinity. Then u is a constant.*

Remark 1.4. (1) If we consider a bounded quasi-harmonic map v from R^m to N , that is, v is a bounded map satisfying

$$\tau(v) = \frac{1}{2}x \cdot \nabla v,$$

then $u(x, t) = v\left(\frac{x}{\sqrt{-t}}\right)$ satisfies the assumption of Theorem 1.3. Thus the result of section 3 in [2] can be deduced by Theorem 1.3.

(2) If $N = R$, Theorem 1.3 is [4, Theorem 1.2].

(3) Let $u(x, t) = x$ for $(x, t) \in R \times (-\infty, 0]$. Clearly it is an ancient solution for the heat equation and it is not constant. So our growth condition in the spatial direction is sharp in this sense.

2. PROOF OF THEOREMS

The quantity we used to do a gradient estimate is $\frac{|\nabla u|^2}{b^2 - \rho^2(u(x, t))}$, which is different from the quantity used in [4].

Proof of Theorem 1.2. Let

$$\phi(x, t) = \frac{|\nabla u|^2}{(b^2 - \rho^2(u(x, t)))^2}.$$

Then

$$\nabla \phi(x) = \frac{\nabla(|\nabla u|^2)}{(b^2 - \rho^2)^2} + 2 \frac{|\nabla u|^2 \nabla \rho^2}{(b^2 - \rho^2)^3}$$

and

$$\Delta \phi(x) = \frac{\Delta(|\nabla u|^2)}{(b^2 - \rho^2)^2} + 4 \frac{\nabla(|\nabla u|^2) \cdot \nabla \rho^2}{(b^2 - \rho^2)^3} + 2 \frac{|\nabla u|^2 \Delta \rho^2}{(b^2 - \rho^2)^3} + 6 \frac{|\nabla \rho^2|^2 |\nabla u|^2}{(b^2 - \rho^2)^4}.$$

Note the following Bochner formula:

$$\frac{1}{2} \Delta(|\nabla u|^2) = |\nabla du|^2 + \langle \nabla u, \nabla \tau(u) \rangle + \langle Ric^M \nabla u, \nabla u \rangle - \langle R^N(u_i, u_j) u_j, u_i \rangle,$$

where $u_i = du(e_i)$ and $e_1, e_2, \dots, e_n$ is an orthonormal frame of M . Using the assumption of Theorem 1.2, we have

$$(2.1) \quad \Delta|\nabla u|^2 \geq 2|\nabla du|^2 + \partial_t |\nabla u|^2 - \kappa |\nabla u|^2.$$

A direct calculation leads to

$$(2.2) \quad \Delta \rho^2(u(x, t)) = H(\rho^2)(\nabla u, \nabla u) + \partial_t(\rho^2(u(x, t))),$$

where $H(\rho^2)$ is the Hessian of ρ^2 . Since the sectional curvature of N , $K_N \leq 0$, the Hessian comparison theorem implies that

$$\Delta(\rho^2(u(x, t))) \geq 2|\nabla u|^2 + \partial_t(\rho^2(u(x, t))).$$

Then we have

$$(2.3) \quad \begin{aligned} \Delta\phi(x) &\geq \frac{2|\nabla du|^2 + \partial_t|\nabla u|^2 - \kappa|\nabla u|^2}{(b^2 - \rho^2)^2} + 4\frac{\nabla\rho^2 \cdot \nabla(|\nabla u|^2)}{(b^2 - \rho^2)^3} \\ &\quad + \frac{4|\nabla u|^4 + 2\partial_t(\rho^2(u(x, t)))|\nabla u|^2}{(b^2 - \rho^2)^3} + \frac{6|\nabla\rho^2|^2|\nabla u|^2}{(b^2 - \rho^2)^4} \end{aligned}$$

and

$$(2.4) \quad \partial_t\phi = \frac{\partial_t|\nabla u|^2}{(b^2 - \rho^2)^2} + 2\frac{|\nabla u|^2\partial_t\rho^2}{(b^2 - \rho^2)^3}.$$

Note that

$$(2.5) \quad \begin{aligned} \frac{2|\nabla du|^2}{(b^2 - \rho^2)^2} + \frac{2|\nabla\rho^2|^2|\nabla u|^2}{(b^2 - \rho^2)^4} &\geq 4\frac{|\nabla du||\nabla u||\nabla\rho^2|}{(b^2 - \rho^2)^3} \\ &\geq 2\frac{|\nabla|\nabla u|^2||\nabla\rho^2|}{(b^2 - \rho^2)^3} \end{aligned}$$

and

$$(2.6) \quad \frac{\nabla\rho^2 \cdot \nabla\phi}{b^2 - \rho^2} = \frac{\nabla\rho^2 \cdot \nabla(|\nabla u|^2)}{(b^2 - \rho^2)^3} + 2\frac{|\nabla\rho^2|^2|\nabla u|^4}{(b^2 - \rho^2)^4}.$$

By (2.3), (2.4), (2.5) and (2.6), we have

$$\Delta\phi - \partial_t\phi \geq \frac{4|\nabla u|^4}{(b^2 - \rho^2)^3} - \kappa\frac{|\nabla u|^2}{(b^2 - \rho^2)^2} + 2\frac{\nabla\phi \cdot \nabla\rho^2}{b^2 - \rho^2}.$$

From here, we will use the well-known cutoff function of Li-Yau [3] and the argument in [4].

Let $\psi = \psi(x, t)$ be a smooth cutoff function supported in $Q_{R,T}$, satisfying the following properties:

- (1) $\psi = \psi(d(x, x_0), t) \equiv \psi(r, t)$; $\psi(x, t) = 1$ in $Q_{R/2, T/4}$, $0 \leq \psi \leq 1$.
- (2) ψ is decreasing as a radial function in the spatial variables.
- (3) $|\partial_r\psi|/\psi^a \leq C_a/R$, $|\partial_r^2\psi|/\psi^a \leq C_a/R^2$ when $0 < a < 1$.
- (4) $|\partial_r\psi|/\psi^{1/2} \leq C/T$.

Let $L = -2\frac{\nabla\rho^2}{b^2 - \rho^2}$. We can calculate

$$(2.7) \quad \begin{aligned} \Delta(\psi\phi) + L \cdot \nabla(\psi\phi) - 2\frac{\nabla\psi}{\psi} \cdot \nabla(\psi\phi) - (\psi\phi)_t \\ = \psi(\Delta\phi - \phi_t) + \phi(\Delta\psi - \psi_t) + \psi L \cdot \nabla\phi + \phi L \cdot \nabla\psi - 2\frac{|\nabla\psi|^2}{\psi}\phi \\ \geq \psi \left(4\frac{|\nabla u|^4}{(b^2 - \rho^2)^3} - \kappa\frac{|\nabla u|^2}{(b^2 - \rho^2)^2} \right) + \phi(\Delta\psi - \psi_t) - 2\frac{\phi\nabla\rho^2 \cdot \nabla\psi}{b^2 - \rho^2} - 2\frac{|\nabla\psi|^2}{\psi}\phi. \end{aligned}$$

Suppose that the maximum of $\psi\phi$ is reached at (x_1, t_1) . By [3], x_1 is not in the cut locus of M . Then at this point, one has

$$\Delta(\psi\phi) \leq 0, \quad (\psi\phi)_t \geq 0, \quad \text{and} \quad \nabla(\psi\phi) = 0.$$

So, we have at (x_1, t_1) ,

$$\begin{aligned}
 (2.8) \quad & 4\psi \frac{|\nabla u|^4}{(b^2 - \rho^2)^3} - \kappa\psi \frac{|\nabla u|^2}{(b^2 - \rho^2)^2} \\
 & \leq 2\phi \frac{\nabla \rho^2 \cdot \nabla \psi}{b^2 - \rho^2} + 2\phi \frac{|\nabla \psi|^2}{\psi} + \phi(\psi_t - \Delta \psi) \\
 & = I + II + III.
 \end{aligned}$$

Using the fact that $|\nabla \rho^2| \leq b|\nabla u|$, we can get

$$\begin{aligned}
 (2.9) \quad & I \leq 2|\phi| \frac{|\nabla \rho^2|}{b^2 - \rho^2} |\nabla \psi| \\
 & \leq 2b|\phi| \frac{|\nabla u|}{b^2 - \rho^2} |\nabla \psi| \\
 & \leq 2b|\phi|^{3/2} |\nabla \psi| \leq b^2 |\psi \phi^2| + C \frac{|\nabla \psi|^4}{b^2 \psi^3}.
 \end{aligned}$$

By the properties of ψ , one has

$$(2.10) \quad II \leq \frac{b^2 \psi \phi^2}{8} + \frac{C}{b^2 R^4}.$$

As for $III = \phi(\psi_t - \Delta \psi)$, by the properties of ψ and the assumption on the Ricci curvature, we have that

$$\begin{aligned}
 (2.11) \quad -\phi \Delta \psi & \leq -\left(\partial_r^2 \psi + (m-1) \frac{\partial_r \psi}{r} + \sqrt{\kappa} |\partial_r \psi|\right) \phi \\
 & \leq \frac{1}{8} b^2 \psi \phi^2 + C \frac{1}{b^2 R^4} + C \frac{\kappa}{b^2 R^2}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.12) \quad |\psi_t| \phi & = \psi^{1/2} \phi \frac{|\psi_t|}{\psi^{1/2}} \\
 & \leq \frac{b^2}{8} \psi \phi^2 + \frac{C}{b^2 T^2}.
 \end{aligned}$$

Finally, we have

$$(2.13) \quad \kappa\psi \frac{|\nabla u|^2}{(b^2 - \rho^2)^2} = \kappa\psi \phi \leq \frac{b^2}{8} \psi \phi^2 + C \frac{\kappa^2}{b^2}.$$

From (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13), we obtain that

$$\psi \phi^2 \leq \frac{C}{b^4} \left(\frac{1}{R^4} + \frac{1}{T^2} + \kappa^2 \right).$$

For $\psi(x, t) = 1$ on $Q_{R/2, T/4}$,

$$\sup_{Q_{R/2, T/4}} \frac{|\nabla u|}{b^2 - \rho^2(u)} \leq \frac{C}{b} \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \kappa \right). \quad \square$$

Proof of Theorem 1.3. Fix (x_0, t_0) in space-time and let $f(R) = \sup_{Q_{R, \sqrt{R}}} \rho(u(x, t))$. Applying Theorem 1.2, we deduce that

$$\frac{|\nabla u(x_0, t_0)|}{4f^2(R) - \rho^2(u(x_0, t_0))} \leq \frac{C}{Rf(R)}.$$

Since $f(R) = o(R)$ by assumption, the result follows after taking R to ∞ . $\square$

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