$p(x)$ -LAPLACIAN WITH INDEFINITE WEIGHT

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ABSTRACT. We consider the eigenvalue problem $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda V(x)|u|^{q(x)-2}u$, in Ω , $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda > 0$, p, q are continuous functions on $\overline{\Omega}$ and V is a given function in a generalized Lebesgue space $L^{s(x)}(\Omega)$ such that $V > 0$ in an open set $\Omega_0 \subset \Omega$, where $|\Omega_0| > 0$. We prove under appropriate conditions on the functions p, q and s that any $\lambda > 0$ sufficiently small is an eigenvalue of the above nonhomogeneous quasilinear problem. The proof relies on simple variational arguments based on Ekeland's variational principle.

1. INTRODUCTION

The study of differential equations and variational problems with nonstandard $p(x)$ -growth conditions has received more and more interest in recent years. The specific attention accorded to such kinds of problems is due to their applications in mathematical physics. More precisely, such equations are used to model phenomenon which arise in elastic mechanics or electrorheological fluids (sometimes referred to as “smart fluids”). In this paper we discuss the existence of the weak solutions for the following problem:

$$(1) \quad \begin{cases} -\Delta_{p(x)} u &= \lambda V(x)|u|^{q(x)-2}u & \text{for } x \in \Omega, \\ u &\neq 0 & \text{for } x \in \Omega, \\ u &= 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a real number, and $p, q, s : \overline{\Omega} \rightarrow \mathbb{R}$ are continuous.

M. Mihăilescu and V. Rădulescu, in [18], studied the same nonhomogeneous eigenvalue problem in the particular case $V(x) = 1$. The authors proved, under the assumption $1 < \min_{\overline{\Omega}} q(x) < \min_{\overline{\Omega}} p(x) < \max_{\overline{\Omega}} q(x)$, that there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for the above problem. The main argument in their paper is related to Ekeland's variational principle (see [14]). Note that their problem has been studied by Ambrosetti and Rabinowitz [1] in the particular case where $p(x) = 2$ and $q(x) = q$ is a constant.

The same problem, for $V(x) = 1$, $p(x) = q(x)$ is studied by Fan, Zhang and Zhao in [13]. The authors established, using the Ljusternik-Schnirelmann critical point theory, the existence of a sequence of eigenvalues. Denoting by Λ the set of all nonnegative eigenvalues, Fan, Zhang and Zhao showed that $\sup \Lambda = +\infty$, and

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they pointed out that only under additional assumptions do we have $\inf \Lambda > 0$. We remark that for the p -Laplace operator (corresponding to $p(x) \equiv p$) we always have $\inf \Lambda > 0$.

In this paper, we study the problem (1) under the assumption:

(H1) $1 < q(x) < p(x) \leq N < s(x)$, $\forall x \in \overline{\Omega}$, $V \in L^{s(x)}(\Omega)$ and $V > 0$ in $\Omega_0 \subset \Omega$, with $|\Omega_0| > 0$.

Our main result established the existence of a continuous family of eigenvalues for problem (1) in a neighborhood of the origin. More precisely, we show under appropriate conditions on p, q , and s that there exists a positive critical real number λ^* under which the problem (1) has a nontrivial positive weak solution.

In the sequel, we denote by $\Delta_{p(x)}$ the $p(x)$ -Laplace operator, i.e.

$$\Delta_{p(x)} u := \operatorname{div} (|\nabla u|^{p(x)-2} \nabla u),$$

where p is a continuous nonconstant function. This differential operator is a natural generalization of the p -Laplace operator $\Delta_p u = \operatorname{div} (|\nabla u|^{p-2} \nabla u)$, where $p > 1$ is a real constant. However, the $p(x)$ -Laplace operator possesses more complicated nonlinearities than the p -Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. Recent qualitative properties of solutions to quasilinear problems in Sobolev spaces with variable exponent have been obtained by Alves and Souto [3], Chabrowski and Fu [6], and Mihăilescu and Rădulescu [17]. Equation (1) will be studied in the framework of the variable Lebesgue and Sobolev spaces $L^{p(x)}$ and $W_0^{1,p(x)}$, which will be briefly described in the next section.

2. PRELIMINARY RESULTS

We recall in what follows some definitions and basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and generalized Sobolev spaces $W^{1,p(x)}(\Omega)$, where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the book of Musielak [16] and the papers of Kováčik and Rákosník [15] and Fan et al. [9, 12]. Set

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$$L^{p(x)}(\Omega) = \{u : u \text{ is a Borel real-valued function on } \Omega \text{ and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty\}.$$

We define on $L^{p(x)}$ the so-called *Luxemburg norm* by the formula

$$|u|_{p(x)} := \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are separable and Banach spaces [15, Theorem 2.5; Corollary 2.7] and the Hölder inequality holds [15, Theorem 2.1]. The inclusions between Lebesgue spaces are also naturally generalized [15, Theorem 2.8]: if $0 < |\Omega| < \infty$ and r_1, r_2 are variable exponents so that $r_1(x) \leq r_2(x)$ almost everywhere in Ω , then there exists the continuous embedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$

the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Hölder type inequality

$$(2) \quad \left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(x)} |v|_{p'(x)}$$

holds.

Moreover, if h_1, h_2 and $h_3 : \overline{\Omega} \rightarrow (1, \infty)$ are three Lipschitz continuous functions such that $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then for any $u \in L^{h_1(x)}(\Omega), v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$, the following inequality holds (see [8], Proposition 2.5):

$$(3) \quad \left| \int_{\Omega} uvw \, dx \right| \leq \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-} \right) |u|_{h_1(x)} |v|_{h_2(x)} |w|_{h_3(x)}.$$

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} \, dx.$$

The space $W^{1,p(x)}(\Omega)$ is equipped by the following norm:

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

If (u_n) , $u \in W^{1,p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations hold:

$$(4) \quad \min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \leq \rho_{p(x)}(u) \leq \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}),$$

$$(5) \quad \min(|\nabla u|_{p(x)}^{p^-}, |\nabla u|_{p(x)}^{p^+}) \leq \rho_{p(x)}(|\nabla u|) \leq \max(|\nabla u|_{p(x)}^{p^-}, |\nabla u|_{p(x)}^{p^+}),$$

$$(6) \quad \begin{aligned} |u|_{p(x)} \rightarrow 0 &\Leftrightarrow \rho_{p(x)}(u) \rightarrow 0, \\ \lim_{n \rightarrow \infty} |u_n - u|_{p(x)} = 0 &\Leftrightarrow \lim_{n \rightarrow \infty} \rho_{p(x)}(u_n - u) = 0, \\ |u|_{p(x)} \rightarrow \infty &\Leftrightarrow \rho_{p(x)}(u) \rightarrow \infty. \end{aligned}$$

We also define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

Using the result of Fan and Zhao [10], we remark that the norm $\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}$ is equivalent to $\|u\| = |\nabla u|_{p(x)}$ in $W_0^{1,p(x)}(\Omega)$.

The space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. Next, we recall some embedding results regarding variable exponent Lebesgue-Sobolev spaces. We note that if $r(x) \in C_+(\overline{\Omega})$ and $r(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ is compact and continuous, where $p^*(x) = Np(x)/(N - p(x))$ if $p(x) < N$ or $p^*(x) = +\infty$ if $p(x) \geq N$.

We recall also the following proposition, which will be needed later:

Proposition 1 ([7]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega), u \neq 0$. Then*

$$\begin{aligned} |u|_{p(x)q(x)} \leq 1 &\Rightarrow |u|_{p(x)q(x)}^{p^+} \leq \|u\|^{p(x)}|u|_{q(x)} \leq |u|_{p(x)q(x)}^{p^-}, \\ |u|_{p(x)q(x)} \geq 1 &\Rightarrow |u|_{p(x)q(x)}^{p^-} \leq \|u\|^{p(x)}|u|_{q(x)} \leq |u|_{p(x)q(x)}^{p^+}. \end{aligned}$$

In particular if $p(x) = p$ is a constant, then

$$\| |u|^p \|_{q(x)} = |u|_{pq(x)}^p.$$

We refer to [15] for more properties of Lebesgue and Sobolev spaces with variable exponent. We also refer to the recent papers [2, 4, 5, 9, 11, 12, 17, 18, 19] for the treatment of nonlinear boundary value problems in Lebesgue-Sobolev spaces with variable exponent.

3. THE MAIN RESULTS AND AN AUXILIARY RESULT

In the sequel, we will denote by $s'(x)$ the conjugate exponent of the function $s(x)$ and put $\alpha(x) := \frac{s(x)q(x)}{s(x)-q(x)}$. Then we have:

Remark 1. Under assumption (H1) on the functions p, q and s , we have $s'(x)q(x) < p^*(x), \forall x \in \overline{\Omega}$, $\alpha(x) < p^*(x), \forall x \in \overline{\Omega}$. Consequently the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ are compact and continuous.

Note that an eigenvalue for problem (1) satisfies the following definition:

Definition 1. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1) if there exists $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} uv dx,$$

for any $v \in W_0^{1,p(x)}(\Omega)$ and we recall that if λ is an eigenvalue of the problem (1), then the corresponding $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ is a weak solution of (1).

Our main result is given by the following theorem:

Theorem 1. Suppose that assumption (H1) holds. Then there exists $\lambda^* > 0$ such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue of the problem (1).

In the sequel, let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. In order to formulate the variational problem (1), let us introduce the functionals Φ and $J : E \rightarrow \mathbb{R}$ defined by

$$\Phi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \text{ and } J(u) = \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} dx.$$

Using Proposition 1, we mention that J is well defined as we have for all $u \in W_0^{1,p(x)}(\Omega)$,

$$|J(u)| \leq \frac{1}{q^-} |V|_{s(x)} \| |u|^{q(x)} \|_{s'(x)} \leq \frac{1}{q^-} |V|_{s(x)} |u|_{s'(x)q(x)}^c,$$

where c is a positive constant.

Let us introduce the energy functional corresponding to problem (1), which is defined as $\Psi_{\lambda} : E \rightarrow \mathbb{R}$, where

$$\Psi_{\lambda}(u) := \Phi(u) - \lambda J(u).$$

We first prove an auxiliary result:

Proposition 2. Under the assumption (H1), $\Psi_{\lambda} \in C^1(E, \mathbb{R})$ and $u \in E$ is a critical point of Ψ_{λ} if and only if u is a weak solution for the problem (1).

Proof. To show that $\Psi_\lambda \in C^1(E, \mathbb{R})$, we show that for all $\varphi \in E$,

$$\lim_{t \rightarrow 0^+} \frac{\Psi_\lambda(u + t\varphi) - \Psi_\lambda(u)}{t} = \langle d\Psi_\lambda(u), \varphi \rangle,$$

and $d\Psi_\lambda : E \rightarrow E^*$ is continuous, where we denote by E^* the dual space of E .

For all $\varphi \in E$ we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{J(u + t\varphi) - J(u)}{t} &= \frac{d}{dt} J(u + t\varphi)|_{t=0} = \frac{d}{dt} \int_{\Omega} \frac{V(x)}{q(x)} |u + t\varphi|^{q(x)} dx|_{t=0} \\ &= \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{V(x)}{q(x)} |u + t\varphi|^{q(x)} \right) |_{t=0} dx = \int_{\Omega} V(x) |u + t\varphi|^{q(x)-1} \operatorname{sgn}(u + t\varphi) \varphi|_{t=0} dx \\ &= \int_{\Omega} V(x) |u + t\varphi|^{q(x)-2} (u + t\varphi) \varphi|_{t=0} dx = \int_{\Omega} V(x) |u|^{q(x)-2} u \varphi dx \\ &= \langle dJ(u), \varphi \rangle. \end{aligned}$$

The differentiation under the integral is allowed for t close to zero. Indeed, for $|t| < 1$, we have

$$|V(x)|u + t\varphi|^{q(x)-2} (u + t\varphi) \varphi| \leq |V(x)|(|u| + |\varphi|)^{q(x)-1} |\varphi| \in L^1(\Omega).$$

Because $u, \varphi \in E$, we have

$$|u|, |\varphi| \in E \hookrightarrow L^{q(x)}(\Omega) \text{ and } |\varphi| \in E \hookrightarrow L^{\alpha(x)}(\Omega).$$

Due to the fact that $v \in L^{s(x)}(\Omega)$, the conclusion is an immediate consequence of inequality (3).

For $u \in E$ chosen, we show that $dJ(u) \in W^{-1, p'(x)}(\Omega) = E^*$, where $1/p(x) + 1/p'(x) = 1$. It is easy to see that $dJ(u)$ is linear.

Since there is a continuous embedding $E \hookrightarrow L^{\alpha(x)}(\Omega)$, there exists a constant $M > 0$ such that

$$(7) \quad |v|_{\alpha(x)} \leq M \|v\|, \quad \text{for all } v \in E.$$

Using (2) and (7) we obtain

$$\begin{aligned} |\langle dJ(u), \varphi \rangle| &= \left| \int_{\Omega} V(x) |u|^{q(x)-2} u \varphi dx \right| \\ &\leq \int_{\Omega} |V(x)| |u|^{q(x)-1} |\varphi| dx \\ &\leq |V|_{s(x)} \|u\|_{\frac{q(x)}{q(x)-1}}^{q(x)-1} \|\varphi\|_{\alpha(x)} \\ &\leq M |V|_{s(x)} \|u\|_{\frac{q(x)}{q(x)-1}}^{q(x)-1} \|\varphi\|. \end{aligned}$$

Hence there exists $M_1 = M |V(x)| \|u\|_{\frac{q(x)}{q(x)-1}}^{q(x)-1} > 0$ such that

$$|\langle dJ(u), \varphi \rangle| \leq M_1 \|\varphi\|.$$

Using the linearity of $dJ(u)$ and the above inequality we deduce that $dJ(u) \in E^* = W^{-1, p'(x)}(\Omega)$. For the Fréchet differentiability, we need the following lemma.

Lemma 1 ([4]). *The map $u \in L^{q(x)}(\Omega) \rightarrow |u|^{q(x)-2} u \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$ is continuous.*

We conclude that J is Fréchet differentiable.

We can also prove (see [5]) that $\Phi \in C^1(E, \mathbb{R})$.

We remark that $\Psi_\lambda \in C^1(E, \mathbb{R})$ because $\Phi, J \in C^1(E, \mathbb{R})$. Moreover,

$$\langle d\Psi_\lambda(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx - \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx, \quad \forall v \in E.$$

Let u be a critical point of Ψ_λ . Then we have $d\Psi_\lambda(u) = 0_{E^*}$, that is,

$$\langle d\Psi_\lambda(u), v \rangle = 0, \quad \text{for all } v \in E,$$

which yields

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx, \quad \forall v \in E.$$

It follows that u is a weak solution for the problem (1). By Definition 1, we have

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx, \quad \forall v \in E;$$

that is, $\langle d\Psi_\lambda(u), v \rangle = 0, \forall v \in E$. We obtain $d\Psi_\lambda(u) = 0_{E^*}$. Hence u is a critical point of Ψ_λ . This completes the proof of Proposition 2. $\square$

The argument of the proof of Theorem 1 is based on Ekeland's variational principle.

Lemma 2. *Suppose we are under the hypotheses of Theorem 1. Then for all $\rho \in (0, 1)$, there exist $\lambda^* > 0$ and $a > 0$ such that for all $u \in E$ with $\|u\| = \rho$,*

$$\Psi_\lambda(u) \geq a > 0, \quad \text{for all } \lambda \in (0, \lambda^*).$$

Proof. Since the embedding $E \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, there exists a positive constant C such that

$$|u|_{s'(x)q(x)} \leq C\|u\|, \quad \forall x \in E.$$

Now, let us assume that $\|u\| < \min(1, 1/C)$, where C is the positive constant from above. Then, we have $|u|_{s'(x)q(x)} < 1$. Using relations (4), (5), and (7) we deduce that for any $u \in E$ with $\|u\| = \rho$ the following inequalities hold true:

$$\begin{aligned} \Psi_\lambda(u) &\geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx - \frac{\lambda}{q^-} \int_{\Omega} V(x) |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} |V|_{s(x)} \|u\|^{q(x)}|_{s'(x)} \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} |V|_{s(x)} |u|_{s'(x)q(x)}^{q^-} \\ &\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} |V|_{s(x)} C^{q^-} \|u\|^{q^-} \\ &= \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{q^-} C^{q^-} |V|_{s(x)} \rho^{q^-} = \rho^{q^-} \left(\frac{1}{p^+} \rho^{p^+ - q^-} - \frac{\lambda}{q^-} C^{q^-} |V|_{s(x)} \right). \end{aligned}$$

By the above inequality we remark that if we define

$$(8) \quad \lambda^* = \frac{\rho^{p^+ - q^-}}{2p^+} \frac{q^-}{C^{q^-} |V|_{s(x)}},$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in E$ with $\|u\| = \rho$ there exists $a > 0$ such that

$$\Psi_\lambda(u) \geq a > 0.$$

The proof of Lemma 2 is complete. $\square$

Lemma 3. *There exists $\varphi \in E$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $\Psi_\lambda(t\varphi) < 0$, for $t > 0$ small enough.*

Proof. Assumption (H1) implies $q(x) < p(x), \forall x \in \overline{\Omega}_0$. In the sequel, we use the notation $q_0^- = \inf_{\Omega_0} q(x)$ and $p_0^- = \inf_{\Omega_0} p(x)$. Let ϵ_0 be such that $q_0^- + \epsilon_0 < p_0^-$. On the other hand, since $q \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q(x) - q_0^-| < \epsilon_0, \forall x \in \Omega_1$. It follows that $q(x) \leq q_0^- + \epsilon_0 < p_0^-, \forall x \in \Omega_1$.

Let $\varphi \in C_0^\infty(\Omega)$ such that $\text{supp}(\varphi) \subset \Omega_1 \subset \Omega_0$, $\varphi = 1$ in a subset $\Omega'_1 \subset \text{supp}(\varphi)$, $0 \leq \varphi \leq 1$ in Ω_1 . Then we have

$$\begin{aligned} \Psi_\lambda(t\varphi) &= \int_{\Omega} \frac{1}{p(x)} |\nabla(t\varphi)|^{p(x)} dx - \lambda \int_{\Omega} \frac{V(x)}{q(x)} |t\varphi|^{q(x)} dx \\ &= \int_{\Omega_0} \frac{t^{p(x)}}{p(x)} |\nabla\varphi|^{p(x)} dx - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} V(x) |\varphi|^{q(x)} dx \\ &\leq \frac{t^{p_0^-}}{p_0^-} \int_{\Omega_0} |\nabla\varphi|^{p(x)} dx - \frac{\lambda t^{q_0^- + \epsilon_0}}{q_0^+} \int_{\Omega_1} V(x) |\varphi|^{q(x)} dx. \end{aligned}$$

Therefore

$$\Psi_\lambda(t\varphi) < 0,$$

for $t < \delta^{1/(p_0^- - q_0^- - \epsilon_0)}$ with

$$0 < \delta < \min \left\{ 1, \frac{\frac{\lambda p_0^-}{q_0^+} \int_{\Omega_1} V(x) |\varphi|^{q(x)} dx}{\int_{\Omega_0} |\nabla\varphi|^{p(x)} dx} \right\}.$$

Finally, we point out that

$$\int_{\Omega_0} |\nabla\varphi|^{p(x)} dx > 0.$$

In fact, if $\int_{\Omega_0} |\nabla\varphi|^{p(x)} dx = 0$, then $\int_{\Omega} |\nabla\varphi|^{p(x)} dx = 0$. Using relation (6), we deduce that $|\nabla\varphi|_{p(x)} = 0$ and consequently $\varphi = 0$ in Ω , which is a contradiction.

The proof of Lemma 3 is complete. $\square$

Proof of Theorem 1. Let $\lambda^* > 0$ be defined as in (8) and $\lambda \in (0, \lambda^*)$. By Lemma 2 it follows that on the boundary of the ball centered at the origin and of radius ρ in E , denoted by $B_\rho(0)$, we have

$$(9) \quad \inf_{\partial B_\rho(0)} \Psi_\lambda > 0.$$

On the other hand, by Lemma 3, there exists $\varphi \in E$ such that $\Psi_\lambda(t\varphi) < 0$ for all $t > 0$ small enough. Moreover, relations (4), (5), and (7) imply that for any $u \in B_\rho(0)$ we have

$$\Psi_\lambda(u) \geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\lambda}{q^-} C^{q^-} |V|_{s(x)} \|u\|^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} \Psi_\lambda < 0.$$

Now let ϵ be such that $0 < \epsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda - \inf_{B_\rho(0)} \Psi_\lambda$. Using the above information, the functional $\Psi_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ is lower bounded on $\overline{B_\rho(0)}$ and

$\Psi_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq \Psi_\lambda(u_\epsilon) \leq \underline{c} + \epsilon, \\ 0 < \Psi_\lambda(u) - \Psi_\lambda(u_\epsilon) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon. \end{cases}$$

Since

$$\Psi_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} \Psi_\lambda + \epsilon \leq \inf_{B_\rho(0)} \Psi_\lambda + \epsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ by $I_\lambda(u) = \Psi_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{\Psi_\lambda(u_\epsilon + t \cdot v) - \Psi_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\| \geq 0.$$

Letting $t \rightarrow 0$ it follows that $\langle d\Psi_\lambda(u_\epsilon), v \rangle + \epsilon \cdot \|v\| \geq 0$ and we infer that $\|d\Psi_\lambda(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$(10) \quad \Psi_\lambda(w_n) \rightarrow \underline{c} \quad \text{and} \quad d\Psi_\lambda(w_n) \rightarrow 0_{E^*}.$$

It is clear that $\{w_n\}$ is bounded in E . Thus, there exists w in E such that, up to a subsequence, $\{w_n\}$ converges weakly to w in E . Since $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, we deduce that there exists a compact embedding $E \hookrightarrow L^{\alpha(x)}(\Omega)$ and consequently $\{w_n\}$ converges strongly in $L^{\alpha(x)}(\Omega)$. For the strong convergence of $\{w_n\}$ in E , we need the following proposition:

Proposition 3.

$$\lim_{n \rightarrow \infty} \int_{\Omega} V(x) |w_n|^{q(x)-2} w_n (w_n - w) dx = 0.$$

Proof. Using (2) we have

$$\begin{aligned} \int_{\Omega} V(x) |w_n|^{q(x)-2} w_n (w_n - w) dx &\leq |V|_{s(x)} \| |w_n|^{q(x)-2} w_n (w_n - w) \|_{s'(x)} \\ &\leq |V|_{s(x)} \| |w_n|^{q(x)-2} w_n \|_{\frac{q(x)}{q(x)-1}} \|w_n - w\|_{\alpha(x)}. \end{aligned}$$

Now if $\| |w_n|^{q(x)-2} w_n \|_{\frac{q(x)}{q(x)-1}} > 1$, by Proposition 1, we get $\| |w_n|^{q(x)-2} w_n \|_{\frac{q(x)}{q(x)-1}} \leq \|w_n\|_{q(x)}^{q^+}$.

The compact embedding $E \hookrightarrow L^{q(x)}(\Omega)$ ends the proof. $\square$

Since $d\Psi_\lambda(w_n) \rightarrow 0$ and w_n is bounded in E we have

$$\begin{aligned} |\langle d\Psi_\lambda(w_n), w_n - w \rangle| &\leq |\langle d\Psi_\lambda(w_n), w_n \rangle| + |\langle d\Psi_\lambda(w_n), w \rangle| \\ &\leq \|d\Psi_\lambda(w_n)\| \|w_n\| + \|d\Psi_\lambda(w_n)\| \|w\|. \end{aligned}$$

So

$$\lim_{n \rightarrow \infty} \langle d\Psi_\lambda(w_n), w_n - w \rangle = 0.$$

Using Proposition 3 and the last relation we deduce that

$$(11) \quad \lim_{n \rightarrow \infty} \int_{\Omega} |\nabla w_n|^{p(x)-2} \nabla w_n \nabla (w_n - w) dx = 0.$$

Now, if we denote $L : E \rightarrow E^*$ such that

$$\langle L(w), v \rangle = \int_{\Omega} |\nabla w|^{p(x)-2} \nabla w \nabla v dx \quad \forall u, v \in E,$$

then

$$\begin{aligned} |\langle L(w_n) - L(w), w_n - w \rangle| &\leq |\langle L(w_n), w_n - w \rangle| + |\langle L(w), w_n - w \rangle| \\ &\leq |\langle L(w_n), w_n - w \rangle| + |\langle L(w), w_n \rangle - \langle L(w), w \rangle|. \end{aligned}$$

From (11) and the fact that $\{w_n\}$ converges weakly to w in E it follows that

$$\lim_{n \rightarrow \infty} \langle L(w_n) - L(w), w_n - w \rangle = 0,$$

and by Theorem 3.1 in Fan and Zhang [11] we deduce that $\{w_n\}$ converges strongly to w in E .

Since $\Psi_{\lambda} \in C^1(E, \mathbb{R})$ we conclude

$$(12) \quad d\Psi_{\lambda}(w_n) \rightarrow d\Psi_{\lambda}(w)$$

as $n \rightarrow \infty$.

Relations (9) and (12) show that $d\Psi_{\lambda}(w) = 0$, and thus w is a weak solution for problem (1). Moreover, by relation (9) it follows that $\Psi_{\lambda}(w) < 0$, and thus w is a nontrivial weak solution for (1). Since $\Psi_{\lambda}(|w|) = \Psi_{\lambda}(w)$, then problem (1) has a nonnegative solution.

The proof of Theorem 1 is complete. $\square$

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