IMPLICATIONS OF THE HASSE PRINCIPLE FOR ZERO CYCLES OF DEGREE ONE ON PRINCIPAL HOMOGENEOUS SPACES

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ABSTRACT. Let k be a perfect field of virtual cohomological dimension ≤ 2 . Let G be a connected linear algebraic group over k such that G^{sc} satisfies a Hasse principle over k . Let X be a principal homogeneous space under G over k . We show that if X admits a zero cycle of degree one, then X has a k -rational point.

INTRODUCTION

The following question of Serre [10, p. 192] is open in general.

Q: Let k be a field and G a connected linear algebraic group defined over k . Let X be a principal homogeneous space under G over k . Suppose X admits a zero cycle of degree one. Does X have a k -rational point?

Let k be a number field, let V be the set of places of k and let k_v denote the completion of k at a place v . We say that a connected linear algebraic group G defined over k satisfies a *Hasse principle* over k if the map $H^1(k, G) \rightarrow \prod_{v \in V} H^1(k_v, G)$ is injective. Let V_r denote the set of real places of k . If G is simply connected, then by a theorem of Kneser, the Hasse principle reduces to injectivity of the maps $H^1(k, G) \rightarrow \prod_{v \in V_r} H^1(k_v, G)$. That this result holds is a theorem due to Kneser, Harder and Chernousov [3], [4], [5]. Sansuc used this Hasse principle to show that **Q** has a positive answer for number fields.

Let k be any field and Ω the set of orderings of k . For $v \in \Omega$ let k_v denote the real closure of k at v . We say that a connected linear algebraic group G defined over k satisfies a *Hasse principle* over k if the map $H^1(k, G) \rightarrow \prod_{v \in \Omega} H^1(k_v, G)$ is injective. It is a conjecture of Colliot-Thélène [1, p. 652] that a simply connected semisimple group satisfies a Hasse principle over a perfect field of virtual cohomological dimension ≤ 2 . Bayer and Parimala [1] have given a proof in the case where G is of classical type, type F_4 and type G_2 .

Our goal in this paper is to extend Sansuc's result by providing a positive answer to **Q** when k is a perfect field of virtual cohomological dimension ≤ 2 and G^{sc} satisfies a Hasse principle over k . More precisely, we prove the following:

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Theorem 0.1. *Let k be a perfect field of virtual cohomological dimension ≤ 2 . Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. Let G be a connected linear algebraic group over k . If G^{sc} satisfies a Hasse principle over k , then the canonical map*

$$H^1(k, G) \rightarrow \prod_{i=1}^m H^1(L_i, G)$$

has trivial kernel.

We obtain the following as a corollary:

Corollary 0.2. *Let k be a perfect field of virtual cohomological dimension ≤ 2 . Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. Let G be a connected linear algebraic group over k . If the simple factors of G^{sc} are of classical type, type F_4 or type G_2 , then the canonical map*

$$H^1(k, G) \rightarrow \prod_{i=1}^m H^1(L_i, G)$$

is injective.

Sansuc's proof of a positive answer to **Q** over number fields relies on the surjectivity of the map $H^1(k, \mu) \rightarrow \prod_{v \in V_r} H^1(k_v, \mu)$ for μ a finite commutative group scheme. This result is a consequence of the Chebotarev density theorem and does not extend to a general field of virtual cohomological dimension ≤ 2 . Even in the case $\mu = \mu_2$, the surjectivity of the map $H^1(k, \mu) \rightarrow \prod_{v \in \Omega} H^1(k_v, \mu)$ imposes severe conditions on k such as the SAP property. The main content of this paper is to replace the arithmetic in Sansuc's paper with a norm principle over a real closed field.

1. ALGEBRAIC GROUPS

In this section, we review some well-known facts from the theory of algebraic groups and define some notation used in the remainder of the work.

Let k be a field. An *algebraic group* G over k is a smooth group scheme of finite type. A surjective morphism of algebraic groups with finite kernel is called an *isogeny* of algebraic groups. An isogeny $G_1 \rightarrow G_2$ is said to be *central* if its kernel is a central subgroup of G_1 .

An *algebraic torus* is an algebraic group T such that $T(\bar{k})$ is isomorphic to a product of multiplicative groups $G_{m, \bar{k}}$. A torus T is said to be *quasitrivial* if it is a product of groups of the form $R_{E_i/k} G_m$, where $\{E_i\}_{1 \leq i \leq r}$ is a family of finite field extensions of k .

An algebraic group G is called *linear* if it is isomorphic to a closed subgroup of GL_n for some n , or equivalently, if its underlying algebraic variety is affine. Of particular interest among connected linear algebraic groups are semisimple groups and reductive groups.

A connected linear algebraic group is called *semisimple* if it has no nontrivial, connected, solvable, normal subgroups. A semisimple group G is said to be *simply connected* if every central isogeny $G' \rightarrow G$ is an isomorphism. We can associate to any semisimple group a simply connected group $\tilde{G}$ (unique up to isomorphism)

such that there is a central isogeny $\tilde{G} \rightarrow G$. We refer to $\tilde{G}$ as the *simply connected cover* of G .

Any simply connected semisimple group is a product of simply connected simple algebraic groups [6, Theorem 26.8]. Any simple algebraic group belongs to one of four infinite families A_n, B_n, C_n, D_n or is of type E_6, E_7, E_8, F_4 or G_2 (see for example [6, §26]). A simple group which is of type A_n, B_n, C_n or D_n but not of type triality D_4 is said to be a *classical group*. All other simple groups are called *exceptional groups*.

A connected linear algebraic group is called *reductive* if it has no nontrivial, connected, unipotent, normal subgroups. Given a connected linear algebraic group G , the *unipotent radical* of G , denoted G^u , is the maximal connected unipotent normal subgroup of G . It is clear that G/G^u is always a reductive group. We denote G/G^u by G^{red} . The commutator subgroup of G^{red} is a semisimple group which we denote G^{ss} . We denote the simply connected cover of G^{ss} by G^{sc} .

A *special covering* of a reductive group G is an isogeny

$$1 \rightarrow \mu \rightarrow G_0 \times S \rightarrow G \rightarrow 1,$$

where G_0 is a simply connected semisimple algebraic k -group and S is a quasitrivial k -torus. Given a reductive group G there exists an integer n and a quasitrivial torus T such that $G^n \times T$ admits a special covering [8, Lemme 1.10].

2. GALOIS COHOMOLOGY AND ZERO CYCLES

For our convenience, we will discuss $\mathbf{Q}$ in the context of Galois Cohomology. We briefly review some of the notions from Galois Cohomology that we will use and then restate $\mathbf{Q}$ in this setting.

Let k be a field and $\Gamma_k = \text{Gal}(\bar{k}/k)$ be the absolute Galois group of k . For an algebraic k -group G , let $H^i(k, G) = H^i(\Gamma_k, G(\bar{k}))$ denote the Galois Cohomology of G with the assumption $i \leq 1$ if G is not abelian. For any k -group G , $H^0(k, G) = G(k)$ and $H^1(k, G)$ is a pointed set which classifies the isomorphism classes of principal homogeneous spaces under G over k . The point in $H^1(k, G)$ corresponds to the principal homogeneous space with rational point. We will interchangeably denote the point in $H^1(k, G)$ by *point* or 1.

Each Γ_k -homomorphism $f : G \rightarrow G'$ induces a functorial map $H^i(k, G) \rightarrow H^i(k, G')$, which we shall also denote by f . Given an exact sequence of k -groups,

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \rightarrow 1,$$

there exists a connecting map $\delta_0 : G_3(k) \rightarrow H^1(k, G_1)$ such that the following is an exact sequence of pointed sets:

$$G_1(k) \xrightarrow{f_1} G_2(k) \xrightarrow{f_2} G_3(k) \xrightarrow{\delta_0} H^1(k, G_1) \xrightarrow{f_1} H^1(k, G_2) \xrightarrow{f_2} H^1(k, G_3).$$

If G_1 is central in G_2 , there is in addition a connecting map $\delta_1 : H^1(k, G_3) \rightarrow H^2(k, G_1)$ such that the following is an exact sequence of pointed sets:

$$G_3(k) \xrightarrow{\delta_0} H^1(k, G_1) \xrightarrow{f_1} H^1(k, G_2) \xrightarrow{f_2} H^1(k, G_3) \xrightarrow{\delta_1} H^2(k, G_1).$$

Given a field extension L of k , $\text{Gal}(\bar{k}/L) \subset \text{Gal}(\bar{k}/k)$ and there is a restriction homomorphism $\text{res} : H^1(k, G) \rightarrow H^1(L, G)$. If G is a commutative group and if the degree of L over k is finite, there is also a corestriction homomorphism

$\text{cores} : H^1(L, G) \rightarrow H^1(k, G)$. The composition $\text{cores} \circ \text{res}$ is multiplication by the degree of L over k .

Let p be any prime number. The p -cohomological dimension of k is less than or equal to r (written $\text{cd}_p(k) \leq r$) if $H^n(k, A) = 0$ for every p -primary torsion Γ_k -module A and $n > r$. The cohomological dimension of k is less than or equal to r (written $\text{cd}(k) \leq r$) if $\text{cd}_p(k) \leq r$ for all primes p . Finally, the virtual cohomological dimension of k , written $\text{vcd}(k)$ is precisely the cohomological dimension of $k(\sqrt{-1})$. If k is a field of positive characteristic, then $\text{vcd}(k) = \text{cd}(k)$.

Let X be a scheme. For any closed point $x \in X$, let $\mathcal{O}_x$ be the local ring at x and let $\mathfrak{M}_x$ be its maximal ideal. The residue field of x , written $k(x)$, is $\mathcal{O}_x/\mathfrak{M}_x$. Zero cycles of X are elements of the free abelian group on closed points $x \in X$. We may associate to any zero cycle $\sum n_i x_i$ on X its degree $\sum n_i [k(x_i) : k]$, where $k(x_i)$ is the residue field of x_i .

A closed point with residue field k is called a *rational point*. It is clear that if x is a closed point of a variety X over k , then it is a rational point of $X_{k(x)}$. We have seen that the point in $H^1(*, G)$ is the principal homogeneous space under G over $*$ with a rational point. Therefore, a principal homogeneous space X under G over k , with zero cycle $\sum n_i x_i$, is an element of the kernel of the product of the restriction maps $H^1(k, G) \rightarrow \prod H^1(k(x_i), G)$. If the zero cycle is of degree one, then the field extensions $k(x_i)$ are necessarily of coprime degree over k .

Guided by this insight, one may restate **Q** as follows.

Q: Let k be a field and let G be a connected, linear algebraic group defined over k . Let $\{L_i\}_{1 \leq i \leq m}$ be a collection of finite extensions of k with $\gcd([L_i : k]) = 1$. Does the canonical map

$$H^1(k, G) \rightarrow \prod_{i=1}^m H^1(L_i, G)$$

have trivial kernel?

3. ORDERINGS OF A FIELD

We recall some basic properties of orderings of a field [9].

An *ordering* v of a field k is given by a binary relation $\leq_v$ such that for all $a, b, c \in k$:

- $a \leq_v a$.
- If $a \leq_v b$ and $b \leq_v c$, then $a \leq_v c$.
- If $a \leq_v b$ and $b \leq_v a$, then $a = b$.
- Either $a \leq_v b$ or $b \leq_v a$.
- If $a \leq_v b$, then $a + c \leq_v b + c$.
- If $a \leq_v b$ and $0 \leq_v c$, then $ca \leq_v cb$.

A field k which admits an ordering is necessarily of characteristic 0. If k is a field with an ordering v , an *algebraic extension* of the ordered field (k, v) is an algebraic field extension L of k together with an ordering v' on L such that v' restricted to k is v . If L is a finite field extension of k of odd degree, then there is always an algebraic extension (L, v') of (k, v) [9, Chapter 3, Theorem 1.10].

A field k is said to be *formally real* if -1 is not a sum of squares in k . A field k is called a *real closed field* if it is a formally real field and no proper algebraic extension is formally real. There is a unique ordering $\square$ on a real closed field. This

ordering is defined by the relation $a \leq b$ if and only if $b - a$ is a square in k . Further, if k is a real closed field, then $k(\sqrt{-1})$ is algebraically closed [9, Theorem 2.3 (iii)].

If L is a finite field extension of k , then $k_v \otimes L$ is isomorphic to a product of the form $\prod k_v \prod k_v(\sqrt{-1})$. Also, since $k_v(\sqrt{-1})$ is an algebraic closure for k there is a natural inclusion $\text{Gal}(\bar{k}, k_v) \subset \Gamma_k$ and thus a restriction map $H^1(k, G) \rightarrow H^1(k_v, G)$.

4. MAIN RESULT

In the discussion which follows we will need the following lemmas.

Lemma 4.1. *Let k be a field and let G be a reductive group over k . Fix an integer n and a quasitrivial torus T such that $G^n \times T$ admits a special covering*

$$1 \rightarrow \mu \rightarrow G_0 \times S \rightarrow G^n \times T \rightarrow 1.$$

Then G^{sc} satisfies a Hasse Principle over k if and only if G_0 satisfies a Hasse principle over k .

Proof. Taking commutator subgroups we have a short exact sequence

$$1 \rightarrow \tilde{\mu} \rightarrow [G_0 \times S : G_0 \times S] \rightarrow [G^n \times T : G^n \times T] \rightarrow 1.$$

Since S and T are tori, $[G_0 \times S : G_0 \times S] \cong [G_0 : G_0]$ and $[G^n \times T : G^n \times T] = [G^n : G^n]$. That G_0 is semisimple gives $[G_0 : G_0] = G_0$. It is clear that $[G^n : G^n] = [G : G]^n$, which in turn is $(G^{\text{ss}})^n$ by the definition of G^{ss} . Therefore, we have the following short exact sequence:

$$1 \rightarrow \tilde{\mu} \rightarrow G_0 \rightarrow (G^{\text{ss}})^n \rightarrow 1,$$

where $\tilde{\mu}$ is some finite group scheme. In particular, G_0 is a simply connected cover of $(G^{\text{ss}})^n$. Since $(G^{\text{sc}})^n$ is certainly a simply connected cover of $(G^{\text{ss}})^n$, uniqueness of the simply connected cover of $(G^{\text{ss}})^n$ gives $(G^{\text{sc}})^n \cong G_0$. In particular, the simple factors of G^{sc} are the same as the simple factors of G_0 , and G^{sc} satisfies the Hasse principle over k if and only if G_0 satisfies the Hasse principle over k . $\square$

Lemma 4.2. *Let k be a real closed field and let G be a reductive group over k which admits a special covering*

$$(4.2.1) \quad 1 \rightarrow \mu \rightarrow G_0 \times S \rightarrow G \rightarrow 1.$$

Let L be a finite étale k -algebra. Let δ be the first connecting map in Galois Cohomology and let $N_{L/k}$ denote the corestriction map $H^1(k \otimes L, \mu) \rightarrow H^1(k, \mu)$. Then

$$N_{L/k}(\text{im}(G(k \otimes L) \xrightarrow{\delta_L} H^1(k \otimes L, \mu))) \subset \text{im}(G(k) \xrightarrow{\delta} H^1(k, \mu)).$$

Proof. Since k is real closed, there exist finite numbers r and s such that $k \otimes L$ is isomorphic to a product of r copies of k and s copies of $k(\sqrt{-1})$. Thus

$$H^1(k \otimes L, \mu) \cong \prod_{r \text{ copies}} H^1(k, \mu) \prod_{s \text{ copies}} H^1(k(\sqrt{-1}), \mu).$$

Since k is real closed, $k(\sqrt{-1})$ is algebraically closed, $H^1(k(\sqrt{-1}), \mu)$ is trivial and $H^1(k \otimes L, \mu)$ is just a product of r copies of $H^1(k, \mu)$. Therefore,

$$N_{L/k} : H^1(k \otimes L, \mu) \rightarrow H^1(k, \mu)$$

is just the product map

$$\prod_{r \text{ copies}} H^1(k, \mu) \rightarrow H^1(k, \mu).$$

That $k \otimes L$ is a product of r copies of k and s copies of $k(\sqrt{-1})$ also gives that

$$G(k \otimes L) \cong \prod_{r \text{ copies}} G(k) \prod_{s \text{ copies}} G(k(\sqrt{-1})).$$

Therefore, the connecting map

$$\prod_{r \text{ copies}} G(k) \prod_{s \text{ copies}} G(k(\sqrt{-1})) \xrightarrow{\delta} \prod_{r \text{ copies}} H^1(k, \mu) \prod_{s \text{ copies}} H^1(k(\sqrt{-1}), \mu)$$

is just the product of the connecting maps

$$G(k) \rightarrow H^1(k, \mu)$$

and

$$G(k(\sqrt{-1})) \rightarrow H^1(k(\sqrt{-1}), \mu),$$

the latter of which is necessarily the trivial map.

So choose

$$(x_1, \dots, x_r, y_1, \dots, y_s) \in G(k \otimes L).$$

Then

$$\begin{aligned} N_{L/k}(\delta(x_1, \dots, x_r, y_1, \dots, y_s)) &= N_{L/k}(\delta(x_1), \dots, \delta(x_r), \delta(y_1), \dots, \delta(y_s)) \\ &= \delta(x_1) \cdots \delta(x_r) \\ &= \delta(x_1 \cdots x_r). \end{aligned}$$

Since the x_i were chosen to be in $G(k)$ for all i , then $x_1 \cdots x_r \in G(k)$ and the desired result holds. $\square$

Lemma 4.3. *Let G be a reductive group and L be a finite field extension of k of odd degree. The kernel of the canonical map $H^1(k, G) \rightarrow H^1(L, G)$ is contained in the kernel of the canonical map $H^1(k, G) \rightarrow \prod_{v \in \Omega} H^1(k_v, G)$.*

Proof. By [9, Chapter 3, Theorem 1.10] each ordering v of k extends to an ordering w of L ; in particular, each real closure k_v is L_w for some ordering w on L . Since the natural map $H^1(k, G) \rightarrow H^1(L_w, G)$ factors through the canonical map $H^1(k, G) \rightarrow H^1(L, G)$, the desired result is immediate. $\square$

We now return to the result which is the main goal of this paper.

Theorem 4.4. *Let k be a perfect field of virtual cohomological dimension ≤ 2 and let G be a connected linear algebraic group over k . Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. If G^{sc} satisfies a Hasse principle over k , then the canonical map*

$$H^1(k, G) \rightarrow \prod_{i=1}^m H^1(L_i, G)$$

has trivial kernel.

Proof. By definition of the groups involved, the following sequence is exact:

$$(4.4.1) \quad 1 \rightarrow G^u \rightarrow G \rightarrow G^{\text{red}} \rightarrow 1.$$

Since G^u is unipotent, $H^i(k, G^u)$ is trivial for $i \geq 1$ and (4.4.1) induces the long exact sequence in Galois Cohomology

$$1 \rightarrow H^1(k, G) \rightarrow H^1(k, G^{\text{red}}) \rightarrow 1,$$

which gives that $H^1(k, G) \cong H^1(k, G^{\text{red}})$. Thus to prove Theorem 4.4 it is sufficient to consider the case where G is a reductive group. Then fix an integer n and quasitrivial torus T such that $G^n \times T$ admits a special covering

$$1 \rightarrow \mu \rightarrow G_0 \times S \rightarrow G^n \times T \rightarrow 1.$$

By functoriality, $H^1(k, G^n \times T) \cong H^1(k, G)^n \times H^1(k, T)$ and since T is quasitrivial, $H^1(k, T) = 1$. It follows that our result holds for G if and only if it holds for $G^n \times T$. Replacing G by $G = G^n \times T$ we assume that G admits a special covering

$$1 \rightarrow \mu \rightarrow G_0 \times S \rightarrow G \rightarrow 1.$$

If k is a field of positive characteristic, $\text{cd}(d) = \text{vcd}(k) = 2$. Since k has no orderings and by hypothesis G^{sc} satisfies a Hasse principle over k , then $H^1(k, G^{\text{sc}}) = \{1\}$. In particular $H^1(k, G_0) = \{1\}$ and the special covering of G above induces the following commutative diagram with exact rows:

$$(4.4.2) \quad \begin{array}{ccccc} 1 & \longrightarrow & H^1(k, G) & \xrightarrow{h} & H^2(k, \mu) \\ & & \downarrow q & & \downarrow r \\ 1 & \longrightarrow & \prod_i H^1(L_i, G) & \longrightarrow & \prod_i H^2(L_i, \mu) \end{array}$$

Choose $\lambda \in \ker(q)$. By commutativity of the diagram, $h(\lambda) \in \ker(r)$. A restriction-corestriction argument gives that r has trivial kernel. Thus $h(\lambda) = \text{point}$. Then by exactness of the top row of the diagram, $\lambda = \text{point}$. (Cf. [2] for the case that k is a “good” field of cohomological dimension 2.)

Therefore, we may assume that the characteristic of k is zero. Fix an index i . The special covering of G above induces the following commutative diagram with exact rows where the vertical maps are the restriction maps:

$$(4.4.3) \quad \begin{array}{ccccccc} H^1(k, \mu) & \xrightarrow{f} & H^1(k, G_0) & \xrightarrow{g} & H^1(k, G) & \xrightarrow{h} & H^2(k, \mu) \\ \downarrow & & \downarrow p & & \downarrow q & & \downarrow r \\ \prod H^1(L_i, \mu) & \xrightarrow{f} & \prod H^1(L_i, G_0) & \xrightarrow{g} & \prod H^1(L_i, G) & \longrightarrow & \prod H^2(L_i, \mu) \end{array}$$

Let λ be in $\ker(q)$. Taking cores $\circ$ res we find that r has trivial kernel and thus by commutativity of (4.4.3), λ is in $\ker(h)$. By exactness of the top row, we choose $\lambda' \in H^1(k, G_0)$ such that $g(\lambda') = \lambda$. Write $p(\lambda') = (\lambda'_{L_i})$. Since $g(\lambda'_{L_i}) = \text{point}$, by exactness of the bottom row of (4.4.3) choose $\eta_{L_i} \in H^1(L_i, \mu)$ such that $f(\eta_{L_i}) = \lambda'_{L_i}$.

For each ordering v of k , the special covering of G above also induces the following commutative diagram with exact rows:

(4.4.4)

$$\begin{array}{ccccccc} H^1(k, \mu) & \xrightarrow{f} & H^1(k, G_0) & \xrightarrow{g} & H^1(k, G) & \xrightarrow{h} & H^2(k, \mu) \\ \downarrow & & \downarrow p' & & \downarrow q' & & \downarrow r' \\ \prod_{v \in \Omega} H^1(k_v, \mu) & \xrightarrow{f} & \prod_{v \in \Omega} H^1(k_v, G_0) & \xrightarrow{g} & \prod_{v \in \Omega} H^1(k_v, G) & \longrightarrow & \prod_{v \in \Omega} H^2(k_v, \mu) \end{array}$$

By Lemma 4.3, λ is in the kernel of q' . Thus by commutativity of (4.4.4), $(\lambda'_v) = p'(\lambda')$ is in $\ker(g)$. Then by exactness of the bottom row of (4.4.4) choose $\alpha_v \in H^1(k_v, \mu)$ such that $f(\alpha_v) = \lambda'_v$. Let $(\alpha_v)_{L_i}$ denote the image of α_v under the canonical map $H^1(k_v, \mu) \rightarrow H^1(k_v \otimes L_i, \mu)$. Let $(\eta_{L_i})_v$ denote the image of η_{L_i} under the canonical map $H^1(L_i, \mu) \rightarrow H^1(k_v \otimes L_i, \mu)$.

By choice of α_v and η_{L_i} , $f((\alpha_v)_{L_i}) = (\lambda'_v)_{L_i} = (\lambda'_{L_i})_v = f((\eta_{L_i})_v)$. In particular, $f((\alpha_v)_{L_i}(\eta_{L_i})_v^{-1})$ is the point in $H^1(k_v \otimes L_i, G_0)$. We have a commutative diagram

$$(4.4.5) \quad \begin{array}{ccccc} G(k_v) & \xrightarrow{\delta} & H^1(k_v, \mu) & \xrightarrow{f} & H^1(k_v, G_0) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_i G(k_v \otimes L_i) & \xrightarrow{\delta_{L_i}} & \prod_i H^1(k_v \otimes L_i, \mu) & \xrightarrow{f} & \prod_i H^1(k_v \otimes L_i, G_0) \end{array}$$

Exactness of the bottom row of (4.4.5) gives that $(\alpha_v)_{L_i}(\eta_{L_i})_v^{-1}$ is in the image of δ_{L_i} . Choose m_i such that $\sum m_i [L_i : k] = 1$. Since δ_{L_i} is multiplicative, it follows that for each index i , $(\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i}$ is in the image of δ_{L_i} .

By Lemma 4.2 above, there exists γ_v in $G(k_v)$ such that

$$\delta(\gamma_v) = \prod_i N_{L_i/k}((\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i}).$$

Since by restriction-corestriction $N_{L_i/k}((\alpha_v)_{L_i}^{m_i}) = \alpha_v^{m_i [L_i:k]}$, it follows that

$$\begin{aligned} \delta(\gamma_v) &= \prod_i N_{L_i/k}((\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i}) \\ &= \alpha_v^{\sum_i m_i [L_i:k]} \prod_i (N_{L_i/k}(\eta_{L_i})_v^{-1})^{m_i} \\ &= \alpha_v \prod_i (N_{L_i/k}(\eta_{L_i})_v^{-1})^{m_i}. \end{aligned}$$

In turn

$$\delta(\gamma_v) \prod_i (N_{L_i/k}(\eta_{L_i})_v)^{m_i} = \alpha_v.$$

Since f is well-defined on the cosets of $G(k_v)$ in $H^1(k_v, \mu)$ [7] and the top row of (4.4.5) is exact, it follows that

$$f\left(\prod_i (N_{L_i/k}(\eta_{L_i})_v)^{m_i}\right) = f(\alpha_v).$$

By choice of α_v the latter is λ'_v . Since G^{sc} satisfies a Hasse principle over k , Lemma 4.1 gives that G_0 satisfies a Hasse principle over k . In particular, the map

$H^1(k, G_0) \rightarrow \prod_v H^1(k_v, G_0)$ is injective, and since $f(\prod_i (N_{L_i/k}(\eta_{L_i}))^{m_i})_v = \lambda'_v$ for all v , then

$$f\left(\prod_i (N_{L_i/k}(\eta_{L_i}))^{m_i}\right) = \lambda'.$$

Taking g as in (4.4.3) above,

$$g\left(f\left(\prod_i (N_{L_i/k}(\eta_{L_i}))^{m_i}\right)\right) = g(\lambda').$$

Then by exactness of the top row of (4.4.3), $\lambda = g(\lambda') = \text{point}$. $\square$

Applying [1, Theorem 10.1] and a Serre twist we obtain the following corollary:

Corollary 4.5. *Let k be a perfect field of virtual cohomological dimension ≤ 2 . Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. Let G be a connected linear algebraic group over k . If the simple factors of G^{sc} are of classical type, type F_4 or type G_2 , then the canonical map*

$$H^1(k, G) \rightarrow \prod_{i=1}^m H^1(L_i, G)$$

is injective.

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