SOBOLEV ESTIMATES FOR THE LOCAL EXTENSION OF $\bar{\partial}_b$ -CLOSED $(0, 1)$ -FORMS ON REAL HYPERSURFACES IN $\mathbb{C}^n$ WITH TWO POSITIVE EIGENVALUES

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ABSTRACT. Let $\mathcal{M}$ be a smooth real hypersurface in complex space of dimension $n \geq 3$, and assume that the Levi-form at z_0 on $\mathcal{M}$ has at least two positive eigenvalues. We estimate solutions of the local $\bar{\partial}$ -closed extension problem near z_0 for $(0, 1)$ -forms in Sobolev spaces. Using this result, we estimate the local solution of tangential Cauchy-Riemann equations near z_0 for $(0, 1)$ -forms in Sobolev spaces.

1. INTRODUCTION

For a set $W \subset \mathbb{C}^n$, we denote the vector space of smooth (p, q) -forms on W by $\Lambda^{p,q}(W)$. Let $\mathcal{M}$ be a smooth real hypersurface in $\mathbb{C}^n$ with a smooth defining function ρ , and let $\mathcal{B}^{p,q}(\mathcal{M})$ be the restriction of $\Lambda^{p,q}(\mathbb{C}^n)$ to $\mathcal{M}$ which is pointwise orthogonal to the ideal generated by $\bar{\partial}\rho$. In the sequel, we let $z_0 \in \mathcal{M}$ be a fixed point and let V be a neighborhood of z_0 in $\mathbb{C}^n$ where ρ is defined. For each open set $U \subset V$, $z_0 \in U$, we set $U^- = \{z \in U; \rho(z) \leq 0\}$ and $U^+ = \{z \in U; \rho(z) \geq 0\}$.

If there exists a neighborhood $U \subset V$, $z_0 \in U$, such that for any $\alpha \in \mathcal{B}^{p,q}(\mathcal{M} \cap U)$ with $\bar{\partial}_b \alpha = 0$ on $\mathcal{M} \cap U$, there exist a smooth (p, q) -form $\tilde{\alpha} \in \Lambda^{p,q}(U^-)$ with $\bar{\partial} \tilde{\alpha} = 0$ in U^- and $(\tilde{\alpha} - \alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$, then we say the one-sided weak $\bar{\partial}$ -closed extension problem is locally solvable.

The $\bar{\partial}$ -closed extension problem and the solvability of the tangential Cauchy-Riemann equation for functions have origins in the two papers by Hans Lewy [17, 18]. In [14], Kohn and Rossi introduced the $\bar{\partial}_b$ -complex and extended these problems to differential forms and solved the global $\bar{\partial}$ -closed extension problem for forms from $\mathcal{M} = b\Omega$ to the domain Ω . Later, these extension problems were studied by several authors [13, 16, 19, 20].

For the local extension problem, Andreotti and Hill [2] solved the local weak $\bar{\partial}$ -closed extension problem when the Levi-form at $z_0 \in \mathcal{M}$ satisfies the condition $Y(q)$. Under the same assumption, Bogges and Shaw [4] proved the same result

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using the integral kernel method. Note that the estimates of the solutions of the local extension problem or local solvability of the $\bar{\partial}_b$ -equation in various spaces such as in C^k , L^p , Lipschitz and in Sobolev spaces, have many applications in the study of complex analysis, for example, the embeddability of abstract CR structures [1, 5, 6, 15, 24].

In [21], Shaw showed the local solvability of $\bar{\partial}_b$ in L^p spaces when $\mathcal{M}$ is the boundary of strongly pseudoconvex domains in $\mathbb{C}^n$ and in the Sobolev spaces when $\mathcal{M}$ is the boundary of pseudoconvex domains of finite type [22]. In [23], Shaw also constructed homotopy formulas for $\bar{\partial}_b$ using integral kernel methods and showed the local solvability for $\bar{\partial}_b$ in Hölder space when the Levi-form at $z_0 \in \mathcal{M}$ has at least $(q+2)$ positive and $(q+2)$ negative eigenvalues. In [10], the author obtained estimates for the local one-sided extension problem and proved the local solvability of $\bar{\partial}_b$ equations for (p, q) -forms with estimates in the Sobolev spaces when the Levi-form has at least $(q+2)$ positive eigenvalues (and hence $n \geq 4$).

It would be much better if we could prove the above result under the assumption that the Levi-form at $z_0 \in \mathcal{M}$ has only $(q+1)$ positive eigenvalues. We are especially interested in the case where $q = 1$ and $n \geq 3$. If $n \geq 4$, and if we have only two positive eigenvalues (not a mixed one), we note that the condition $Y(q)$, for $q = 1$, is not satisfied.

In this paper, we prove the above extension problem for $(0, 1)$ -forms with estimates in Sobolev spaces when the Levi-form at $z_0 \in \mathcal{M}$ has at least 2 positive eigenvalues. For a set $W \subset \mathbb{C}^n$, we denote the Sobolev norm of order s on W by $\|\cdot\|_{s,W}$.

Theorem 1.1. *Let $\mathcal{M}$ be a smooth hypersurface in $\mathbb{C}^n$, $n \geq 3$, with smooth defining function ρ , and suppose that the Levi-form at $z_0 \in \mathcal{M}$ has at least 2 positive eigenvalues. Then there is a neighborhood U of z_0 such that for any neighborhood $U_0 \supset \supset U$, and for any $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U_0)$ satisfying $\bar{\partial}_b \alpha = 0$ on $\mathcal{M} \cap U$, there exists $\tilde{\alpha} \in \Lambda^{0,1}(U^-)$ such that $\bar{\partial} \tilde{\alpha} = 0$ on U^- and $(\tilde{\alpha} - \alpha) \wedge \bar{\partial} \rho = 0$ on $\mathcal{M} \cap U$. Also, if we let $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U , $\tilde{\alpha}$ satisfies the estimate*

$$(1.1) \quad \|\tilde{\alpha}\|_{s,U^-} \leq C_s \|\chi \alpha\|_{s+1,\mathcal{M}},$$

for each real $0 \leq s < 1/2$, and

$$(1.2) \quad \|\tilde{\alpha}\|_{s,U^-} \leq C_s \|\chi \alpha\|_{s+1/2,\mathcal{M}},$$

for each real $s \geq 1/2$, where C_s does not depend on α .

Note that the local $\bar{\partial}$ -closed extension problem and the local solvability of the $\bar{\partial}_b$ equation are closely related [19, 20]. We also note that there are well-known nonsolvability results of the tangential Cauchy Riemann equation for $n = 2$ [18] and for $q = n - 1$ [11]. Using the results of Theorem 1.1, we solve the local $\bar{\partial}_b$ equation in Sobolev spaces.

Theorem 1.2. *Let $\mathcal{M}$ be a smooth hypersurface in $\mathbb{C}^n$, $n \geq 3$, and suppose that the Levi-form at $z_0 \in \mathcal{M}$ has at least 2 positive eigenvalues, and $\mathcal{M}$ is pseudoconvex near z_0 . Then there is a neighborhood U of z_0 such that for any $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U)$ such that $\bar{\partial}_b \alpha = 0$ on $\mathcal{M} \cap U$, and for any $W \subset \subset U$, and for each real $s \geq 0$, there exists $u_s \in \mathcal{B}^{(0,0)}(\mathcal{M} \cap W)$ such that $\bar{\partial}_b u_s = \alpha$ on $\mathcal{M} \cap W$ and satisfies the estimate*

$$\|u_s\|_{s,\mathcal{M} \cap W} \leq C_s \|\alpha\|_{s+1,\mathcal{M} \cap U}.$$

Remark 1.3. In Theorem 1.1, the differentiability assumption $\alpha \in C^\infty$ can be weakened to $\alpha \in H^{s+1}(\mathcal{M} \cap U)$ (or $\alpha \in H^{s+1/2}(\mathcal{M} \cap U)$ if $s \geq 1/2$) to get $\tilde{\alpha} \in H^s(M \cap U^-)$, and similarly for Theorem 1.2.

2. PRELIMINARIES

Let Ω be a domain in $\mathbb{C}^n$ with smooth boundary, let $\bar{\partial}$ be the Cauchy-Riemann operator on Ω , and let ϑ be the formal adjoint of $\bar{\partial}$. Also, we let $N_{(p,q)}$ denote the Neumann operator for (p, q) -forms. The Hodge star operator $*$: $\bigwedge^{p,q}(\Omega) \rightarrow \bigwedge^{n-p,n-q}(\Omega)$ is defined by the equation $\psi \wedge * \phi = \langle \psi, \phi \rangle dV$ where dV is the volume form on Ω . We set

$$\mathcal{C}^{p,q}(\Omega) = \{\phi \in \bigwedge^{p,q}(\Omega); \phi \wedge \bar{\partial}\rho = 0 \text{ on } b\Omega\},$$

where ρ is a defining function for Ω .

Let I be an open ball in $\mathbb{R}^d$, let $|I|$ denote the diameter of I , and let $H_{s,l}(\Omega \times I)$ be the Sobolev space of order s on Ω and of order l in I with the norm denoted by $\|\cdot\|_{s,l}$. We state a theorem for smooth dependence of solutions of the $\bar{\partial}$ -Neumann problem with respect to a parameter $\tau \in I$ [9].

Theorem 2.1. *Let $\{\Omega_\tau\}_{\tau \in I}$ be a smooth family of diffeomorphic strongly pseudoconvex domains in $\mathbb{C}^n$ and suppose that $\{\alpha_\tau\}_{\tau \in I}$ is a family of (p, q) -forms on $\{\Omega_\tau\}_{\tau \in I}$ such that $\alpha_\tau \in R(\bar{\partial}_\tau)$, the range of $\bar{\partial}_\tau$, for each $\tau \in I$, where $|I|$ is sufficiently small. Then for each real number $s \geq -1/2$ and for each nonnegative integer l , there is $C_{s,l} > 0$ such that the Neumann solution U of $\square U = \alpha$ and the canonical solution $u = \bar{\partial}^* U$ of $\bar{\partial} u = \alpha$ on $\{\Omega_\tau\}_{\tau \in I}$ satisfy*

$$(2.1) \quad \|U\|_{s+1,l}, \|u\|_{s+1/2,l} \leq C_{s,l} \sum_{r=0}^l \|\alpha\|_{s+l-r,r},$$

where $\alpha \in H_{s+l-r,r}(\Omega \times I)$, $0 \leq r \leq l$.

Remark 2.2. Note that the weak extension problem is a Cauchy problem to preserve the boundary values in a tangential direction. This means that we have to solve the $\bar{\partial}^*$ -equation instead of the $\bar{\partial}$ -equation. Let Ω be a smoothly bounded pseudoconvex domain in $\mathbb{C}^k$ and let $\alpha \in \mathcal{B}^{p,q}(b\Omega)$, where $0 \leq p \leq k$ and $1 \leq q \leq k-1$. Then a necessary and sufficient condition for the extension problem to be solved is

$$(2.2) \quad \int_{b\Omega} \alpha \wedge h = 0,$$

for every $h \in \bigwedge^{k-p,k-q-1}(\Omega) \cap \text{Ker}(\bar{\partial})$, and (2.2) is equivalent to the condition $\bar{\partial}_b \alpha = 0$ for $q \leq k-2$. Then the explicit formula for the extension is given by $\tilde{\alpha} = \alpha' - u$, where α' is a smooth extension of α onto Ω , and $u = -*\bar{\partial}N_{(n-p,n-q)}*\bar{\partial}\alpha'$.

In the sequel, we let $B_\epsilon(z)$ be a ball of radius $\epsilon > 0$ centered at z . Also, for $z \in \mathbb{C}^n$, we write $z = (z', z'')$ where $z' \in \mathbb{C}^2$ and $z'' \in \mathbb{C}^{n-2}$. In [8], we filled a neighborhood of $z_0 = (z'_0, z''_0) \in \mathcal{M}$ by strongly convex domains $\{\Omega_{z''}\}_{B_\sigma(z''_0)}$ whose diameters converge to zero as $z'' \rightarrow z''_0$. Therefore the estimates of the Neumann operators are not uniform in Sobolev spaces. For a remedy of this non-uniformity, we shall fill a neighborhood of z_0 in a different manner. That is, we slice the domain by subdomains of complex dimension two starting at $z_0 \in \mathcal{M}$ as follows:

Following the holomorphic changes of coordinates as in Section 2 of [10], we have special coordinates $z = (z', z'')$, $z' = (z_1, z_2)$, $z'' = (z_3, \dots, z_n)$, $z(z_0) = (0, \dots, 0, 1)$, and a new local defining function $\tilde{\rho}(z)$, which can be written as

$$(2.3) \quad \tilde{\rho}(z) = |z_n|^2 - 1 + \sum_{i=1}^2 |z_i|^2 + \sum_{i=3}^{n-1} \lambda_i |z_i|^2 + \mathcal{O}(|z - z_0|^3),$$

near $z_0 = (0, \dots, 0, 1)$. Let $\sigma > 0$ be a small constant to be determined. We regard $z'' := t'' = (t_3, \dots, t_n) \in \mathbb{C}^{n-2}$ as a parameter variable near $t''_0 := z''_0 = (0, \dots, 0, 1) \in \mathbb{C}^{n-2}$ and we construct a family of strongly convex domains:

For each $|z'| < \sigma^{1/4}$ and $|t'' - t''_0| < \sigma^{1/4}$, set $\rho(z', t'') = \tilde{\rho}(z', t_3, \dots, t_{n-1}, t_n - \sigma^{1/3} z_1)$. In view of (2.3), we can write

$$\begin{aligned} \rho(z', t'') &= (|t_n|^2 - 1) - 2\sigma^{1/3} \operatorname{Re}(z_1 \bar{t}_n) + \sigma^{2/3} |z_1|^2 + \sum_{i=1}^2 |z_i|^2 \\ &\quad + \sum_{i=3}^{n-1} \lambda_i |t_i|^2 + \mathcal{O}(|z - z_0|^3). \end{aligned}$$

For $t'' \in B_\sigma(t''_0)$, we set

$$(2.4) \quad r_{t''}^\sigma(z') := (1 - |t_n|^2) + 2\sigma^{1/3} \operatorname{Re}(z_1 \bar{t}_n) - \sum_{i=3}^{n-1} \lambda_i |t_i|^2 + \mathcal{O}(|z - z_0|^3).$$

At $t'' = t''_0 = (0, \dots, 0, 1)$, i.e., at the center of the ball $B_\sigma(t''_0)$, we have

$$r_{t''_0}^\sigma(z') = 2\sigma^{1/3} \operatorname{Re}(z_1) + \mathcal{O}(|z'|^3),$$

because $t_n = 1$. Therefore it follows that

$$\Omega_{t''_0} := \{z' \in B_{\sigma^{1/4}}(z'_0); \sigma^{2/3} |z_1|^2 + \sum_{i=1}^2 |z_i|^2 < r_{t''_0}^\sigma(z')\}$$

is a nonempty strongly convex domain contained in the side of $\rho \leq 0$ because, for example, $(\sigma^{3/8}, 0) \in \Omega_{t''_0}$ provided $\sigma > 0$ is sufficiently small. Note that $\Omega_{t''_0}$ is a small deformation of a ball whose radius is larger than or equal to $\sigma^{17/48}$. Also we see that $z_0 \in b\Omega_{t''_0} \subset \mathcal{M}$ and $\Omega_{t''_0}$ is the central slice of the side $\rho \leq 0$.

Similarly, for each $t'' \in B_\sigma(t''_0)$, we set

$$\Omega_{t''} := \{z' \in B_{\sigma^{1/4}}(z'_0); \sigma^{2/3} |z_1|^2 + \sum_{i=1}^2 |z_i|^2 < r_{t''}^\sigma(z')\}.$$

Since $|t_n - 1| \leq \sigma$, we see that the second term in the right-hand side of (2.4), $2\sigma^{1/3} \operatorname{Re}(z_1 \bar{t}_n)$, is the major term. As above, $\Omega_{t''} \subset \{z' \in B_{\sigma^{1/4}}(z'_0); \rho(z', t'') \leq 0\}$ and $b\Omega_{t''} \subset \mathcal{M}$, and $\Omega_{t''}$ is a small deformation of a ball whose radius is larger than or equal to $\sigma^{17/46}$.

Remark 2.3. Note that $\operatorname{Re} z_1 \lesssim \sigma^{1/3}$ if $z' \in \Omega_{t''}$ and this necessitates that $|z'| \lesssim \sigma^{1/3}$; that is, $\Omega_{t''} \subset B_{\sigma^{7/24}}(z_0) \subset B_{\sigma^{1/4}}(z_0)$ provided σ is sufficiently small.

Remark 2.4. With the special coordinates $z = (z', z'')$ defined in (2.3), set $D_\sigma = \{z \in \mathbb{C}^n; |z_j| < \sigma^{7/24}, 1 \leq j \leq n-1, |z_n - 1| < \sigma^{1/4}\}$. Then $U^- \subset D_\sigma^-$. Also note that $r_{t''}^\sigma < 0$, and hence $\Omega_{z''}$ is an empty set for appropriate $t'' = z'' = (z_3, \dots, z_n)$,

say, when $\sigma^{7/24} < |z_n - 1| < \sigma^{1/4}$ and $|z_j| < \sigma$, $j = 3, \dots, n-1$, provided $\sigma > 0$ is sufficiently small.

Remark 2.5. Let us fix $\sigma = \sigma_0$ satisfying the above conditions and set $I := B_{\sigma_0}(t_0'')$. Then $\{\Omega_{z''}\}_{z'' \in I}$ is a family of bounded strongly convex domains in $\mathbb{C}^2$ and they are a small deformation of a ball of radius bigger than $\sigma_0^{17/48}$ (and hence its diameters are bounded uniformly from below). Also, $\Omega_{z''}$ is diffeomorphic to $\Omega_{z_0''}$ for each $z'' \in I$ and foliate in the part $\rho \leq 0$. Define a tubular neighborhood U^- of z_0 by

$$(2.5) \quad U^- := \bigcup_{z'' \in I} \bar{\Omega}_{z''} \times \{z''\}.$$

3. $\bar{\partial}$ -CLOSED EXTENSION FOR $(p, 1)$ -FORMS

To prove the local extension theorem, we use the local decomposition of the set U^- considered in Remark 2.5. We will solve the ϑ -equation and use the estimates (2.1) on parameter variables $z'' \in I$, where I is defined as in Remark 2.5. Set $\mathcal{K} = \{1, 2\}$ and $\mathcal{K}^c = \{3, \dots, n\}$. For a smooth function f defined in $\mathbb{C}^n$, we define

$$\bar{\partial}_{\mathcal{K}} f = \sum_{j=1}^2 \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \quad \text{and} \quad \bar{\partial}_{\mathcal{K}^c} f = \sum_{j=3}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

We can extend this definition for arbitrary smooth forms.

Since p does not play an important role in the estimates, we set $p = 0$; i.e., we consider only the cases $\bigwedge^{0,1}(W)$ where W is an appropriate set. We recall that $\|\cdot\|_{s,k,W}$ is the Sobolev norm of order s in the $z' \in \mathbb{C}^2$ variable and of order k in the $z'' \in I = B_{\sigma_0}(z'') \subset \mathbb{C}^{n-2}$ variable. In the sequel, every summation will be over strictly increasing indices, and the constants such as C_s , $C_{s,k}$ will depend only on s or k and can vary line to line while we estimate.

Proposition 3.1. *Let $\mathcal{M}$ be a smooth real hypersurface in $\mathbb{C}^n$, $n \geq 3$, with smooth defining function ρ defined in a neighborhood of $z_0 \in \mathcal{M}$, and suppose that the Levi-form at z_0 has at least 2 positive eigenvalues. Then there is a neighborhood U of z_0 such that for any neighborhood $U_0 \supset \supset U$ and for any $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U_0)$ satisfying $\bar{\partial}_b \alpha = 0$ on $\mathcal{M} \cap U$, there exist $\tilde{\alpha}_l \in \Lambda^{0,1}(U_0^-)$, $l = 0, 1, 2$, such that $(\tilde{\alpha}_l - \alpha) \wedge \bar{\partial} \rho = 0$ on $\mathcal{M} \cap U$ and $\bar{\partial} \tilde{\alpha}_l$ can be written as*

$$(3.1) \quad \bar{\partial} \tilde{\alpha}_l = \sum_{\substack{I \subset \mathcal{K}, J \subset \mathcal{K}^c \\ |I|+|J|=2, |J| \geq l}} \alpha_{IJ}^l d\bar{z}^I \wedge d\bar{z}^J$$

on U^- for some smooth functions α_{IJ}^l . Also, if we let $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U , then for each real $s \geq 0$, $\tilde{\alpha}_l$ satisfies the estimates

$$(3.2) \quad \begin{aligned} \|\tilde{\alpha}_l\|_{s,U^-} &\leq C_s \|\chi \alpha\|_{s+\frac{l-1}{2}, \mathcal{M}}, \quad \text{for } l = 0, 1, \text{ and} \\ \|\tilde{\alpha}_2\|_{s,U^-} &\leq C_s \|\chi \alpha\|_{s+1, \mathcal{M}}. \end{aligned}$$

Also if $s \geq \frac{1}{2}$, we have

$$(3.3) \quad \|\tilde{\alpha}_2\|_{s,U^-} \leq C_s \|\chi \alpha\|_{s+1/2, \mathcal{M}}.$$

Proof. Let V be a neighborhood of $z_0 \in \mathcal{M}$ where special frames are defined, and let $I = B_{\sigma_0}(z_0'')$ be the neighborhood of $z_0'' \in \mathbb{C}^{n-2}$ defined in Remark 2.5. By shrinking $\sigma_0 > 0$ if necessary, we may assume that there is a neighborhood U of $z_0 \in \mathcal{M}$ such that Theorem 2.1 holds on U where $U \subset D_{\sigma_0} \subset B_{\sigma_0^{1/4}}(z_0) \subset V$, and

$U^- = \bigcup_{z'' \in I} \bar{\Omega}_{z''} \times \{z''\}$, where D_{σ_0} is defined in Remark 2.4. For any U_0 with $U \subset\subset U_0 \subset D_{\sigma_0}$, we choose a smooth cutoff function $\chi \in C_0^\infty(U_0)$ with $\chi = 1$ on U .

Assume $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U_0)$ and $\bar{\partial}_b \alpha = 0$ on $\mathcal{M} \cap U$. Using Theorem 2.1, we shall construct $\tilde{\alpha}_l$ inductively so that $\tilde{\alpha}_l$ satisfies (3.1) and (3.2). If we replace α by $\chi\alpha$, then we may assume that $\alpha \in C_0^\infty(\mathcal{M} \cap U_0)$. From Lemma 9.3.4 in [7], there is $\tilde{\alpha}_0 := E\alpha \in \bigwedge^{0,1}(U_0)$, $E\alpha = \alpha$ and $\bar{\partial}E\alpha = \mathcal{O}(\rho^\infty)$ on $\mathcal{M} \cap U$ such that

$$(3.4) \quad \|\tilde{\alpha}_0\|_{s,U^-} \leq C_s \|\chi\alpha\|_{s-1/2,\mathcal{M}},$$

for each real $s \geq 0$. This proves (3.1) and (3.2) for $l = 0$.

Replacing $\tilde{\alpha}_0$ by $\chi\tilde{\alpha}_0$, we assume that $\tilde{\alpha}_0 \in C_0^\infty(U_0)$. Let us write

$$\bar{\partial}\tilde{\alpha}_0 = \sum_{1 \leq j < k \leq n} \alpha_{jk}^0 d\bar{z}_j \wedge d\bar{z}_k,$$

and set

$$g^0 = \alpha_{12}^0 d\bar{z}_1 \wedge d\bar{z}_2 \text{ on } U^-, \text{ and } g^0 = 0 \text{ on } U^+.$$

Then it follows that $g^0 \in C^\infty(U)$ because $\bar{\partial}\tilde{\alpha}_0 = \mathcal{O}(\rho^\infty)$ on $\mathcal{M} \cap U$. Also note that g^0 comes from the components of $\bar{\partial}_K \tilde{\alpha}_0$. Hence for each real $s \geq -1$, and nonnegative integer k , there are $\tilde{C}_{s,k}$ and $C_{s,k}$ such that

$$(3.5) \quad \|g^0\|_{s,k,U^-} \leq \tilde{C}_{s,k} \|\tilde{\alpha}_0\|_{s+1,k,U^-} \leq C_{s,k} \|\chi\alpha\|_{s+k+1/2,\mathcal{M}}.$$

To remove the g^0 term in $\bar{\partial}\tilde{\alpha}_0$, we try to solve $\bar{\partial}_K u = g^0(\cdot, z'')$, for each $z'' \in I$, and set $\alpha_1 = \tilde{\alpha}_0 - u$. However, to preserve the boundary condition, it is required that $u \in C^{0,1}(\Omega_{z''})$ for each $z'' \in I$. This means that we have to solve the $\bar{\partial}^*$ -equation rather than the $\bar{\partial}$ -equation.

Since $g^0(\cdot, z'')$ is a $(0, 2)$ -form, it becomes a top degree problem in $\mathbb{C}^2$, and hence it is required to satisfy (2.2); that is,

$$(3.6) \quad F_h(z'') := \int_{\Omega_{z''}} g^0(\cdot, z'') \wedge h = 0, \text{ for every } h \in C_{(2,0)}^\infty(\mathbb{C}^2) \cap \ker(\bar{\partial}_K).$$

To prove (3.6), we consider the coefficients of the term $d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_k$ in $\bar{\partial}^2 \tilde{\alpha}_0$. Then we see that

$$(3.7) \quad \frac{\partial \alpha_{12}^0}{\partial \bar{z}_k} - \frac{\partial \alpha_{1k}^0}{\partial \bar{z}_2} + \frac{\partial \alpha_{2k}^0}{\partial \bar{z}_1} = 0,$$

for $3 \leq k \leq n$. Since $g^0(\cdot, z'') = 0$ for z'' with $(z', z'') \in U^+ \subset D_\sigma$, it follows that F_h has compact support in $\mathbb{C}^{n-2}$. Therefore it follows from (3.7) that

$$\begin{aligned} \frac{\partial F_h(z'')}{\partial \bar{z}_k} &= \int_{\mathbb{C}^2} \frac{\partial g^0(\cdot, z'')}{\partial \bar{z}_k} \wedge h = \int_{\mathbb{C}^2} \left(\frac{\partial \alpha_{1k}^0}{\partial \bar{z}_2} - \frac{\partial \alpha_{2k}^0}{\partial \bar{z}_1} \right) \wedge h \\ &= - \int_{\mathbb{C}^2} \left(\alpha_{1k}^0 \frac{\partial h}{\partial \bar{z}_2} - \alpha_{2k}^0 \frac{\partial h}{\partial \bar{z}_1} \right) = 0 \end{aligned}$$

for $3 \leq k \leq n$. In view of Remark 2.4, it follows that $F_h(z'') = 0$ for $z'' = (z_3, \dots, z_n)$ with $\sigma^{7/24} < |z_n - 1| < \sigma^{1/4}$, provided σ is sufficiently small. Therefore we see that (3.6) holds.

Set $u_0 = - *_{\mathcal{K}} \bar{\partial}_K N_{(2,0)}^{\mathcal{K}} *_{\mathcal{K}} g^0(\cdot, z'')$ where $N_{(p,q)}^{\mathcal{K}}$ is the Neumann operator for (p, q) -forms, and $*_{\mathcal{K}}$ is the Hodge star operator on $\Omega_{z''}$. Then we have $\bar{\partial}_K u_0 = g^0$

and $u_0 \in \mathcal{C}^{0,1}(\Omega_{z''})$ for each z'' . Also for each real $s \geq 0$ and nonnegative integer k , it follows from (2.1), (3.4) and (3.5) that

$$(3.8) \quad \|u_0\|_{s,k,U^-} \lesssim \sum_{r=0}^k \|g^0\|_{s-1/2+k-r,r,U^-} \lesssim \|\tilde{\alpha}_0\|_{s+1/2+k,U^-} \lesssim \|\chi\alpha\|_{s+k,\mathcal{M}}.$$

Note that $u_0 \wedge \bar{\partial}_{\mathcal{K}}\rho = 0$ on $\mathcal{M} \cap U$ because $u_0 \in \mathcal{C}^{0,1}(\Omega_{z''})$ for each z'' . We have to correct u_0 so that the corrected one, $\tilde{u}_0$, belongs to $\mathcal{C}^{0,1}(\Omega)$; that is, $\tilde{u}_0 \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$. Since $u_0 \wedge \bar{\partial}_{\mathcal{K}}\rho = 0$ on $\mathcal{M} \cap U$, we can write

$$u_0 = \delta^0 \wedge \bar{\partial}_{\mathcal{K}}\rho + \rho\gamma^0$$

for some $\delta^0 \in \mathcal{C}^{0,0}(\Omega_{z''})$ and $\gamma^0 \in \mathcal{C}^{0,1}(\Omega_{z''})$. We may assume that $\delta^0 \wedge \bar{\partial}_{\mathcal{K}}\rho$ and $\rho\gamma^0$ are disjoint, and hence it follows from the estimate in (3.8) that

$$(3.9) \quad \|\delta^0\|_{s,k,U^-} \lesssim \|u_0\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+k,\mathcal{M}}$$

for each real $s \geq 0$ and nonnegative integer k .

Set $\tilde{u}_0 = u_0 + \delta^0 \wedge \bar{\partial}_{\mathcal{K}^c}\rho$. Then one obtains that

$$\tilde{u}_0 \wedge \bar{\partial}\rho = (\delta^0 \wedge \bar{\partial}_{\mathcal{K}}\rho + \rho\gamma^0 + \delta^0 \wedge \bar{\partial}_{\mathcal{K}^c}\rho) \wedge \bar{\partial}\rho = 0$$

on $\mathcal{M} \cap U$ and hence $\tilde{u}_0 \in \mathcal{C}^{0,1}(U^-)$. Now we set $\tilde{\alpha}_1 = \tilde{\alpha}_0 - \tilde{u}_0$. Then it follows that $(\tilde{\alpha}_1 - \alpha) \wedge \partial\rho = -\tilde{u}_0 \wedge \partial\rho = 0$ on $U \cap \mathcal{M}$, and we can write

$$\bar{\partial}\tilde{\alpha}_1 = \sum_{3 \leq k \leq n} (\alpha_{1k}^1 d\bar{z}_1 + \alpha_{2k}^1 d\bar{z}_2) \wedge d\bar{z}_k + \sum_{3 \leq i < j \leq n} \alpha_{ij}^1 d\bar{z}_i \wedge d\bar{z}_j,$$

for some smooth functions α_{ij}^1 .

In view of (3.4), (3.8) and (3.9) one obtains that

$$(3.10) \quad \|\tilde{\alpha}_1\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+k,\mathcal{M}}$$

for each real $s \geq 0$ and nonnegative integer k . Thus (3.2) holds for $l = 1$ by the Sobolev interpolation theorem.

For each $3 \leq k \leq n$, set $g_k = \alpha_{1k}^1 d\bar{z}_1 + \alpha_{2k}^1 d\bar{z}_2$. If we consider the coefficients of $\bar{\partial}^2 \tilde{\alpha}_1 = 0$, we see that $\bar{\partial}_{\mathcal{K}} g_k = 0$; that is, g_k is a $\bar{\partial}_{\mathcal{K}}$ closed $(0, 1)$ -form in $\mathbb{C}^2$. Since $\vartheta_{\mathcal{K}} = -*_{\mathcal{K}} \bar{\partial}_{\mathcal{K}} *_{\mathcal{K}}$, it follows that $*_{\mathcal{K}} g_k$ is a $\vartheta_{\mathcal{K}}$ -closed $(2, 1)$ -form in $\mathbb{C}^2$. Set $u^k(\cdot, z'') = -*_{\mathcal{K}} \bar{\partial}_{\mathcal{K}} N_{(2,1)}^{\mathcal{K}} *_{\mathcal{K}} g_k(\cdot, z'')$. Then we have $\bar{\partial}_{\mathcal{K}} u^k = g_k$ and $u^k \in \mathcal{C}^{0,0}(\Omega_{z''})$ for each $z'' \in I$, and hence $u^k = 0$ on $U \cap \mathcal{M}$.

Set $\tilde{u}_1 = \sum_{k=3}^n u^k d\bar{z}_k$. Note that each g_k comes from the components of $\bar{\partial}\tilde{\alpha}_1$. Thus, for each real $s \geq 0$ and nonnegative integer k , it follows from (2.1) and (3.10) that

$$(3.11) \quad \|\tilde{u}_1\|_{s,k,U^-} \lesssim \sum_{r=0}^k (\|\tilde{\alpha}_1\|_{s+1/2+k-r,r,U^-} + \|\tilde{\alpha}_1\|_{s-1/2+k-r,r+1,U^-}) \lesssim \|\chi\alpha\|_{s+1+k,\mathcal{M}}.$$

Here we lose $1/2$ derivative because we need $s - 1/2 + k - r \geq 0$ to use (3.10). On the other hand, if $s \geq 1/2$, we have

$$(3.12) \quad \|\tilde{u}_1\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+1/2+k,\mathcal{M}}.$$

Set $\tilde{\alpha}_2 = \tilde{\alpha}_1 - \tilde{u}_1$. Then we can write

$$\bar{\partial}\tilde{\alpha}_2 = \sum_{3 \leq i < j \leq n} \alpha_{ij}^2 d\bar{z}_i \wedge d\bar{z}_j,$$

for some smooth functions α_{ij}^2 . It also follows that $(\tilde{\alpha}_2 - \tilde{\alpha}_1) \wedge \bar{\partial}\rho = -\tilde{u}_1 \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$ because $\tilde{u}_1 = 0$ on $\mathcal{M} \cap U$. In view of (3.11), we obtain that

$$(3.13) \quad \|\tilde{\alpha}_2\|_{s,k,U^-} \lesssim \|\chi\alpha\|_{s+1+k,\mathcal{M}},$$

for each real $s \geq 0$ and nonnegative integer k .

Using (3.13) and the interpolation theorem in Sobolev spaces, we see that (3.2) holds for each real $s \geq 0$. Also, if $s \geq 1/2$, we see from (3.12) that (3.3) holds. This proves Proposition 3.1. $\square$

Now we are ready to prove the weak $\bar{\partial}$ -closed extension problem (Theorem 1.1). We recall that $U^- = \bigcup_{z'' \in I} \bar{\Omega}_{z''} \times \{z''\}$ as defined in (2.5).

Proof of Theorem 1.1. In view of Proposition 3.1, there exists $\tilde{\alpha}_2 \in \Lambda^{0,1}(U^-)$ such that

$$(3.14) \quad \bar{\partial}\tilde{\alpha}_2 = \sum_{3 \leq i < j \leq n} \tilde{\alpha}_{ij} d\bar{z}_i \wedge d\bar{z}_j,$$

for some smooth functions $\tilde{\alpha}_{ij}$. If we consider the coefficients of $d\bar{z}_l \wedge d\bar{z}_i \wedge d\bar{z}_j$ of $\bar{\partial}^2\tilde{\alpha}_2 = 0$, we see that

$$\frac{\partial \tilde{\alpha}_{ij}}{\partial \bar{z}_l} = 0,$$

for $l = 1, 2$. Hence $\tilde{\alpha}_{ij}$ is a holomorphic function on $\Omega_{z''}$ for each z'' .

Also, note that $(\tilde{\alpha}_2 - \alpha) \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$. Therefore $\bar{\partial}(\tilde{\alpha}_2 - \alpha) \wedge \bar{\partial}\rho = 0$ and hence $\bar{\partial}\tilde{\alpha}_2 \wedge \bar{\partial}\rho = 0$ on $\mathcal{M} \cap U$ because $\bar{\partial}_b\alpha = 0$ on $\mathcal{M} \cap U$. Considering the coefficients of $d\bar{z}_l \wedge d\bar{z}_i \wedge d\bar{z}_j$ of $\bar{\partial}\tilde{\alpha}_2 \wedge \bar{\partial}\rho$, $3 \leq i < j \leq n$, one obtains, from (3.14), that

$$\frac{\partial \rho}{\partial \bar{z}_l} \tilde{\alpha}_{ij} = 0,$$

for $l = 1, 2$ on $\mathcal{M} \cap U$. Since $\bar{\partial}_K\rho \neq 0$ on $b\Omega_{z''}$, at least one of $\partial\rho/\partial\bar{z}_l$ for $l = 1, 2$ is not equal to zero and hence $\tilde{\alpha}_{ij} = 0$ on $b\Omega_{z''}$. Since $\tilde{\alpha}_{ij}$ is a holomorphic function on $\Omega_{z''}$ for each $z'' \in I$, we see that $\tilde{\alpha}_{ij} \equiv 0$ and hence $\bar{\partial}\tilde{\alpha}_2 = 0$ on U^- by (3.14). If we set $\tilde{\alpha} = \tilde{\alpha}_2$, then $\tilde{\alpha}$ satisfies the estimates (1.1) and (1.2) by (3.2). This proves Theorem 1.1. $\square$

Proof of Theorem 1.2. Let U^- be the neighborhood constructed in Theorem 1.1 and $\alpha \in \mathcal{B}^{0,1}(\mathcal{M} \cap U)$ satisfies $\bar{\partial}_b\alpha = 0$ on $\mathcal{M} \cap U$. By shrinking U if necessary, we may assume that $\mathcal{M} \cap U$ is pseudoconvex. Assume $W \subset\subset U$, $z_0 \in W$, and choose U_1 such that $W \subset\subset U_1 \subset\subset U$. In view of Theorem 1.1, there is a weak $\bar{\partial}$ -closed extension $\tilde{\alpha}$ of α onto U_1^- satisfying the estimate (1.1) and (1.2) on U_1^- . By a minor modification of the proof in the lemma in section 4 of [3], we can construct a small pseudoconvex domain $B \subset U_1^-$ such that $W \cap \mathcal{M} \subset B \cap \mathcal{M}$.

For each real $s \geq 0$, set $\tilde{u}_s = \bar{\partial}^* N_s^B \tilde{\alpha}$ where N_s^B denotes the weighted $\bar{\partial}$ -Neumann operator in B with weight $e^{-t_s|z|^2}$ for sufficiently large $t_s > 0$ depending on s . Then we have $\bar{\partial}\tilde{u}_s = \tilde{\alpha}$ in B , and it follows that

$$(3.15) \quad \|\tilde{u}_s\|_{s,B} \leq C_s \|\tilde{\alpha}\|_{s,B}.$$

Set $u_s = \tau\tilde{u}_s$ where τ is the projection in $\bigwedge^{0,1}(\mathcal{M})$ onto $\mathcal{B}^{0,1}(\mathcal{M})$ defined by first restricting a $(0,1)$ -form ϕ in $\mathbf{C}^n$ to $\mathcal{M}$, then projecting the restriction to $\mathcal{B}^{p,1}(\mathcal{M})$.

Then $\bar{\partial}_b u_s = \alpha$ on W , and if we use the estimates (1.1), (1.2), (3.15) and the trace theorem in Sobolev spaces, we obtain that

$$\|u_s\|_{s,W} \leq C_s \|D\tilde{u}_s\|_{s-1/2,B} \leq C_s \|\tilde{\alpha}\|_{s+1/2,B} \leq C_s \|\chi\alpha\|_{s+1,\mathcal{M}},$$

for each real $s \geq 0$, where $\chi \in C_0^\infty(U)$ satisfies $\chi = 1$ on $U_1 \subset\subset U$. This completes the proof of Theorem 1.2. $\square$

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