FACTOR MAP, DIAMOND AND DENSITY OF PRESSURE FUNCTIONS

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ABSTRACT. Letting $\pi : X \rightarrow Y$ be a one-block factor map and Φ be an almost-additive potential function on X , we prove that if π has diamond, then the pressure $P(X, \Phi)$ is strictly larger than $P(Y, \pi\Phi)$. Furthermore, if we define the ratio $\rho(\Phi) = P(X, \Phi)/P(Y, \pi\Phi)$, then $\rho(\Phi) > 1$ and it can be proved that there exists a family of pairs $\{(\pi_i, X_i)\}_{i=1}^k$ such that $\pi_i : X_i \rightarrow Y$ is a factor map between X_i and Y , $X_i \subseteq X$ is a subshift of finite type such that $\rho(\pi_i, \Phi|_{X_i})$ (the ratio of the pressure function for $P(X_i, \Phi|_{X_i})$ and $P(Y, \pi\Phi)$) is dense in $[1, \rho(\Phi)]$. This extends the result of Quas and Trow for the entropy case.

1. INTRODUCTION

The present paper is devoted to studying the topic that for a given one-block factor map, how the existence of diamond and different kinds of potential functions affect the pressure function, and what is the *density* of the pressure. This is mainly motivated by the related works concerning entropy [2] and the dense entropy property [3]. Before formulating our results, we give some notation and background first. Let $\pi : X \rightarrow Y$ be a 1-block factor map between two one-dimensional mixing subshifts of finite type X and Y . Then the following result is well-known:

Theorem 1.1 (Theorem 4.1.7 of [4]). *Suppose $\pi : X \rightarrow Y$ is a one-block factor map between mixing subshifts of finite type (SFTs for short) and that X has positive entropy. Then either*

- (1) $\pi : X \rightarrow Y$ is uniformly bounded-to-one,
- (2) π has no diamond,
- (3) $h_{top}(X) = h_{top}(Y)$

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or

- (4) $\pi : X \rightarrow Y$ is uncountable-to-one on some points,
- (5) π has diamond,
- (6) $h_{\text{top}}(X) > h_{\text{top}}(Y)$.

We remark here that Theorem 1.1 also holds for higher-dimensional SFTs (Theorem 3.6 of [2]). However, unlike the one-dimensional case, some stronger specification property is needed for the higher-dimensional case. Let $\Phi = (\log \phi_n)_{n=1}^\infty$ be a real-valued potential function on X , i.e., $\log \phi_n : X \rightarrow \mathbb{R}$ for all $n \in \mathbb{N}$. We define the *push-forward* potential in Y by

$$(1.1) \quad \pi\Phi(y) = \left(\max_{x \in X: \pi(x)=y} \log \phi_n(x) \right)_{n=1}^\infty = \left(\max_{x \in X: \pi(x)=y} \log \phi_n \circ \pi^{-1}(x) \right)_{n=1}^\infty,$$

and define the pressure function on X by

$$(1.2) \quad P(X, \Phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{I \in X_n} \sup_{x \in I} \phi_n(x)$$

whenever the limit exists and X_n stands for the collection of n -cylinders in X . It is of interest to know whether Theorem 1.1 holds for the pressure function. Precisely, we consider the following.

Problem 1.2. If $\pi : X \rightarrow Y$ is a one-block factor map with diamond between X and Y , which potential functions $\Phi = (\log \phi_n)_{n=1}^\infty$ on X make $P(X, \Phi) > P(Y, \pi\Phi)$?

Problem 1.3. Under the same assumption of Problem 1.2, what is the difference $P(X, \Phi) - P(Y, \pi\Phi)$?

In this investigation, we have the following results:

Theorem A. Let $\pi : X \rightarrow Y$ be a one-block factor map between two mixing shift spaces X and Y . Assume $\Phi = (\log \phi_n) \in \mathbf{C}_{aa}(X, T)$ (defined in (2.2)) and satisfies the bounded distortion property (defined in (2.4)). Then either

- (1) $P(X, \Phi) > P(Y, \pi\Phi)$,
- (2) π has diamonds,

or

- (3) $P(X, \Phi) = P(Y, \pi\Phi)$,
- (4) π has no diamond.

For the case π has diamond, Theorem A shows that $P(X, \Phi) - P(Y, \pi\Phi) > 0$ if and only if π has diamond. This extends Theorem 1.1 to pressure for $\Phi \in \mathbf{C}_{aa}(X, T)$. For the difference $P(X, \Phi) - P(Y, \pi\Phi)$ of Problem 1.3, we have the following result.

Theorem B. Under the same assumption of Theorem A, let $\nu \in \mathcal{M}(Y, S)$ be the equilibrium measure on Y with respect to the push-forward potential $\pi\Phi(y) = (\max_{x \in X: \pi(x)=y} \log \phi_n \circ \pi^{-1}(x))_{n=1}^\infty$ and $\mu \in \mathcal{M}(X, T)$ be the conditional equilibrium state of Φ with respect to ν (see (2.13)) and Proposition 2.6). Then

$$(1.3) \quad P(X, \Phi) - P(Y, \pi\Phi) = h_\mu(T) - h_\nu(S).$$

Theorem B indicates that the difference of $P(X, \Phi) - P(Y, \pi\Phi)$ equals $h_\mu(T) - h_\nu(S)$, and it is useful for characterizing the positivity of $P(X, \Phi) - P(Y, \pi\Phi)$ by showing $h_\mu(T) > h_\nu(S)$ (see Theorem 3.1).

On the other hand, for a dynamical system (X, T) , it is natural to ask what are the subsystems of X and what are the possible values of the entropies (resp. pressure) of the subsystems of X . If X is an n -dimensional SFT for $n \in \mathbb{N}$, Quas and Trow [3] show that for $\varepsilon > 0$, there exists a proper subshift $\hat{X}$ of X which is also an SFT with the property that

$$h_{\text{top}}(X) - \varepsilon < h_{\text{top}}(\hat{X}) < h_{\text{top}}(X).$$

If $\pi : X \rightarrow Y$ is a one-block factor map with diamond and $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathbf{C}_{aa}(X, T)$ with the bounded distortion property, we define the ratio of $P(X, \Phi)$ and $P(Y, \pi\Phi)$ by

$$(1.4) \quad \rho(\pi, \Phi) = P(X, \Phi)/P(Y, \pi\Phi) \quad \text{if } P(Y, \pi\Phi) \neq 0.$$

It follows from Theorem A that we have $\rho(\pi, \Phi) > 1$. We ask the following questions:

Problem 1.4. Under the same assumptions of Theorem A, does there exist a family $\pi_i : X_i \rightarrow Y$ where X_i is a subsystem of X and $\pi_i = \pi|_{X_i}$ is a one-block factor for all $i \in \mathbb{N}$ such that

$$\rho(\pi_i, \Phi|_{X_i}) = P(X_i, \Phi|_{X_i})/P(Y, \pi\Phi)$$

is dense in $[1, \rho(\pi, \Phi)]$, where $\Phi|_{X_i}$ stands for the restriction of Φ to X_i ?

For Problem 1.4, we have the following result.

Theorem C. *Under the same assumptions of Theorem A, there exists a family of pairs $\{(\pi_i, X_i)\}_{i=1}^{\infty}$ such that*

- (1) X_i is a subsystem of X , $\forall i \in \mathbb{N}$;
- (2) X_i is an SFT, $\forall i \in \mathbb{N}$;
- (3) $\pi_i : X_i \rightarrow Y$ is a one-block factor map for all $i \in \mathbb{N}$ such that $\rho(\pi_i, \Phi|_{X_i}) \neq 0$ are dense in $[1, \rho(\Phi)]$. That is, for $\varepsilon > 0$, there exists an integer $k = k(\varepsilon)$ and a monotone decreasing sequence $\{P(X_i, \Phi|_{X_i})\}_{i=1}^k$ such that

$$P(X_i, \Phi|_{X_i}) - P(X_{i+1}, \Phi|_{X_{i+1}}) < \varepsilon,$$

and, for all $p \in [P(Y, \pi\Phi), P(X, \Phi)]$, there exists a $1 \leq j \leq k$ with

$$P(X_{j+1}, \Phi|_{X_{j+1}}) < p < P(X_j, \Phi|_{X_j}).$$

The content of this paper is the following. In Section 2, we introduce the so-called **a-weighted thermodynamic formalism** developed recently by Barral and Feng [1]. This tool is useful for the proofs of Theorem B and Theorem A, and we leave their proofs to Section 3 and give the proof of Theorem C in Section 4.

2. PRELIMINARIES AND **a**-WEIGHTED THERMODYNAMIC FORMALISM

For the reader's convenience we recall some definitions and known results in this section.

2.1. Sub-additive thermodynamic formalism. The following definitions and notation come from the recent works of Barral and Feng [1].

Definition 2.1. (1) We say that $\Phi = (\log \phi_n)_{n=1}^{\infty}$ is *sub-additive* on X and write $\Phi \in \mathbf{C}_s(X, T)$ if there exists $C_1 > 0$ such that

$$(2.1) \quad \phi_{n+m}(x) \leq C_1 \phi_n(x) \phi_m(T^n x) \quad \forall x \in X \text{ and } n, m \in \mathbb{N}.$$

- (2) We say that $\Phi = (\log \phi_n)_{n=1}^\infty$ is *asymptotically sub-additive* on X and write $\Phi \in \mathbf{C}_{ass}(X, T)$ if for any $\varepsilon > 0$ there exists a sub-additive potential $\Psi = (\log \psi_n(x))_{n=1}^\infty$ on X such that

$$(2.2) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{x \in X} |\log \phi_n(x) - \log \psi_n(x)| \leq \varepsilon.$$

- (3) $\Phi = (\log \phi_n)_{n=1}^\infty$ is called *almost additive* on X and we write $\Phi \in \mathbf{C}_{aa}(X, T)$ if ϕ_n is positive and continuous on X for all $n \in \mathbb{N}$ and there exists $C_2 > 0$ such that

$$(2.3) \quad C_2^{-1} \phi_n(x) \phi_m(T^n(x)) \leq \phi_{n+m}(x) \leq C_2 \phi_n(x) \phi_m(T^n(x)),$$

$\forall x \in X$ and $n, m \in \mathbb{N}$.

- (4) $\Phi = (\log \phi_n)_{n=1}^\infty$ is called the *bounded distortion property* if there exists a constant $C_3 > 0$ such that

$$(2.4) \quad C_3^{-1} \phi_n(y) \leq \phi_n(x) \leq C_3 \phi_n(y) \quad \forall x, y \in I \in X_n.$$

We introduce the following result of the variational principle for the asymptotic sub-additive potential Φ on X .

Theorem 2.2 (Feng and Huang, [9]). *Let $\Phi \in \mathbf{C}_{ass}(X, T)$ and $T : X \rightarrow X$ be a mixing continuous transformation. Then*

$$(2.5) \quad P(X, \Phi) = \sup \{h_\eta(T) + \Phi_*(\eta) : \eta \in \mathcal{M}(X, T)\},$$

where $\mathcal{M}(X, T)$ denotes the collection of T -invariant probability measures on X endowed with the weak-star topology, $h_\eta(T)$ denote the measure-theoretic entropies of η and $\Phi_*(\eta)$ is given by

$$(2.6) \quad \Phi_*(\eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \phi_n(x) d\eta(x).$$

A measure $\mu \in \mathcal{M}(X, T)$ attaining the supremum of (2.5) is called the *equilibrium measure* of Φ . A measure $\mu \in \mathcal{M}(X, T)$ is called a *Gibbs measure* with respect to Φ if there exists a $Q_1 > 0$ such that

$$(2.7) \quad Q_1^{-1} \leq \frac{\mu([I])}{\exp(-nP(X, \Phi))\phi_n([I])} \leq Q_1, \quad \forall I \in X_n, \quad n \in \mathbb{N},$$

where

$$(2.8) \quad \phi_n([I]) = \sup_{x \in [I]} \phi_n(x).$$

It follows from Theorem 2.2 that we can construct the variational principle for $P(X, \Phi)$ and $P(Y, \pi\Phi)$.

Proposition 2.3. *Let $\Phi \in \mathbf{C}_{ass}(X, T)$ and let $\pi\Phi$ be defined as in (1.1) on Y . Then:*

- (1) $\pi\Phi \in \mathbf{C}_{ass}(Y, S)$.
 (2) The two variational principles hold:

$$(2.9) \quad P(X, \Phi) = \sup \{h_\eta(S) + \Phi_*(\eta) : \eta \in \mathcal{M}(X, T)\},$$

$$(2.10) \quad P(Y, \pi\Phi) = \sup \{h_\xi(S) + (\pi\Phi)_*(\xi) : \xi \in \mathcal{M}(Y, S)\}.$$

Furthermore, if we assume $\Phi \in \mathbf{C}_{aa}(X, T)$ and satisfies the bounded distortion property, then

- (3) $\pi\Phi \in \mathbf{C}_{aa}(Y, S)$.

- (4) *There exist unique equilibrium measures $\mu \in \mathcal{M}(X, T)$, $\nu \in \mathcal{M}(Y, S)$ attaining the supremums of (2.9) and (2.10) respectively.*
- (5) *Both μ and ν satisfy the Gibbs property; i.e., (2.7) holds for μ and ν .*

2.2. $\mathbf{a}$ -weighted thermodynamic formalism. Let (X, T) and (Y, S) be mixing shift spaces. Assume $\Phi \in \mathbf{C}_{ass}(X, T)$ and $\mathbf{a} = (a, b) \in \mathbb{R}^2$ so that $a > 0$ and $b \geq 0$. Barral and Feng [1] introduce the $\mathbf{a}$ -weighted topological pressure of Φ :

$$(2.11) \quad P^{\mathbf{a}}(X, \Phi) = \sup \{ \Phi_*(\eta) + ah_\eta(T) + bh_{\eta \circ \pi^{-1}}(S) : \eta \in \mathcal{M}(X, T) \}.$$

A measure $\mu \in \mathcal{M}(X, T)$ attaining the supremum of (2.11) is called the $\mathbf{a}$ -weighted equilibrium state of Φ . Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathbf{C}_{ass}(X, T)$, define a sequence $\Psi = (\log \psi_n)_{n=1}^\infty$ of potentials on Y by

$$(2.12) \quad \psi_n(y) = \sum_{I \in X_n: [I] \cap \pi^{-1}(y) \neq \emptyset} \sup_{x \in [I] \cap \pi^{-1}(y)} \phi_n(x)^{\frac{1}{a}}, \quad y \in Y,$$

and set $\frac{a}{a+b}\Psi = (\log(\psi_n^{\frac{a}{a+b}}))_{n=1}^\infty$. For $\nu \in \mathcal{M}(Y, S)$, a measure $\mu \in \mathcal{M}(X, T)$ is called a conditional equilibrium state of Φ with respect to ν if $\mu \circ \pi^{-1} = \nu$ and

$$(2.13) \quad \Phi_*(\mu) + h_\mu(T) - h_\nu(S) = \sup \{ \Phi_*(\eta) + h_\eta(T) - h_\nu(S) : \eta \in \mathcal{M}(X, T), \eta \circ \pi^{-1} = \nu \}.$$

Barral and Feng [1] developed the following results.

Theorem 2.4. *Let $\mathbf{a} = (a, b) \in \mathbb{R}^2$ so that $a > 0$ and $b \geq 0$. If $\Phi \in \mathbf{C}_{ass}(X, T)$ (resp. $\Phi \in \mathbf{C}_{aa}(X, T)$), then*

- (1) Ψ and $\frac{a}{a+b}\Psi \in \mathbf{C}_{ass}(X, T)$ (resp. Ψ and $\frac{a}{a+b}\Psi \in \mathbf{C}_{aa}(X, T)$);
- (2) $P^{\mathbf{a}}(X, \Phi) = (a+b)P(Y, \frac{a}{a+b}\Psi)$ ($P(Y, \Psi)$ is defined in (1.2));
- (3) μ is an $\mathbf{a}$ -weighted equilibrium state of Φ iff $\nu = \mu \circ \pi^{-1}$ is an equilibrium state of $\frac{a}{a+b}\Psi$ and μ is a conditional equilibrium state of $\frac{1}{a}\Phi$ with respect to ν , where $\frac{1}{a}\Phi = (\log(\phi_n^{\frac{1}{a}}))_{n=1}^\infty$.

Letting $\mathbf{a} = (a, b) \in \mathbb{R}^2$ so that $a > 0$ and $b \geq 0$, a measure $\mu \in \mathcal{M}(X, T)$ is called an $\mathbf{a}$ -weighted Gibbs measure if there exists $Q_2 > 0$ such that

$$(2.14) \quad Q_2^{-1} \leq \frac{\mu([I])\psi_n(\pi[I])^{\frac{b}{a+b}}}{\exp(-\frac{n}{a+b}P^{\mathbf{a}}(X, \Phi))\phi_n(I)^{\frac{1}{a}}} \leq Q_2,$$

where

$$(2.15) \quad \psi(J) = \sum_{I \in X_n: \pi I = J} \phi_n([I])^{\frac{1}{a}}, \quad \forall J \in Y_n.$$

The following theorem was also proved in [1]. It shows that the $\mathbf{a}$ -weighted Gibbs measure exists uniquely for $\Phi \in \mathbf{C}_{aa}(X, T)$ with the bounded distortion property.

Theorem 2.5. *Let $\pi : X \rightarrow Y$ be a one-block factor. Let $\mathbf{a} = (a, b) \in \mathbb{R}^2$ so that $a > 0$ and $b \geq 0$. Let $\Phi \in \mathbf{C}_{aa}(X, T)$ and satisfy the bounded distortion property. Then*

- (1) Φ has a unique $\mathbf{a}$ -weighted equilibrium measure, say μ ;
- (2) μ is also the unique $\mathbf{a}$ -weighted Gibbs measure of Φ ;

- (3) if we define $\nu = \mu \circ \pi^{-1}$, then there exists Q_3 and $Q_4 > 0$ such that for all $J \in Y_n, n \in \mathbb{N}$ and $I = \pi(J)$,

$$(2.16) \quad Q_3^{-1} \leq \frac{\nu([J])}{\exp(-\frac{n}{a+b}P^{\mathbf{a}}(X, \Phi))\psi_n([J])^{\frac{a}{a+b}}} \leq Q_3$$

and

$$(2.17) \quad Q_4^{-1} \leq \frac{\mu([I])^a \nu(\pi[J])^b}{\exp(-nP^{\mathbf{a}}(X, \Phi))\phi_n([I])} \leq Q_4.$$

Combining Theorem 2.4 and Theorem 2.5 we have the following.

Proposition 2.6. *Let $\Phi \in \mathbf{C}_{aa}(X, T)$ and satisfy the bounded distortion property and let $\nu \in \mathcal{M}(Y, S)$ be the Gibbs measure of $\Psi = (\log \psi_n)_{n=1}^{\infty}$ as defined in (2.22); it is thus an equilibrium measure of Ψ . Then Φ has a unique conditional equilibrium measure μ with respect to ν and there exists a constant $C_4 > 0$ such that*

$$(2.18) \quad C_4^{-1} \leq \frac{\mu([I])\psi_n(\pi([I]))}{\nu(\pi([I]))\phi_n([I])} \leq C_4 \quad \forall I \in X_n \text{ and } n \in \mathbb{N},$$

where $\psi(J) = \sum_{I \in X_n: \pi(I)=J} \phi_n([I])$ and $\phi_n([I])$ is defined in (2.8).

Next we show that $P(X, \Phi) = P(Y, \Psi)$.

Proposition 2.7. *Let $\pi : X \rightarrow Y$ be a one-block factor map and $\Phi \in \mathbf{C}_{ass}(X, T)$. Let $\Psi \in \mathbf{C}_{ass}(Y, S)$ be defined in (2.22). Then*

$$(2.19) \quad P(X, \Phi) = P(Y, \Psi).$$

Proof. Taking $\mathbf{a} = (1, 0) \in \mathbb{R}^2$, it follows from (2.11) and Theorem 2.4 that

$$(2.20) \quad \begin{aligned} P^{\mathbf{a}}(X, \Phi) &= \sup \{ \Phi_*(\eta) + h_{\eta}(T) : \eta \in \mathcal{M}(X, T) \} \\ &= P(Y, \Psi). \end{aligned}$$

Combining (2.20) and Theorem 2.2 with the fact that $\Phi \in \mathbf{C}_{ass}(X, T)$ yields

$$P(X, \Phi) = P^{\mathbf{a}}(X, \Phi) = \sup \{ \Phi_*(\eta) + h_{\eta}(T) : \eta \in \mathcal{M}(X, T) \} = P(Y, \Psi).$$

This completes the proof. $\square$

We end this subsection by introducing the *relativised variational principle*, which was developed by Ledrappier, Walters, Cao, Zhao, Feng and Huang (cf. [5], [8], [9] and [1]). This will be useful in the study of the relationship between $P(X, \Phi)$ and $P(Y, \pi\Phi)$.

Proposition 2.8 (Lemma 3.1 of [1]). *Let $\Phi \in \mathbf{C}_{aa}(X, T)$ and satisfy the bounded distortion property. If $\nu \in \mathcal{M}(Y, S)$, then:*

- (1) *The relativised variational principle holds:*

$$(2.21) \quad \int_Y P(X, \Phi, \pi^{-1}(y)) d\nu(y) = \sup \{ h_{\mu}(T) - h_{\nu}(S) + \Phi_*(\mu) \},$$

where the supremum is taken over all $\mu \in \mathcal{M}(X, T)$ with $\mu \circ \pi^{-1} = \nu \in \mathcal{M}(Y, S)$.

(2) Define $\Psi = (\log \psi_n)_{n=1}^\infty$ on Y , where

$$(2.22) \quad \psi_n(y) = \sum_{I \in X_n: [I] \cap \pi^{-1}(y) \neq \emptyset} \sup_{x \in [I] \cap \pi^{-1}(y)} \phi_n(x).$$

Then

$$\begin{aligned} \int_Y P(X, \Phi, \pi^{-1}(y)) d\nu(y) &= \sup \{h_\mu(T) - h_\nu(S) + \Phi_*(\mu)\} \\ &= \Psi_*(\nu), \end{aligned}$$

where the supremum is taken over all $\mu \in \mathcal{M}(X, T)$ with $\mu \circ \pi^{-1} = \nu \in \mathcal{M}(Y, S)$ and $\Psi_*(\nu)$ is defined in (2.6).

3. PROOFS OF THEOREM A AND THEOREM B

In this subsection we give the proofs of Theorem B and Theorem A. In the following, we assume $\Psi = (\log \psi_n)_{n=1}^\infty$, as defined in (2.22).

Proof of Theorem B. Take $\mathbf{a} = (1, 0) \in \mathbb{R}^2$. It follows from Theorem 2.4 and Proposition 2.3 that we have $\pi\Phi, \Psi \in \mathbf{C}_{aa}(Y, S)$. Define the partition functions

$$(3.1) \quad Z_n(X, \Phi) = \sum_{I \in X_n} \phi_n(I), \quad Z_n(Y, \Psi) = \sum_{J \in Y_n} \psi_n(J),$$

and

$$(3.2) \quad Z_n(Y, \pi\Phi) = \sum_{J \in Y_n} (\pi\phi)_n(J).$$

One can easily check that $(\log Z_n(X, \Phi))_{n=1}^\infty \in \mathbf{C}_{ass}(X, T)$, and $(\log Z_n(Y, \Psi))_{n=1}^\infty, (\log Z_n(Y, \pi\Phi))_{n=1}^\infty \in \mathbf{C}_{ass}(Y, S)$. By the standard argument, we conclude that $P(X, \Phi)$, $P(Y, \Psi)$ and $P(Y, \pi\Phi)$ exist. Since $P(X, \Phi) = P(Y, \Psi)$ (Proposition 2.7), for $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ we have

$$(3.3) \quad \exp(-n\varepsilon) \leq \frac{Z_n(X, \Phi)}{Z_n(Y, \Psi)} \leq \exp(n\varepsilon).$$

Let $\nu \in \mathcal{M}(Y, S)$ be the Gibbs measure of Ψ as in Proposition 2.3 and $\mu \in \mathcal{M}(X, T)$ be its unique conditional equilibrium measure according to Proposition 2.6. Then there exist Q_3 and $Q_4 > 0$ such that for all $I \in X_n$ and $J = \pi I \in Y_n$ with $n \in \mathbb{N}$,

$$(3.4) \quad Q_3^{-1} \exp(-nP(X, \Phi)) \phi_n([I]) \leq \mu([I]) \leq Q_3 \exp(-nP(X, \Phi)) \phi_n([I])$$

and

$$(3.5) \quad Q_4^{-1} \exp(-nP(Y, \Psi)) \psi_n([J]) \leq \nu([J]) \leq Q_4 \exp(-nP(Y, \Psi)) \psi_n([J]).$$

Since μ and ν are Gibbs on X and Y , they are also ergodic. It follows from the Shannon-McMillian-Brieman Theorem, for $\varepsilon > 0$ there exists $N_2 > 0$ such that if $n \geq N_2$ and μ -a.e. $I \in X_n$ with $J = \pi([I]) \in Y_n$,

$$(3.6) \quad \exp(-n(h_\mu(T) + \varepsilon)) \leq \mu([I]) \leq \exp(-n(h_\mu(T) - \varepsilon))$$

and

$$(3.7) \quad \exp(-n(h_\nu(S) + \varepsilon)) \leq \nu([J]) \leq \exp(-n(h_\nu(S) - \varepsilon)).$$

Let $n \geq \max\{N_1, N_2\}$. Combining (3.3), (3.4), (3.5), (3.6) and (3.7) with the fact that $P(Y, \Psi) = P(X, \Phi)$ we have

$$\begin{aligned}
 Z_n(X, \Phi) &\leq Z_n(Y, \Psi) \exp(n\varepsilon) \\
 &= \exp(n\varepsilon) \sum_{J \in Y_n} \psi_n([J]) = \exp(n\varepsilon) \sum_{J \in Y_n} \psi_n([J]) \phi_n([I])^{-1} (\pi\phi)_n([J]) \\
 &\leq Q_3 Q_4 \exp(n\varepsilon) \exp(n(P(Y, \Psi) - P(X, \Phi))) \sum_{J \in Y_n} \nu([J]) \mu^{-1}([I]) (\pi\phi)_n([J]) \\
 (3.8) \quad &\leq Q_3 Q_4 \exp(n(h_\mu(T) - h_\nu(S) + 3\varepsilon)) \sum_{J \in Y_n} (\pi\phi)_n([J]).
 \end{aligned}$$

For the opposite inequality, we have

$$\begin{aligned}
 Z_n(X, \Phi) &\geq \exp(-n\varepsilon) Z_n(Y, \Psi) = \exp(-n\varepsilon) \sum_{J \in Y_n} \psi_n([J]) \phi_n([I])^{-1} (\pi\phi)_n([J]) \\
 &\geq Q_3^{-1} Q_4^{-1} \exp(-n\varepsilon) \exp(n(P(Y, \Psi) - P(X, \Phi))) \\
 &\quad \times \sum_{J \in Y_n} \nu([J]) \mu^{-1}([I]) (\pi\phi)_n([J]) \\
 (3.9) \quad &\geq Q_3^{-1} Q_4^{-1} \exp(n(h_\mu(T) - h_\nu(S) - 3\varepsilon)) \sum_{J \in Y_n} (\pi\phi)_n([J]).
 \end{aligned}$$

Then (1.3) follows by dividing both sides of (3.8) and (3.9) with n and taking n to infinity. This completes the proof of Theorem B. $\square$

For the proof of Theorem A, we need the following results. They show that under the same assumption of Theorem A with the fact that π has diamond, $P(X, \Phi)$ is strictly larger than $P(Y, \pi\Phi)$.

Theorem 3.1. *Let $\pi : X \rightarrow Y$ be a one-block factor with diamond. Let $\Phi \in \mathbf{C}_{aa}(X, T)$ and satisfy the bounded distortion property. If $\nu \in \mathcal{M}(Y, S)$ is the equilibrium state $\Psi = (\log \psi_n)_{n=1}^\infty$ as defined in (2.22) and μ is the conditional equilibrium state of Φ with respect to ν , then*

$$(3.10) \quad h_\mu(T) > h_\nu(S).$$

Furthermore, $P(X, \Phi) > P(Y, \pi\Phi)$.

Proof. Since π has diamond, then by Theorem 3.6 of [2] there exists $y \in Y$, $C_5 > 0$ and $C_6 > 1$ such that

$$(3.11) \quad \#\{I \in X_n : I \cap \pi^{-1}(y) \neq \emptyset\} \geq C_5 C_6^n.$$

It follows from Proposition 2.6 that there exists $C_4 > 0$ such that $\forall I \in X_n$ and $J = \pi([I]) \in Y_n$,

$$(3.12) \quad C_4^{-1} \frac{\phi_n([I])}{\psi_n([J])} \leq \frac{\mu([I])}{\nu([J])} \leq C_4 \frac{\phi_n([I])}{\psi_n([J])}.$$

For μ -a.e. $I \in X_n$ with $y \in J = \pi(I)$, it follows from (3.11), (3.12) and the Shannon-McMillian-Brieman Theorem that

$$\begin{aligned} h_\mu(T) &= \lim_{n \rightarrow \infty} \frac{-1}{n} \log \mu([I]) \\ &\geq \lim_{n \rightarrow \infty} \frac{\log C_4}{n} + \lim_{n \rightarrow \infty} \frac{-1}{n} \log \left\{ \frac{\phi_n([I])}{\psi_n([J])} \nu([J]) \right\} \quad (\text{by (3.12)}) \\ &= \lim_{n \rightarrow \infty} \frac{-1}{n} \left[\log \frac{\phi_n([I])}{\psi_n([J])} \right] - \frac{1}{n} \log \nu([J]) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \sum_{I \in X_n: I \cap \pi^{-1}(y) \neq \emptyset} \phi_n([I]) - \log \phi_n([I]) \right] + \lim_{n \rightarrow \infty} \frac{-1}{n} \log \nu([J]) \\ &\geq \lim_{n \rightarrow \infty} \frac{1}{n} [\log C_3^{-1} C_5 C_6^n \phi_n([I]) - \log \phi_n([I])] + \lim_{n \rightarrow \infty} \frac{-1}{n} \log \nu([J]) \\ &= \log C_6 + h_\nu(S). \end{aligned}$$

The constant C_3 comes from the bounded distortion property for Φ ((2.4) in Definition 2.1). Since $C_6 > 1$, we have $h_\mu(T) > h_\nu(S)$. Combining Theorem B and (3.10) we have $P(X, \Phi) > P(Y, \pi\Phi)$, and the proof is completed. $\square$

We continue the proof of Theorem A.

Proof of Theorem A. By the variational principle of $P(X, \Phi)$ and $P(Y, \pi\Phi)$ and Theorem 3.1, we only need to show that if $P(X, \Phi) > P(Y, \pi\Phi)$, then π has diamond. Assume π has no diamond. Then by Theorem 3.6 of [2], $h_{\text{top}}(X) = h_{\text{top}}(Y)$. Then $\pi : X \rightarrow Y$ is almost everywhere bounded-to-one. Using the identical argument of Theorem 3.1, we can also derive that $P(X, \Phi) = P(Y, \pi\Phi)$, a contradiction. This completes the proof of Theorem A $\square$

4. PROOF OF THEOREM C

Let $\rho(\pi, \Phi)$ be as defined in (1.4). It follows from Theorem A that we have

Proposition 4.1. *Under the same assumptions of Theorem A:*

- (1) *If π has no diamond, then $\rho(\pi, \Phi) = 1$.*
- (2) *If π has diamond, then $\rho(\pi, \Phi) > 1$.*

We are ready to give the proof of Theorem C. Some auxiliary results are needed. First we define $X \setminus I = X \setminus \bigcup_{i \in \mathbb{N}} T^{-i}(I)$. The following result comes from [3].

Theorem 4.2 (Theorem 2.9 of [3]). *Let X be an SFT with positive topological entropy and let $\mu \in \mathcal{M}(X, T)$ be the measure of maximal entropy. Then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ and $I \in X_n$, then*

$$(4.1) \quad h_\mu(X) - \varepsilon \leq h_\mu(X \setminus I) \leq h_\mu(X).$$

Remark 4.3. We remark here that in [3], Quas and Trow derived that

$$h_{\text{top}}(X) - \varepsilon \leq h_{\text{top}}(X \setminus I) \leq h_{\text{top}}(X).$$

It is not hard to extend this result to (4.1) for μ is Gibbs for some potential Φ from their proof.

We now deduce the pressure from Theorem 4.2.

Theorem 4.4. *Let X be a mixing SFT, and let $\Phi \in \mathbf{C}_{aa}(X, T)$ and satisfy the bounded distortion property. For $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$ and for all $I \in X_n$ where $X \setminus I$ is mixing, we have*

$$P(X, \Phi) - \varepsilon \leq P(X \setminus I, \Phi|_{X \setminus I}) < P(X, \Phi).$$

Proof. Since $\Phi \in \mathbf{C}_{aa}(X, T)$ and satisfies the bounded distortion property, it follows from Proposition 2.3 that we may assume μ is the Gibbs measure for Φ on X . Let $\frac{\varepsilon}{2} > 0$ and $I_1 \in X_n$. According to the variational principle and Theorem 4.2,

$$(4.2) \quad P(X, \Phi) - \Phi_*(\mu) - \varepsilon/2 = h_\mu(X) - \varepsilon/2 \leq h_\mu(X \setminus I_1).$$

This shows that

$$(4.3) \quad P(X, \Phi) - \varepsilon/2 \leq h_\mu(X \setminus I_1) + \Phi_*(\mu).$$

We claim that $\Phi_*(\mu) \leq (\Phi|_{X \setminus I_1})_*(\mu) + \varepsilon/2$. Indeed, since $\Phi \in \mathbf{C}_{aa}(X, T)$ we have

$$\begin{aligned} \Phi_*(\mu) &= \lim_{n \rightarrow \infty} \int_X \frac{1}{n} \log \phi_n(x) d\mu(x) \leq \int_X \frac{1}{n} \log \phi_n(x) d\mu(x) \\ &\leq \sum_{I \in X_n} \int_I \frac{1}{n} \log \phi_n(x) d\mu(x) \leq \sum_{I \in X_n} \frac{1}{n} \log \phi_n([I]) \mu([I]) \\ &\leq \sum_{I \in (X \setminus I_1)_n} \frac{1}{n} \log \phi_n([I]) \mu([I]) + \frac{1}{n} \log \phi_n([I_1]) \mu([I_1]) \\ &\leq (\Phi|_{X \setminus I_1})_*(\mu) + \Phi_*(\mu) \mu([I_1]) + \delta, \end{aligned}$$

for some small $\delta > 0$. We note here that the 4th inequality follows from the Birkhoff ergodic theorem and the fact that $\Phi \in \mathbf{C}_{aa}(X, T)$. Then the claim follows by taking $\mu([I_1])$ small enough and the fact that $\delta \rightarrow 0$ as $n \rightarrow \infty$. Therefore, it follows from (4.3) that we have

$$\begin{aligned} P(X, \Phi) &\leq h_\mu(X \setminus I_1) + \Phi_*(\mu) + \varepsilon/2 \\ &\leq h_\mu(X \setminus I_1) + (\Phi|_{X \setminus I_1})_*(\mu) + \varepsilon. \\ &\leq P(X \setminus I_1, \Phi|_{X \setminus I_1}) + \varepsilon. \end{aligned}$$

This completes the proof. $\square$

Lemma 4.5. *Under the same assumptions of Theorem A, Theorem C holds if and only if for all $\varepsilon > 0$ there exists a family of pairs $\{(\pi_i, X_i)\}_{i=1}^k$ such that*

- (1) X_i is a subsystem of X , $\forall i = 1, \dots, k$,
- (2) X_i is an SFT, $\forall i = 1, \dots, k$, and
- (3) $\{B(P(X_i, \Phi|_{X_i}), \varepsilon)\}_{i=1}^k$ forms an ε -cover of $[P(Y, \pi\Phi), P(X, \Phi)]$.

Proof. Let $\rho \in [1, \rho(\pi, \Phi)]$ and let $P(Y, \pi\Phi) = p > 0$. For $\varepsilon p > 0$ we assume that there exists a family of pairs $\{(\pi_i, X_i)\}_{i=1}^k$ where $\pi_i : X_i \rightarrow Y$ is a factor map and X_i is an SFT $\forall i = 1, \dots, k$, and $\{B(P(T_i, \Phi|_{X_i}), \varepsilon p)\}_{i=1}^k$ forms an εp -cover of $[P(Y, \pi\Phi), P(X, \Phi)]$. Since $\rho \in [1, \rho(\pi, \Phi)]$ and

$$P(Y, \pi\Phi) = p \leq \rho p \leq \rho(\pi, \Phi) P(Y, \pi\Phi) = P(X, \Phi),$$

i.e., $\rho p \in [P(Y, \pi\Phi), P(X, \Phi)]$, there exists an $1 \leq i \leq k$ such that $\rho p \in B(P(X_i, \Phi|_{X_i}), \varepsilon p)$, i.e.,

$$|\rho p - P(T_i, \Phi|_{X_i})| \leq \varepsilon p.$$

This means that $|\rho - \rho(\pi_i, \Phi|_{X_i})| \leq \varepsilon$. On the other hand, for $\varepsilon > 0$, take a sequence $1 \leq \rho_1 = 1, \rho_2, \dots, \rho_{k-1}, \rho_k = \rho(\pi, \Phi) \leq \rho(\pi, \Phi)$ with

$$(4.4) \quad \frac{\varepsilon}{2p} \leq |\rho_{i+1} - \rho_i| \leq \frac{\varepsilon}{p} \text{ for all } i \in 1, \dots, k-1.$$

Define $I_i = [\rho_i, \rho_{i+1}]$ for $i \in 1, \dots, k-1$. It follows from Theorem C that there exists a sequence $\{(\pi_i, X_i)\}_{i=1}^k$ where π_i is a factor from X_i to Y , $X_i \subseteq X$ is an SFT and $\rho(\pi_i, \Phi|_{X_i}) \in [\rho_i, \rho_{i+1}]$. If $i = k$, $P(X_k, \Phi|_{X_k}) \leq \rho_k P(Y, \pi\Phi) = P(T, \Phi)$, and if $i = 1$, $P(X_1, \Phi|_{X_1}) \geq \rho_1 P(Y, \pi\Phi) = P(Y, \pi\Phi)$. If $P(Y, \pi\Phi) \leq q \leq P(X, \Phi)$, then $1 \leq \frac{q}{p} \leq \rho(\pi, \Phi)$; thus $\frac{q}{p} \in I_i$ for some i . Therefore

$$\left| \frac{q}{p} - \rho(\pi_i, X_i) \right| \leq \varepsilon.$$

This implies that $|q - P(X_i, \Phi|_{X_i})| \leq p\varepsilon$, and this means that $\{B(P(X_i, \Phi|_{X_i}), p\varepsilon)\}_{i=1}^k$ forms a $p\varepsilon$ -cover of $[P(Y, \pi\Phi), P(X\Phi)]$. This completes the proof. $\square$

Lemma 4.6. *Let $\pi : X \rightarrow Y$ be a one-block factor with diamond. Then properties (1), (2) and (3) of Lemma 4.5 hold.*

Proof. Without loss of generality we assume that X is full shift. Since $\Phi \in \mathbf{C}_{aa}(X, T)$, we conclude that $Z_n(X, \Phi) = \sum_{I \in X_n} \phi_n([I])$ is sub-additive. Then we have

$$(4.5) \quad P(X, \Phi) \leq \frac{1}{n} \log Z_n(X, \Phi), \quad \forall n \in \mathbb{N},$$

and for $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

$$(4.6) \quad P(X, \Phi) \geq \frac{1}{n} \log Z_n(X, \Phi) - \varepsilon.$$

By Theorem 4.4 we choose $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ and for all $I \in X_n$, $X \setminus I$ is mixing, we have

$$P(X, \Phi) - \varepsilon \leq P(X \setminus I, \Phi|_{X \setminus I}) < P(X, \Phi).$$

Since $\pi : X \rightarrow Y$ has diamond, then for all $J \in Y_n$ with $n \in \mathbb{N}$,

$$\#\{I \in X_n : \pi(I) = J\} \geq 1.$$

We define $S_J = \{I \in X_n : \pi(I) = J\}$ and arrange $\{J : J \in Y_n\}$ in the lexicographic order with $J_1 < J_2 < \dots < J_m$ and define $S_1 = S_{J_1}, S_2 = S_{J_2}, \dots, S_m = S_{J_m}$. We also arrange S_i in the lexicographic order, i.e.,

$$S_i = \left\{ I_j^{(i)} \right\}_{j=1}^{|S_i|}, \text{ for } i = 1, \dots, m,$$

where $|A|$ denotes the number of elements of A . For all i , define

$$\hat{S}_i = S_i \setminus I_1^{(i)};$$

i.e., drop the first pattern in S_i for all $i = 1, \dots, m$. Therefore $|\hat{S}_i| = |S_i| - 1$ for $i = 1, \dots, m$. Letting $|\hat{S}_1| + |\hat{S}_2| + \dots + |\hat{S}_m| = r(m)$, we put all elements of

$\hat{S}_1, \dots, \hat{S}_m$ together in order and renumber all its elements by $\{I_j\}_{j=1}^{r(m)}$, i.e.,

$$\begin{aligned} \{\hat{S}_1, \dots, \hat{S}_m\} &= \left\{ \left\{ I_2^{(1)}, \dots, I_{|\hat{S}_1|}^{(1)} \right\}, \dots, \left\{ I_2^{(1)}, \dots, I_{|\hat{S}_1|}^{(1)} \right\} \right\} \\ &= \{I_1, I_2, \dots, I_{r(m)}\}. \end{aligned}$$

Define $S = \{I_1, I_2, \dots, I_{r(m)}\}$. We construct a family of subsystems of X as follows:

- (1) Let $X_0 = X$ and $X_1 = X \setminus I_1$.
- (2) $X_j = X_{j-1} \setminus I_j$ for all $1 \leq j \leq r(m)$.
- (3) Finally, $X_{r(m)} = X_0 \setminus \bigcup_{j=1}^{r(m)} I_j$.

For $1 \leq j \leq r(m)$, define $\pi_i = \pi|_{X_i} : X_i \rightarrow Y$. According to the construction, it can be easily checked that π_i is a factor for all $i \in [1, r(m)]$, and it follows from Theorem 4.4 that

$$(4.7) \quad P(X, \Phi) > P(X_1, \Phi|_{X_1}) > \dots > P(X_{r(m)}, \Phi|_{X_{r(m)}})$$

and

$$(4.8) \quad P(X_{i+1}, \Phi|_{X_{i+1}}) \geq P(X_i, \Phi|_{X_i}) - \varepsilon \text{ for all } i \in [1, r(m) - 1].$$

Finally, we claim that

$$(4.9) \quad P(X_{r(m)}, \Phi|_{X_{r(m)}}) \leq P(Y, \pi\Phi) + \varepsilon.$$

Indeed, it follows from (4.6) and Proposition 2.7 that

$$(4.10) \quad P(Y, \Psi) = P(X, \Phi) \geq \frac{1}{n} \log Z_n(X, \Phi) - \varepsilon.$$

Since

$$Z_n(Y, \Psi) = \sum_{J \in Y_n} \sum_{I: \pi(I)=J} \phi_n(I)$$

is also sub-additive, there exists $N_3 \in \mathbb{N}$ such that if $n \geq N_3$, then

$$(4.11) \quad P(Y, \Psi) \geq \frac{1}{n} \log Z_n(Y, \Psi) - \varepsilon.$$

By the construction of $X_{r(m)}$, we have

$$(4.12) \quad Z_n(Y, \Psi) = Z_n(Y, \pi\Phi) = Z_n(X_{r(m)}, \Phi),$$

Combining (4.11), (4.5) and (4.12) we have that if $n \geq \max\{N_1, N_2, N_3\}$, then

$$\begin{aligned} P(Y, \pi\Phi) &\geq \frac{1}{n} \log Z_n(Y, \pi\Phi) - \varepsilon = \frac{1}{n} \log Z_n(Y, \Psi) - \varepsilon \\ &= \frac{1}{n} \log Z_n(X_{r(m)}, \Phi|_{X_{r(m)}}) - \varepsilon \geq P(X_{r(m)}, \Phi|_{X_{r(m)}}) - \varepsilon. \end{aligned}$$

Thus (4.9) holds, and it follows from (4.7), (4.8) and (4.9) that

$$\bigcup_{i=1}^{r(m)} B(P(X_i, \Phi|_{X_i}), \varepsilon)$$

forms an ε -cover of $[P(Y, \pi\Phi), P(X, \Phi)]$. The proof is completed. $\square$

Finally, we finish the proof of Theorem C.

Proof of Theorem C. The proof is obtained by combining Lemma 4.5 and Lemma 4.6. The proof is completed. $\square$

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