VARIOUS CHARACTERIZATIONS OF PRODUCT HARDY SPACE

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ABSTRACT. This article deals with the characterizations of Hardy space H^1 on $\mathbb{R}^n \times \mathbb{R}^m$ using different norms on distinct variables. This result can be applied to the boundedness of certain operators on $H^1(\mathbb{R}^n \times \mathbb{R}^m)$.

1. INTRODUCTION

The product Hardy space theory was developed by R. Gundy and E.M. Stein [7], S.-Y. Chang and R. Fefferman [1, 2, 4, 5], as well as J. Journé [11, 12]. See [13, 15, 16] for some recent progress on multiparameter theory. As we all know, the space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ has a variety of different norms, in terms of maximal functions, square functions and Riesz transforms (see Theorem 3.4 of [13]).

Theorem A (Equivalent forms of H^1 norm). *All the norms below are equivalent and can be used as a definition of $H^1(\mathbb{R}^n \times \mathbb{R}^m)$:*

$$\|f^+\|_{L^1} \simeq \|f^*\|_{L^1} \simeq \|f\|_{L^1} + \|S(f)\|_{L^1} \simeq \sum_{i=0}^n \sum_{j=0}^m \|R_{1,i} R_{2,j} f\|_{L^1},$$

where $S(f)$ is the Littlewood–Paley square function and we use f^+ and f^* to denote the vertical and nontangential maximal function, respectively, $R_{1,i}$ to denote the i th Riesz transform for the first variable $x_1 \in \mathbb{R}^n$ and $R_{2,j}$ the j th Riesz transform for the second variable $x_2 \in \mathbb{R}^m$, and the 0th is the identity operator.

Moreover, $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ adapts an atomic decomposition as follows. See Section 2 for the denotation.

Definition B. An $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ function a is called an $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ -atom if there exists an open set Ω with finite measure satisfying:

- (1) $\text{supp } a \subset \Omega$;
- (2) a has a further decomposition as $a = \sum_{R \in m(\Omega)} a_R$, where a_R satisfy
 - (i) $\text{supp } a_R \subset 3R$;

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- (ii) $\int_{\mathbb{R}^n} a_R(x, y) dx = 0, \quad \forall y \in \mathbb{R}^m, \quad \int_{\mathbb{R}^m} a_R(x, y) dy = 0, \quad \forall x \in \mathbb{R}^n;$
 (3) $\|a\|_2 \leq |\Omega|^{-1/2}$ and $\sum_{R \in m(\Omega)} \|a_R\|_2^2 \leq |\Omega|^{-1}.$

Theorem C. $f \in H^1(\mathbb{R}^n \times \mathbb{R}^m)$ if and only if f has a representation via atoms, i.e.

$$f = \sum_j \lambda_j a_j,$$

where the a_j are atoms as in Definition B and $\sum_j |\lambda_j| \leq C \|f\|_{H^1(\mathbb{R}^n \times \mathbb{R}^m)}.$

In this paper, we develop several characterizations of Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ using different norms on distinct variables.

To begin with, we consider the following $(\mathcal{N}, S)$ -function which appears as a nontangential maximal function on the first variable and as a Littlewood-Paley square function on the second variable. The notation $\Gamma(x_1) = \{(y_1, t_1) \in \mathbb{R}_+^{n+1} : |x_1 - y_1| < t_1\}$ denotes the standard cone with vertex $x_1 \in \mathbb{R}^n$. Similarly, $\Gamma(x_2) = \{(y_2, t_2) \in \mathbb{R}_+^{m+1} : |x_2 - y_2| < t_2\}$ denotes the standard cone with vertex $x_2 \in \mathbb{R}^m$:

$$(1.1) \quad f_{\mathcal{N}, S}(x_1, x_2) := \sup_{|y_1 - x_1| < t_1} \left(\iint_{\Gamma(x_2)} |\phi_{t_1} \psi_{t_2} * f(y_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \right)^{1/2}.$$

Here and throughout the whole paper, $\phi \in C^\infty(\mathbb{R}^n)$, $|\phi(x)| \leq C(1 + |x|)^{-n-1}$, $|\nabla \phi(x)| \leq C(1 + |x|)^{-n-2}$, $\int \phi(x) dx = 1$ and $\phi_{t_1}(x) = t_1^{-n} \phi(\frac{x}{t_1})$, $\psi \in C^\infty(\mathbb{R}^m)$, $|\psi(y)| \leq C(1 + |y|)^{-m-1}$, $|\nabla \psi(y)| \leq C(1 + |y|)^{-m-2}$, $\int \psi(y) dy = 0$, $\int_0^\infty |\hat{\psi}(\xi t)|^2 dt/t \leq C$ and $\psi_{t_2}(y) = t_2^{-m} \psi(\frac{y}{t_2})$, where $t_1 > 0$, $t_2 > 0$ and C is a positive constant.

Then we will prove that $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ can be characterized in terms of the $(\mathcal{N}, S)$ -function. To see this, define

$$(1.2) \quad H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m) := \{f \mid f \in L^1(\mathbb{R}^n \times \mathbb{R}^m), f_{\mathcal{N}, S} \in L^1(\mathbb{R}^n \times \mathbb{R}^m)\}$$

with the norm

$$\|f\|_{H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)} := \|f\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} + \|f_{\mathcal{N}, S}\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)}.$$

We then have the first main result of this paper as follows.

Theorem 1.1. $H^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m).$

Note that Theorem A implies that $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ can be characterized by the product type nontangential maximal function or the Littlewood-Paley square function, which can be considered as the nontangential maximal function or the Littlewood-Paley square function (in the single parameter case) acting on both the first and second variables. Our result shows that one can also use the combination of the nontangential maximal function and the Littlewood-Paley square function (in the single parameter case) to give an equivalent characterization of $H^1(\mathbb{R}^n \times \mathbb{R}^m)$. Thus, we can further consider the combinations among the Littlewood-Paley g -function, vertical maximal function, as well as the nontangential maximal function and the Littlewood-Paley square function in the single parameter case. That is, we

consider the $(\mathcal{N}, g)$ -function and $(S, \mathcal{N})$ -function as follows:

$$\begin{aligned} f_{\mathcal{N},g}(x_1, x_2) &:= \sup_{|y_1 - x_1| < t_1} \left(\int_0^\infty |\phi_{t_1} \psi_{t_2} * f(y_1, x_2)|^2 \frac{dt_2}{t_2} \right)^{1/2}; \\ f_{S,\mathcal{N}}(x_1, x_2) &:= \left(\iint_{\Gamma(x_2)} \sup_{|y_1 - x_1| < t_1} |\phi_{t_1} \psi_{t_2} * f(y_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \right)^{1/2}. \end{aligned}$$

Then we define

$$(1.3) \quad H_{\mathcal{N},g}^1(\mathbb{R}^n \times \mathbb{R}^m) := \{f \mid f \in L^1(\mathbb{R}^n \times \mathbb{R}^m), f_{\mathcal{N},g} \in L^1(\mathbb{R}^n \times \mathbb{R}^m)\},$$

with the norm

$$\|f\|_{H_{\mathcal{N},g}^1(\mathbb{R}^n \times \mathbb{R}^m)} := \|f\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} + \|f_{\mathcal{N},g}\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)},$$

and

$$(1.4) \quad H_{S,\mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m) := \{f \mid f \in L^1(\mathbb{R}^n \times \mathbb{R}^m), f_{S,\mathcal{N}} \in L^1(\mathbb{R}^n \times \mathbb{R}^m)\},$$

with the norm

$$\|f\|_{H_{S,\mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m)} := \|f\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} + \|f_{S,\mathcal{N}}\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)}.$$

Similarly, we can define spaces $H_{g,\mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m)$, $H_{+,S}^1(\mathbb{R}^n \times \mathbb{R}^m)$, $H_{+,g}^1(\mathbb{R}^n \times \mathbb{R}^m)$, $H_{S,+}^1(\mathbb{R}^n \times \mathbb{R}^m)$ and $H_{g,+}^1(\mathbb{R}^n \times \mathbb{R}^m)$ in the norms given by the L^1 norm of the corresponding function plus the L^1 norm of the function itself, e.g.

$$\|f\|_{H_{+,S}^1(\mathbb{R}^n \times \mathbb{R}^m)} := \|f\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} + \|f_{+,S}\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)},$$

where $f_{+,S}(x_1, x_2) =: \sup_{t_1 > 0} \left(\iint_{\Gamma(x_2)} |\phi_{t_1} \psi_{t_2} * f(x_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \right)^{1/2}$.

Now we state the second main result of this paper as follows.

Theorem 1.2. *Let all the notation be the same as above. Then we have*

$$\begin{aligned} H^1(\mathbb{R}^n \times \mathbb{R}^m) &\simeq H_{\mathcal{N},g}^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{S,\mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{g,\mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m) \\ &\simeq H_{+,S}^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{+,g}^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{S,+}^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{g,+}^1(\mathbb{R}^n \times \mathbb{R}^m). \end{aligned}$$

The paper is organized as follows. In Section 2, we introduce some definitions and preliminary results. In Section 3, we prove Theorem 1.1 using two approaches. In the last section we prove Theorem 1.2.

2. PRELIMINARIES

For any $x_2 \in \mathbb{R}^m$, define $\mathbb{B}(x_2) := \{F_{t_2, y_2}(y_1) : t_2 > 0, y_1 \in \mathbb{R}^n, y_2 \in \Gamma(x_2)\}$ with the norm

$$(2.1) \quad \|F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)} = \left(\iint_{\Gamma(x_2)} |F_{t_2, y_2}(y_1)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \right)^{1/2}.$$

Next, we use $*_2$ to denote the convolution on the second variable; that is,

$$f *_2 g(x, y) = \int_{\mathbb{R}^m} f(z) g(x, y - z) dz$$

for any functions f on $\mathbb{R}^m$ and g on $\mathbb{R}^n \times \mathbb{R}^m$. Similarly, we can define the convolution $*_1$ on the first variable.

Now we recall Journé's covering lemma ([10]) as follows.

Suppose $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ is open of finite measure. Denote by $m(\Omega)$ the maximal dyadic subrectangles of Ω . Let $m_1(\Omega)$ denote those dyadic subrectangles $R \subseteq \Omega$, $R = I \times J$ that are maximal in the x_1 direction. In other words if $S = I' \times J \supseteq R$ is a dyadic subrectangle of Ω , then $I = I'$. Define $m_2(\Omega)$ similarly.

Lemma 2.1. *For any $R = I \times J \in m_1(\Omega)$, we set $\gamma_1(R) = \sup_{\substack{I' \subset I \\ I' \times J \subset \Omega^*}} |I'|/|I|$, where $\Omega^* = \{x \in \mathbb{R}^{n+m}, \mathcal{M}(\chi_\Omega)(x) > \frac{1}{2}\}$. Define γ_2 similarly. Then for any $\delta > 0$,*

$$\sum_{R \in m_2(\Omega)} |R| \gamma_1^{-\delta}(R) \leq c_\delta |\Omega| \quad \text{and} \quad \sum_{R \in m_1(\Omega)} |R| \gamma_2^{-\delta}(R) \leq c_\delta |\Omega|,$$

where c_δ is a constant depending only on δ , not on Ω .

Moreover, we need the following tent space theory which has been studied by [3]. $\mathbb{R}_+^{n+1}$ will denote the usual upper half-space in $\mathbb{R}^{n+1}$. If O is an open subset of $\mathbb{R}^n$, then the “tent” over O , denoted by $\widehat{O}$, is given as $\widehat{O} = \{(x, t) \in \mathbb{R}_+^{n+1} : B(x, t) \subset O\}$. For any function $f(y, t)$ defined on $\mathbb{R}_+^{n+1}$ we will denote

$$\mathcal{A}(f)(x) = \left(\iint_{\Gamma(x)} |f(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}.$$

As in [3], the “tent space” T_2^p is defined as the space of functions f such that $\mathcal{A}(f) \in L^p(\mathbb{R}^n)$, when $p < \infty$. The resulting equivalence classes are then equipped with the norm $\|f\|_{T_2^p} = \|\mathcal{A}(f)\|_p$. We recall that a T_2^1 atom is a function $a(x, t)$ supported in $\widehat{Q}$ (for some cube $Q \subset \mathbb{R}^n$), with $\int_{\widehat{Q}} |a(x, t)|^2 dx dt / t \leq 1/|Q|$.

Proposition 2.2. *Every element $F \in T_2^1(\mathbb{R}_+^{n+1})$ can be written as $F = \sum_j \lambda_j a_j$, where the a_j are T_2^1 atoms, and $\sum_j |\lambda_j| \leq C \|F\|_{T_2^1}$.*

In the following Lemma 2.3 (see [14] for the proof), we shall assume that $\Phi \in C_0^1(\mathbb{R}^n)$ is nonnegative, radial and nonincreasing, $\Phi = 1$ on $B(0, 1/2)$, $\text{supp } \Phi \subset B(0, 1)$ and $\int \Phi(x) dx = 1$. We may sometimes use capital letters to denote points in $\mathbb{R}_+^{n+1}$, for example, $X = (x, t)$, and set $u(x, t) = P_t f(x)$,

$$\nabla_X u(X) = (\nabla_x u, \partial_t u) \quad \text{and} \quad |\nabla_X u|^2 = |\nabla_x u|^2 + |\partial_t u|^2.$$

Lemma 2.3. *For every $f, g \in L^2(\mathbb{R}^n)$, we have*

$$\begin{aligned} & \iint_{\mathbb{R}_+^{n+1}} |t \nabla_X u(x, t)|^2 |\Phi_t * g(x)|^2 \frac{dx dt}{t} \\ (2.2) \quad & \leq \int_{\mathbb{R}^n} |f(x)|^2 |g(x)|^2 dx + \iint_{\mathbb{R}_+^{n+1}} |u(x, t)|^2 |\Psi_t * g(x)|^2 \frac{dx dt}{t}, \end{aligned}$$

where Ψ is a vector-valued function with the same support as Φ and mean value 0.

3. PROOF OF THEOREM 1.1

First we point out that the definition of the space $H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)$ is independent of the choices of functions ϕ and ψ which satisfy the conditions as in (1.1).

In fact, we first fix a function ψ satisfying the conditions as in (1.1), and then we prove that the definition of $H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)$ is independent of the choices of functions

ϕ . To see this, we set $F_{t_2, y_2}(y_1) = \psi_{t_2} * f(y_1, y_2)$, where $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$. Then, define

$$M_\phi(f)(x_1, x_2) := \sup_{t_1 > 0} \|F_{t_2, y_2} * \phi_{t_1}(x_1)\|_{\mathbb{B}(x_2)},$$

$$F^*(x_1, x_2) := \sup_{|x_1 - y_1| < t_1} \|F_{t_2, y_2} * P_{t_1}(y_1)\|_{\mathbb{B}(x_2)},$$

where P_{t_1} is the Poisson kernel on $\mathbb{R}^n$. Here and throughout this section, we use $\|\cdot\|_{\mathbb{B}(x_2)}$ to denote the vector-valued norm defined as in (2.1). It is easy to see that P_{t_1} satisfies all conditions as ϕ in (1.1). Moreover, let $\mathcal{F} = \{\|\cdot\|_{\alpha, \beta}\}$ be any finite collection of seminorms on the test function space $\mathcal{S}$, where $\|\cdot\|_{\alpha, \beta}$ is given by $\|\phi\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \phi(x)|$ and α and β are n -tuples of natural numbers. Define $\mathcal{S}_{\mathcal{F}} = \{\phi \in \mathcal{S} : \|\phi\|_{\alpha, \beta} \leq 1, \text{ for all } \|\cdot\|_{\alpha, \beta} \in \mathcal{F}\}$ and

$$M_{\mathcal{F}}(f)(x_1, x_2) = \sup_{\phi \in \mathcal{S}_{\mathcal{F}}} M_\phi(f)(x_1, x_2).$$

Then we have the following result:

Proposition 3.1. *Let $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$. Then the following conditions are equivalent:*

- (i) *There is a $\phi \in \mathcal{S}$ with $\int \phi \, dx \neq 0$ so that $M_\phi(f) \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$.*
- (ii) *There is a collection $\mathcal{F}$ so that $M_{\mathcal{F}}(f) \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$.*
- (iii) *$F^* \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$.*

We can obtain this proposition by following the proof of Theorem 1 as in Chapter 3, §1.2, [17] with only minor modifications.

Proposition 3.1 implies that for any fixed ψ , the definition of $H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)$ is independent of the choices of the functions ϕ . Then in the proofs of Propositions 3.2 and 3.3, we fix a function ψ and then use the Poisson kernel in the definition of $H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)$ instead of ϕ to prove Theorem 1.1. This yields that the definition of $H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)$ is also independent of the choices of ψ .

Now we will prove Theorem 1.1 by showing the following two propositions.

Proposition 3.2. $H^1(\mathbb{R}^n \times \mathbb{R}^m) \subseteq H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)$.

Proposition 3.3. $H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m) \subseteq H^1(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof of Proposition 3.2. In this part, we give two approaches: one relies on the atomic decomposition for the product Hardy space $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ (see [2, 8] for the idea); the other is due to the result of C. Fefferman and E.M. Stein ([6]), using the distribution inequality.

Method 1 (via atomic decomposition). Let $f(x) = \sum_j \lambda_j a_j(x)$, where the a_j 's are product atoms and $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. We note that the operator $T_{\mathcal{N}, S} : f \mapsto f_{\mathcal{N}, S}$ is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$, where $f_{\mathcal{N}, S}$ was defined in (1.1). Recently, Han et al. have showed that a linear operator T , which is bounded on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$, is bounded from $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^1(\mathbb{R}^n \times \mathbb{R}^m)$ if T is uniformly bounded on all product atoms in $L^1(\mathbb{R}^n \times \mathbb{R}^m)$. See Theorem 1.3 of [9]. The proof is also applicable to sublinear operators. Thus, to prove Proposition 3.2, it suffices to show that there exists some constant C such that for each product atom a ,

$$(3.3) \quad \|a_{\mathcal{N}, S}\|_{L^1(\mathbb{R}^n \times \mathbb{R}^m)} \leq C.$$

Assume that $a = \sum_{R \in m(\Omega)} a_R$ is a product atom supported in some open set Ω of $\mathbb{R}^{n+m}$ with finite measure. For any $R = I \times J \in m(\Omega)$, let $\ell(I), \ell(J)$ be the side-lengths of cubes I and J , and I' be the longest dyadic interval containing I so that $I' \times J \subset \Omega^* = \{x \in \mathbb{R}^{n+m} : \mathcal{M}_s(\chi_\Omega)(x) > 1/2\}$. Then $I' \times J$ is in $m_1(\Omega^*)$, and let J' be the longest dyadic interval so that $J' \supseteq J$ and $I' \times J' \subset \Omega^{**}$, where $\Omega^{**} = \{x \in \mathbb{R}^{n+m} : \mathcal{M}_s(\chi_\Omega^*)(x) > 1/2\}$. Let $\tilde{R}$ be the 10-fold dilate of $I' \times J'$ concentric with $I' \times J'$. Clearly, the boundedness of the strong maximal function shows that $|\bigcup \tilde{R}| \leq c|\Omega^{**}| \leq c|\Omega^*| \leq c|\Omega|$. We have

$$\int_{\bigcup \tilde{R}} a_{\mathcal{N},S}(x) dx \leq C |\bigcup \tilde{R}|^{1/2} \|a_{\mathcal{N},S}\|_2 \leq C |\Omega|^{1/2} \|a_{\mathcal{N},S}\|_2.$$

We claim that $\|a_{\mathcal{N},S}\|_2 \leq C|\Omega|^{-1/2}$. In fact, observe that for each $|x_1 - y_1| < t_1$, $y_2 \in \mathbb{R}^m$,

$$|\phi_{t_1} * f(y_1, y_2)| \leq C \mathcal{M}^{(1)}(f(\cdot, y_2))(x_1),$$

where we use $\mathcal{M}^{(1)}$ to denote the Hardy-Littlewood maximal operator on the first variable. By using the L^2 -boundedness of $\mathcal{M}^{(1)}$ and the square function, we obtain

$$\begin{aligned} & \iint (a_{\mathcal{N},S}(x_1, x_2))^2 dx_1 dx_2 \\ & \leq C \iint \iint_{\Gamma(x_2)} \left(\mathcal{M}^{(1)}(\psi_{t_2} * a(\cdot, y_2))(x_1) \right)^2 \frac{dy_2 dt_2}{t_2^{m+1}} dx_1 dx_2 \\ & \leq C \iint \iint_{\Gamma(x_2)} |\psi_{t_2} * a(x_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} dx_1 dx_2 \\ & \leq C \|a\|_2 \leq C |\Omega|^{-1/2}. \end{aligned}$$

Therefore, we have proved $\int_{\bigcup \tilde{R}} a_{\mathcal{N},S}(x) dx \leq C$. It suffices to prove

$$(3.4) \quad \int_{(\bigcup \tilde{R})^c} a_{\mathcal{N},S}(x) dx \leq C.$$

One writes

$$\begin{aligned} \int_{(\bigcup \tilde{R})^c} a_{\mathcal{N},S}(x) dx & \leq \sum_{R \in m(\Omega)} \int_{(\tilde{R})^c} (a_R)_{\mathcal{N},S}(x) dx \\ & \leq \sum_{R \in m(\Omega)} \int_{x_1 \notin 10I'} (a_R)_{\mathcal{N},S}(x) dx + \int_{x_2 \notin 10J'} (a_R)_{\mathcal{N},S}(x) dx \\ & =: D + E. \end{aligned}$$

We first estimate the term D. Observe that

$$D = \sum_{R \in m(\Omega)} \left(\int_{x_1 \notin 10I'} \int_{x_2 \in 10J} + \int_{x_1 \notin 10I'} \int_{x_2 \notin 10J} \right) (a_R)_{\mathcal{N},S}(x) dx =: D_1 + D_2.$$

Denote $F_{t_2, y_2}(y_1) := \psi_{t_2} * a_R(y_1, y_2)$. By Hölder's inequality,

$$(3.5) \quad D_1 \leq C \sum_{R \in m(\Omega)} |J|^{1/2} \int_{x_1 \notin 10I'} \left(\int_{|x_1 - y_1| < t_1} \sup \|\phi_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 dx_2 \right)^{1/2} dx_1.$$

Denote the center of I and J by x_I and x_J , respectively. By noting that $|x_1 - y_1| < t_1$, $z_1 \in 3I$ and $x_1 \notin 10I'$, we can apply the cancellation of atoms

(i.e. (ii) of (2) in Definition B) to obtain

$$\begin{aligned}
 |\phi_{t_1} \psi_{t_2} * (a_R)(y_1, y_2)| &= \left| \int_{3I} (\phi_{t_1}(y_1 - z_1) - \phi_{t_1}(y_1 - x_I)) \psi_{t_2} *_2 (a_R)(z_1, y_2) dz_1 \right| \\
 &\leq C \frac{t_1}{(t_1 + |x_1 - x_I|)^{n+1}} \left(1 \wedge \frac{\ell(I)}{t_1}\right) \int_{3I} |\psi_{t_2} *_2 a_R(z_1, y_2)| dz_1 \\
 (3.6) \quad &\leq C \frac{\ell(I)}{|x_1 - x_I|^{n+1}} |I|^{1/2} \left(\int_{3I} |\psi_{t_2} *_2 a_R(z_1, y_2)|^2 dz_1 \right)^{1/2},
 \end{aligned}$$

where $\left(1 \wedge (\ell(I)/t_1)\right) := \min(1, \ell(I)/t_1)$. By substituting (3.6) back into (3.5), we have

$$\begin{aligned}
 D_1 &\leq C \sum_{R \in m(\Omega)} |R|^{1/2} \int_{x_1 \notin 10I'} \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \left(\iint |a_R(z_1, y_2)|^2 dz_1 dy_2 \right)^{1/2} dx_1 \\
 &\leq C \sum_{R \in m(\Omega)} |R|^{1/2} \frac{\ell(I)}{\ell(I')} \|a_R\|_2 \\
 &\leq C,
 \end{aligned}$$

where in the last inequality we have used Hölder's inequality and Lemma 2.1.

Consider the term D_2 . By $\int a_R(z_1, z_2) dz_1 = 0$ and $\int a_R(z_1, z_2) dz_2 = 0$, we have

(3.7)

$$\begin{aligned}
 &|\phi_{t_1} \psi_{t_2} * a_R(y_1, y_2)| \\
 &= \left| \iint (\phi_{t_1}(y_1 - z_1) - \phi_{t_1}(y_1 - x_I)) (\psi_{t_2}(y_2 - z_2) - \psi_{t_2}(y_2 - x_J)) a_R(z_1, z_2) dz_1 dz_2 \right| \\
 &\leq C \frac{t_1}{(t_1 + |x_1 - x_I|)^{n+1}} \left(1 \wedge \frac{\ell(I)}{t_1}\right) \frac{t_2}{(t_2 + |x_2 - x_J|)^{m+1}} \left(1 \wedge \frac{\ell(J)}{t_2}\right) \|a_R\|_2 |R|^{1/2} \\
 &\leq C \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \frac{t_2^{1/2}}{(t_2 + |x_2 - x_J|)^{m+1/2}} \left(1 \wedge \frac{\ell(J)}{t_2}\right) \|a_R\|_2 |R|^{1/2}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 D_2 &\leq C \sum_{R \in m(\Omega)} \|a_R\|_2 |R|^{1/2} \int_{x_1 \notin 10I'} \int_{x_2 \notin 10J} \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \\
 &\quad \times \left(\int_0^\infty \frac{t_2}{(t_2 + |x_2 - x_J|)^{2m+1}} \left(1 \wedge \frac{\ell(J)}{t_2}\right)^2 \frac{dt_2}{t_2} \right)^{1/2} dx_2 dx_1 \\
 &\leq C \sum_{R \in m(\Omega)} \|a_R\|_2 |R|^{1/2} \int_{x_1 \notin 10I'} \int_{x_2 \notin 10J} \frac{\ell(I)}{|x_1 - x_I|^{n+1}} \frac{\ell(J)^{1/2}}{|x_2 - x_J|^{m+1/2}} dx_2 dx_1 \\
 &\leq C \sum_{R \in m(\Omega)} \|a_R\|_2 |R|^{1/2} \frac{\ell(I)}{\ell(I')} \leq C,
 \end{aligned}$$

where the last inequality follows from Hölder's inequality and Lemma 2.1.

We then estimate the term E. Observe that

$$E = \sum_{R \in m(\Omega)} \left(\int_{x_1 \in 10I} \int_{x_2 \notin 10J'} + \int_{x_1 \notin 10I} \int_{x_2 \notin 10J'} \right) (a_R)_{\mathcal{N}, S}(x) dx =: E_1 + E_2.$$

We only estimate E_1 since E_2 can be estimated by the similar argument to that of D_2 . Applying Hölder's inequality, we have

$$E_1 \leq C|I|^{1/2} \sum_{R \in m(\Omega)_{x_2} \not\subset 10J'} \int_{x_1 \in 10I} \left(\int_{|x_1 - y_1| < t_1} \sup_{|x_1 - y_1| < t_1} \|\phi_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 dx_1 \right)^{1/2} dx_2.$$

Observe that $|\phi_{t_1} \psi_{t_2} * a_R(y_1, y_2)| \leq \mathcal{M}^{(1)}(\psi_{t_2} * a_R(\cdot, y_2))(x_1)$ whenever $|x_1 - y_1| < t_1$. This, together with the L^2 -boundedness of $\mathcal{M}^{(1)}$, yields

$$(3.8) \quad E_1 \leq C|I|^{1/2} \sum_{R \in m(\Omega)_{x_2} \not\subset 10J'} \int_{x_1 \in 10I} \left(\int_{\Gamma(x_2)} |\psi_{t_2} * a_R(x_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} dx_1 \right)^{1/2} dx_2.$$

By the similar argument to (3.7), one writes

$$(3.9) \quad |\psi_{t_2} * a_R(x_1, y_2)| \leq \int_{\mathbb{R}^m} \left(1 \wedge \frac{\ell(J)}{t_2} \right) \frac{t_2^{1/2}}{(t_2 + |x_2 - x_J|)^{m+1/2}} |a_R(x_1, z_2)| dz_2.$$

Substituting (3.9) back into (3.8) and then using Lemma 2.1, we can obtain

$$E_1 \leq C \sum_{R \in m(\Omega)} |R|^{1/2} \left(\frac{\ell(J)}{\ell(J')} \right)^{1/2} \|a_R\|_2 \leq C.$$

Therefore, we obtain the required estimate (3.4), which implies (3.3).

Method 2 (via distribution inequality). It suffices to prove that

(3.10)

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^m} \sup_{|x_1 - y_1| < t_1} \left(\iint_{\Gamma(x_2)} |P_{t_1} \psi_{t_2} * f(y_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \right)^{1/2} dx_1 dx_2 \\ & \leq C \int_{\mathbb{R}^n \times \mathbb{R}^m} \left(\iint_{\Gamma_\beta(x_1)} \iint_{\Gamma(x_2)} |t_1 \nabla P_{t_1} \psi_{t_2} * f(y_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \frac{dy_1 dt_1}{t_1^{n+1}} \right)^{1/2} dx_1 dx_2 \end{aligned}$$

for any $f \in H^1(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^n \times \mathbb{R}^m)$, where P_{t_1} is the Poisson kernel in $\mathbb{R}^n$, C is a constant independent of f and $\Gamma_\beta(x_1) = \{(y_1, t_1) : |x_1 - y_1| < \beta t_1\}$ is the cone with aperture $\beta > 1$.

Now set $F_{t_2, y_2}(y_1) = \psi_{t_2} * f(y_1, y_2)$. Then we can further define the vector-valued square function and nontangential maximal function of $F_{t_2, y_2}(y_1)$ as follows:

$$(3.11) \quad S_\beta(F)(x_1, x_2) = \left(\iint_{\Gamma_\beta(x_1)} \|t_1 \nabla P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 \frac{dy_1 dt_1}{t_1^{n+1}} \right)^{1/2},$$

$$(3.12) \quad F^*(x_1, x_2) = \sup_{|x_1 - y_1| < t_1} \|P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)},$$

where $\mathbb{B}(x_2)$ is as in (2.1).

Thus, to prove (3.10), it suffices to prove that

$$(3.13) \quad \int_{\mathbb{R}^n \times \mathbb{R}^m} F^*(x_1, x_2) dx_1 dx_2 \leq C \int_{\mathbb{R}^n \times \mathbb{R}^m} S_\beta(F)(x_1, x_2) dx_1 dx_2.$$

We let E be the closed set $\{x_1 \in \mathbb{R}^n : S_\beta(F)(x_1, x_2) \leq \alpha\}$ and $A := E^c$. Thus, $\lambda_{S_\beta(F)}(\alpha) = |\{x_1 \in \mathbb{R}^n : S_\beta(F)(x_1, x_2) > \alpha\}| = |A|$. Similarly, let $\lambda_{F^*}(\alpha) = |\{x_1 \in \mathbb{R}^n : F^*(x_1, x_2) > \alpha\}|$.

To show (3.13), we will prove that for all $\alpha > 0$ and a.e. $x_2 \in \mathbb{R}^m$,

$$(3.14) \quad \lambda_{F^*}(\alpha) \leq C \left\{ \lambda_{S_\beta(F)}(\alpha) + \alpha^{-2} \int_0^\alpha s \lambda_{S_\beta(F)}(s) ds \right\}.$$

Now let E_0 be those points at which E has relative density at least $1/2$; more precisely, set $E_0 = \{x_1 \in \mathbb{R}^n : \text{for every cube } Q, \text{ such that } x_1 \in Q, |E \cap Q| \geq \frac{1}{2}|Q|\}$. Observe that since E is closed, $E_0 \subset E$; clearly E_0 is closed. If χ_A is the characteristic function of $A = E^c$, then $E_0^c = A^* = \{x_1 \in \mathbb{R}^n : \mathcal{M}^{(1)}(\chi_A)(x_1, x_2) > 1/2\}$. Thus $|A^*| \leq C|A| = C\lambda_{S_\beta(F)}(\alpha)$.

We now form the region $\mathcal{R} = \bigcup_{x_1 \in E_0} \Gamma(x_1)$ with the corresponding approximating regions $\mathcal{R}_\epsilon$ whose boundaries, $\mathcal{B}_\epsilon$, are given as hypersurfaces $t = c\delta_\epsilon(x)$ with $\delta_\epsilon(x)$ smooth and $|\partial\delta_\epsilon/\partial x_j| \leq 1$, $j = 1, \dots, n$. (See [18], page 206.) We have

$$\begin{aligned} & \int_E S_\beta(F)(x_1, x_2)^2 dx_1 \\ &= \int_{\bigcup_{x_1 \in E} \Gamma_\beta(x_1)} \left\| \nabla P_{t_1} * F_{t_2, y_2}(y_1) \right\|_{\mathbb{B}(x_2)}^2 |\{x_1 \in E : (y, t) \in \Gamma_\beta(x_1)\}| t_1^{1-n} dy_1 dt_1. \end{aligned}$$

In the second integral we restrict integration over $(y, t) \in \mathcal{R}$, which implies that for some $\bar{x} \in E_0$, $(y, t) \in \Gamma(\bar{x})$, i.e., $|\bar{x} - y| < t$. Then $(y, t) \in \Gamma_\beta(x_1)$ whenever $|x_1 - \bar{x}| < (\beta - 1)t$. Thus $|\{x_1 \in E : (y, t) \in \Gamma_\beta(x_1)\}| \geq |E \cap B|$, where B is the ball centered at $\bar{x} \in E_0$ with radius $(\beta - 1)t$. In view of the definition of E_0 , the latter quantity exceeds ct^n , and so

$$\begin{aligned} \int_E S_\beta(F)(x_1, x_2)^2 dx_1 &\geq C \int_{\mathcal{R}} \left\| \nabla P_{t_1} * F_{t_2, y_2}(y_1) \right\|_{\mathbb{B}(x_2)}^2 t_1 dy_1 dt_1 \\ &\geq C \int_{\mathcal{R}_\epsilon} \left\| \nabla P_{t_1} * F_{t_2, y_2}(y_1) \right\|_{\mathbb{B}(x_2)}^2 t_1 dy_1 dt_1. \end{aligned}$$

We write the last integral as

$$\begin{aligned} & C \int_{\mathcal{R}_\epsilon} \iint_{\Gamma(x_2)} |\nabla P_{t_1} * F_{t_2, y_2}(y_1)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} t_1 dy_1 dt_1 \\ &= C \iint_{\Gamma(x_2)} \int_{\mathcal{R}_\epsilon} |\nabla P_{t_1} * F_{t_2, y_2}(y_1)|^2 t_1 dy_1 dt_1 \frac{dy_2 dt_2}{t_2^{m+1}}. \end{aligned}$$

Then, transform the integral by Green's theorem, obtaining

$$(3.15) \quad \begin{aligned} \int_E S_\beta(F)(x_1, x_2)^2 dx_1 &\geq C_1 \iint_{\Gamma(x_2)} \int_{\mathcal{B}_\epsilon} |P_{t_1} * F_{t_2, y_2}(y_1)|^2 d\sigma \frac{dy_2 dt_2}{t_2^{m+1}} \\ &\quad - C_2 \iint_{\Gamma(x_2)} \int_{\mathcal{B}_\epsilon} |P_{t_1} * F_{t_2, y_2}(y_1)| t_1 |\nabla P_{t_1} * F_{t_2, y_2}(y_1)| d\sigma \frac{dy_2 dt_2}{t_2^{m+1}}, \end{aligned}$$

where C_1 and C_2 are two positive constants independent of ϵ .

Let

$$\mathcal{J}_\epsilon^2 = \iint_{\Gamma(x_2)} \int_{\mathcal{B}_\epsilon} |P_{t_1} * F_{t_2, y_2}(y_1)|^2 d\sigma \frac{dy_2 dt_2}{t_2^{m+1}} = \int_{\mathcal{B}_\epsilon} \|P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 d\sigma.$$

We have

$$\int_{\mathcal{B}_\epsilon} \|P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 d\sigma \leq C \int_{\mathbb{R}^n} (F^*)^2 d\sigma \leq C \int_{\mathbb{R}^n} (F^*)^2 dx_1 < \infty$$

for a.e. $x_2 \in \mathbb{R}^m$. Hence $\mathcal{J}_\epsilon$ is finite for every ϵ .

We now divide the boundary $\mathcal{B}_\epsilon$ into two parts $\mathcal{B}_\epsilon = \mathcal{B}_\epsilon^{E_0} \cup \mathcal{B}_\epsilon^{B^*}$, where $\mathcal{B}_\epsilon^{E_0}$ is the part above the set E_0 , and $\mathcal{B}_\epsilon^{B^*}$ is the part lying above the set B^* . Next,

$$\begin{aligned} & \iint_{\Gamma(x_2)} \int_{\mathcal{B}_\epsilon} |P_{t_1} * F_{t_2, y_2}(y_1)| t_1 |\nabla P_{t_1} * F_{t_2, y_2}(y_1)| d\sigma \frac{dy_2 dt_2}{t_2^{m+1}} \\ &= \iint_{\Gamma(x_2)} \left(\int_{\mathcal{B}_\epsilon^{B^*}} + \int_{\mathcal{B}_\epsilon^{E_0}} \right) |P_{t_1} * F_{t_2, y_2}(y_1)| t_1 |\nabla P_{t_1} * F_{t_2, y_2}(y_1)| d\sigma \frac{dy_2 dt_2}{t_2^{m+1}}. \end{aligned}$$

To estimate the above two terms, we recall the following result on harmonic functions.

Lemma 3.4 ([18]). *Suppose u is harmonic in Γ_β^k . If $\iint_{\Gamma_\beta^k} |\nabla u|^2 y^{1-n} dx dy \leq 1$, then $|y \nabla u| \leq C$ in Γ_α^h . Here Γ_β^k and Γ_α^h are truncated cones with $\beta > \alpha$, $k > h$. C is a constant depending only on α, β, h, k and the dimension n .*

Thus, by using Lemma 3.4, we have that $t_1 \|\nabla P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)} \leq C\alpha$ since $S_\beta(F)(x_1, x_2) \leq \alpha$ for $x_1 \in E$. By Schwarz's inequality, we get that

$$\begin{aligned} & \iint_{\Gamma(x_2)} \int_{\mathcal{B}_\epsilon^{B^*}} |P_{t_1} * F_{t_2, y_2}(y_1)| t_1 |\nabla P_{t_1} * F_{t_2, y_2}(y_1)| d\sigma \frac{dy_2 dt_2}{t_2^{m+1}} \\ & \leq C \mathcal{J}_\epsilon \alpha \sigma(\mathcal{B}_\epsilon^{B^*})^{1/2} \leq C \mathcal{J}_\epsilon \alpha |B^*|^{1/2} \leq C \mathcal{J}_\epsilon (\alpha^2 \lambda_{S_\beta(F)}(\alpha))^{1/2}. \end{aligned}$$

Moreover, $\mathcal{I}_\epsilon = \iint_{\Gamma(x_2)} \int_{\mathcal{B}_\epsilon^{E_0}} |P_{t_1} * F_{t_2, y_2}(y_1)| t_1 |\nabla P_{t_1} * F_{t_2, y_2}(y_1)| d\sigma \frac{dy_2 dt_2}{t_2^{m+1}} \rightarrow 0$ as $\epsilon \rightarrow 0$. Hence (3.15) gives $\mathcal{J}_\epsilon^2 \leq C \int_E S_\beta(F)(x_1, x_2)^2 dx_1 + C \mathcal{J}_\epsilon (\alpha^2 \lambda_{S_\beta(F)}(\alpha))^{1/2} + C \mathcal{I}_\epsilon$ and therefore

$$(3.16) \quad \mathcal{J}_\epsilon^2 \leq C \left\{ \int_E S_\beta(F)(x_1, x_2)^2 dx_1 + \alpha^2 \lambda_{S_\beta(F)}(\alpha) \right\}$$

for ϵ small enough.

Next, for each $\epsilon > 0$, define a function f^ϵ on $\mathbb{R}^n$ by setting

$$f^\epsilon(x_1) = C \|P_{\delta_\epsilon(x_1)} * F_{t_2, y_2}(x_1)\|_{\mathbb{B}(x_2)} + C \alpha \chi_{B^*}(x_1),$$

where χ_{B^*} is the characteristic function of B^* , and $t = \delta_\epsilon(x_1)$ is the equation of the hypersurface $\mathcal{B}_\epsilon = \partial \mathcal{R}_\epsilon$. Let $U_\epsilon(x_1, t_1)$ be the Poisson integral of the function f^ϵ , i.e., $U_\epsilon(x_1, t_1) = P_{t_1} * f^\epsilon(x_1)$. We claim that

$$(3.17) \quad \|P_{t_1} * F_{t_2, y_2}(x_1)\|_{\mathbb{B}(x_2)} \leq C U_\epsilon(x_1, t_1), \quad (x_1, t_1) \in \mathcal{B}_\epsilon.$$

In fact, choose β^* satisfying $1 < \beta^* < \beta$. Then there exists a positive constant C such that $B'((y_1, t_1), Ct_1) \subset \bigcap_{x_0 \in E_0} \Gamma_{\beta^*}(x_0)$ whenever $(y_1, t_1) \in \mathcal{B}_\epsilon$, where $B'((y_1, t_1), Ct_1)$ is the ball centered at (y_1, t_1) with radius Ct_1 . Now from Lemma 3.4, we have $t_1 \|\nabla P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)} \leq C\alpha$ whenever $(y_1, t_1) \in B'$. Set $Q_1 = (y_1, t_1)$ and for any distinct point $Q_2 \in B'$, we have $|Q_1 - Q_2| \leq Ct_1$. Moreover,

$$\begin{aligned} & \|P_{t_1} * F_{t_2, y_2}(\cdot)(Q_1) - P_{t_1} * F_{t_2, y_2}(\cdot)(Q_2)\|_{\mathbb{B}(x_2)} \\ & \leq Ct_1 \sup_{Q' \in B'} \|P_{t_1} * F_{t_2, y_2}(\cdot)(Q')\|_{\mathbb{B}(x_2)} \leq C \alpha \chi_{B^*}(y_1). \end{aligned}$$

Set $S_\epsilon = \mathcal{B}_\epsilon \cap B'$. Then

$$\begin{aligned}
 \|P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)} &= \|P_{t_1} * F_{t_2, y_2}(\cdot)(Q_1)\|_{\mathbb{B}(x_2)} \\
 &\leq \frac{1}{|S_\epsilon|} \int_{S_\epsilon} \|P_{t_1} * F_{t_2, y_2}(\cdot)(Q_2)\|_{\mathbb{B}(x_2)} d\sigma + C\alpha\chi_{B^*}(y_1) \\
 &\leq \frac{C}{t_1^n} \int_{|z-y_1| < Ct_1} |f^\epsilon(z)| dz \\
 &\leq C \int_{|z-y_1| < Ct_1} \frac{t_1}{(t_1^2 + |z-y_1|^2)^{\frac{n+1}{2}}} |f^\epsilon(z)| dz \\
 &\leq CP_{t_1} * (f^\epsilon)(y_1) = U_\epsilon(y_1, t_1).
 \end{aligned}$$

We then select a subsequence of the f^ϵ which converges weakly to $\bar{f} \in L^2(\mathbb{R}^n)$. From (3.16), we have

$$(3.18) \quad \int_{\mathbb{R}^n} |\bar{f}(x_1)|^2 dx_1 \leq C \left\{ \int_E S_\beta(F)(x_1, x_2)^2 dx_1 + \alpha^2 \lambda_{S_\beta(F)}(\alpha) \right\}.$$

Passing to the limit, we obtain that

$$\|P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)} \leq CU(y_1, t_1), \quad (y_1, t_1) \in \mathcal{R},$$

where U is the Poisson kernel of $\bar{f}$, and $F^*(x_1) \leq U^*(x_1)$ for $x_1 \in E_0$. Thus, we have

$$\begin{aligned}
 \lambda_{F^*}(\alpha) &\leq |E_0^c| + |\{x_1 \in E_0 : F^*(x_1) > \alpha\}| \\
 &\leq C\lambda_{S_\beta(F)}(\alpha) + \alpha^{-2} \int_{E_0} F^*(x_1)^2 dx_1 \\
 &\leq C\lambda_{S_\beta(F)}(\alpha) + \alpha^{-2} \int_{E_0} U^*(x_1)^2 dx_1 \\
 &\leq C\lambda_{S_\beta(F)}(\alpha) + \alpha^{-2} \int_{E_0} \bar{f}(x_1)^2 dx_1 \\
 &\leq C\lambda_{S_\beta(F)}(\alpha) + C\alpha^{-2} \int_E S_\beta(F)(x_1, x_2)^2 dx_1 \\
 &\leq C\lambda_{S_\beta(F)}(\alpha) + C\alpha^{-2} \int_0^\alpha s\lambda_{S_\beta(F)}(s) ds,
 \end{aligned}$$

which implies (3.14). $\square$

Proof of Proposition 3.3. It suffices to prove that there exists a constant $C > 0$ such that

$$\begin{aligned}
 (3.19) \quad &\int_{\mathbb{R}^n \times \mathbb{R}^m} \left(\iint_{\Gamma(x_1)} \iint_{\Gamma(x_2)} |t_1 \nabla P_{t_1} \psi_{t_2} * f(y_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \frac{dy_1 dt_1}{t_1^{n+1}} \right)^{1/2} dx_1 dx_2 \\
 &\leq C \int_{\mathbb{R}^n \times \mathbb{R}^m} \sup_{|x_1 - y_1| < t_1} \left(\iint_{\Gamma(x_2)} |P_{t_1} \psi_{t_2} * f(y_1, y_2)|^2 \frac{dy_2 dt_2}{t_2^{m+1}} \right)^{1/2} dx_1 dx_2
 \end{aligned}$$

for any $f \in H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m) \cap L^2(\mathbb{R}^n \times \mathbb{R}^m)$.

Let $S_\beta(F)(x_1, x_2)$ and $F^*(x_1, x_2)$ be the same as (3.11) and (3.12), respectively. Set $S(F)(x_1, x_2) = S_1(F)(x_1, x_2)$. Thus, to prove (3.19), it suffices to prove that

$$(3.20) \quad \int_{\mathbb{R}^n \times \mathbb{R}^m} S(F)(x_1, x_2) dx_1 dx_2 \leq C \int_{\mathbb{R}^n \times \mathbb{R}^m} F^*(x_1, x_2) dx_1 dx_2.$$

To prove (3.20), it is enough to show that for all $\alpha > 0$ and almost everywhere $x_2 \in \mathbb{R}^m$,

$$(3.21) \quad \int_{\{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}} S(F)(x_1, x_2)^2 dx_1 \\ \leq C\alpha^2 |\{x_1 : F^*(x_1, x_2) > \alpha\}| + C \int_{\{x_1 : F^*(x_1, x_2) \leq \alpha\}} F^*(x_1, x_2)^2 dx_1.$$

In fact, note that for almost everywhere $x_2 \in \mathbb{R}^m$, we have

$$(3.22) \quad \int_{\mathbb{R}^n} S(F)(x_1, x_2) dx_1 = \int_0^\infty |\{x_1 : S(F)(x_1, x_2) > \alpha\}| d\alpha.$$

Suppose (3.21) holds. Then

$$\begin{aligned} |\{x_1 : S(F)(x_1, x_2) > \alpha\}| &\leq |\{x_1 : S(F)(x_1, x_2) > \alpha, \mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}| \\ &\quad + |\{x_1 : \mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) \geq 2^{-(n+1)}\}| \\ &\leq \frac{1}{\alpha^2} \int_{\{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}} S(F)(x_1, x_2)^2 dx_1 \\ &\quad + C2^{(n+1)} |\{x_1 : F^*(x_1, x_2) > \alpha\}| \\ &\leq C \frac{1}{\alpha^2} \int_{\{x_1 : F^*(x_1, x_2) \leq \alpha\}} F^*(x_1, x_2)^2 dx_1 \\ &\quad + C |\{x_1 : F^*(x_1, x_2) > \alpha\}|. \end{aligned}$$

By integrating α and x_2 on both sides of the inequality above, we then obtain (3.20).

Now let us prove (3.21):

$$\begin{aligned} &\int_{\{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}} S(F)(x_1, x_2)^2 dx_1 \\ &= \iint_{\Gamma(x_2)} \int_{\{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}} \iint_{\Gamma(x_1)} |t_1 \nabla P_{t_1} * F_{t_2, y_2}(y_1)|^2 dx_1 \frac{dy_1 dt_1}{t_1^{n+1}} \frac{dy_2 dt_2}{t_2^{m+1}}. \end{aligned}$$

Note that

$$\begin{aligned} &\int_{\{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}} \iint_{\Gamma(x_1)} |t_1 \nabla P_{t_1} * F_{t_2, y_2}(y_1)|^2 \frac{dy_1 dt_1}{t_1^{n+1}} dx_1 \\ &= \int_{R^*} |\nabla P_{t_1} * F_{t_2, y_2}(y_1)|^2 t_1^{1-n} |B(y_1, t_1) \cap \{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}| dy_1 dt_1 \\ &\leq \int_{R^*} |\nabla P_{t_1} * F_{t_2, y_2}(y_1)|^2 t_1 dy_1 dt_1, \end{aligned}$$

where we use $B(y_1, t_1)$ to denote the ball in $\mathbb{R}^n$ centered at y_1 with radius t_1 , and

$$R^* = \{(y_1, t_1) : |B(y_1, t_1) \cap \{z : F^*(z, x_2) > \alpha\}| \leq 2^{-(n+1)} |B(y_1, t_1)|\}.$$

Thus, we have obtained that

$$\begin{aligned} &\int_{\{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}} S(F)(x_1, x_2)^2 dx_1 \\ &\leq \int_{R^*} \|\nabla P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 t_1 dy_1 dt_1. \end{aligned}$$

It is easy to check that if $|B(y_1, t_1) \cap \{z : F^*(z, x_2) > \alpha\}| \leq 2^{-(n+1)}|B(y_1, t_1)|$, then $g * \Phi_{t_1}(y_1) > C$ for some constant $C > 0$, where $\Phi \in C_0^1(\mathbb{R}^n)$ is as in Lemma 2.3 and $g(x) = \chi_{\{F^*(x, x_2) \leq \alpha\}}(x)$. This, together with Lemma 2.3, implies

$$\begin{aligned}
& \int_{\{\mathcal{M}^{(1)}(\chi_{F^* > \alpha})(x_1) < 2^{-(n+1)}\}} S(F)(x_1, x_2)^2 dx_1 \\
& \leq C \int_{\mathbb{R}^n} \|\nabla P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 |\Phi_{t_1} * g(y_1)| t_1 dy_1 dt_1 \\
& \leq C \iint_{\Gamma(x_2)} \int_{\mathbb{R}_+^{n+1}} |\nabla P_{t_1} * F_{t_2, y_2}(y_1)|^2 |\Phi_{t_1} * g(y_1)| t_1 dy_1 dt_1 \frac{dy_2 dt_2}{t_2^{m+1}} \\
& \leq C \iint_{\Gamma(x_2)} \int_{\mathbb{R}^n} |F_{t_2, y_2}(x_1)|^2 |g(x_1)|^2 dx_1 \frac{dy_2 dt_2}{t_2^{m+1}} \\
& \quad + C \iint_{\Gamma(x_2)} \int_{\mathbb{R}_+^{n+1}} |P_{t_1} * F_{t_2, y_2}(y_1)|^2 |\Psi_{t_1} * g(y_1)|^2 \frac{dy_1 dt_1}{t_1} \frac{dy_2 dt_2}{t_2^{m+1}} \\
& = C \int_{\mathbb{R}^n} \|F_{t_2, y_2}(x_1)\|_{\mathbb{B}(x_2)}^2 |g(x_1)|^2 dx_1 \\
& \quad + C \int_{\mathbb{R}_+^{n+1}} \|P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)}^2 |\Psi_{t_1} * g(y_1)|^2 \frac{dy_1 dt_1}{t_1} \\
& =: \text{I} + \text{II},
\end{aligned}$$

where the third inequality follows from Lemma 2.3.

For the term I, from the definitions of $g(x)$ and $F^*(x_1, x_2)$, we have

$$\text{I} \leq C \int_{\{x_1 : F^*(x_1, x_2) \leq \alpha\}} F^*(x_1, x_2)^2 dx_1.$$

As for the term II, we only need to consider $\Psi_{t_1} * (g)(y_1) \neq 0$. In this case, $B(y_1, t_1) \cap \{z_1, F^*(z_1, x_2) \leq \alpha\} \neq \emptyset$. Thus, there exists a point $z_1^0 \in \mathbb{R}^n$ such that $|z_1^0 - y_1| < t_1$ and $F^*(z_1^0, x_2) \leq \alpha$. By the definition of $F^*(z_1^0, x_2)$, we have

$$(3.23) \quad \|P_{t_1} * F_{t_2, y_2}(y_1)\|_{\mathbb{B}(x_2)} \leq \sup_{|z_1^0 - z_1| < s_1} \|P_{s_1} * F_{t_2, y_2}(z_1)\|_{\mathbb{B}(x_2)} = F^*(z_1^0, x_2) \leq \alpha.$$

Therefore, by (3.23) and the cancellation condition of Ψ , we obtain

$$\begin{aligned}
\text{II} & \leq C\alpha^2 \int_{\mathbb{R}_+^{n+1}} |\Psi_{t_1} * \chi_{F^*(\cdot, x_2) \leq \alpha}(y_1)|^2 \frac{dy_1 dt_1}{t_1} \\
& = C\alpha^2 \int_{\mathbb{R}_+^{n+1}} |\Psi_{t_1} * \chi_{F^*(\cdot, x_2) > \alpha}(y_1)|^2 \frac{dy_1 dt_1}{t_1} \\
& \leq C\alpha^2 |\{y_1 : F^*(y_1, x_2) > \alpha\}|.
\end{aligned}$$

Combining the estimates of I and II, we can see that (3.21) holds, which implies that (3.20) holds. This completes the proof of Theorem 1.1. $\square$

4. PROOF OF THEOREM 1.2

4.1. Proof of $H^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{\mathcal{N}, g}^1(\mathbb{R}^n \times \mathbb{R}^m)$. By the similar argument to Method 1 of Proposition 3.2, we can easily check that

$$H^1(\mathbb{R}^n \times \mathbb{R}^m) \subset H_{\mathcal{N}, g}^1(\mathbb{R}^n \times \mathbb{R}^m) \quad \text{with} \quad \|f\|_{H_{\mathcal{N}, g}^1(\mathbb{R}^n \times \mathbb{R}^m)} \leq C \|f\|_{H^1(\mathbb{R}^n \times \mathbb{R}^m)}.$$

To prove the converse part, it suffices to prove that

$$(4.24) \quad \iint \left(\int_0^\infty \int_0^\infty |\psi_{t_1} \psi_{t_2} * f(x_1, x_2)|^2 \frac{dt_2}{t_2} \frac{dt_1}{t_1} \right)^{1/2} dx_1 dx_2 \\ \leq C \iint \sup_{|x_1 - y_1| < t_1} \left(\int_0^\infty |\phi_{t_1} \psi_{t_2} * f(y_1, x_2)|^2 \frac{dt_2}{t_2} \right)^{1/2} dx_1 dx_2.$$

Denote by $\mathcal{B}$ the vector-valued function space $\{F_{t_2, x_2}(y_1) : y_1 \in \mathbb{R}^n, t_2 \in (0, \infty), x_2 \in \mathbb{R}^m\}$ with the norm

$$\|F_{t_2, x_2}(y_1)\|_{\mathcal{B}} = \left(\int_0^\infty |F_{t_2}(y_1, x_2)|^2 \frac{dt_2}{t_2} \right)^{1/2}.$$

We will prove (4.24) by two steps. Firstly, we can prove

$$(4.25) \quad \iint \left(\iint_{\Gamma(x_1)} \|\psi_{t_1} \psi_{t_2} * f(y_1, x_2)\|_{\mathcal{B}}^2 \frac{dy_1 dt_1}{t_1^{n+1}} \right)^{1/2} dx_1 dx_2 \\ \leq C \iint \sup_{|x_1 - y_1| < t_1} \left(\int_0^\infty \|\phi_{t_1} \psi_{t_2} * f(y_1, x_2)\|_{\mathcal{B}}^2 \frac{dt_1}{t_1} \right)^{1/2} dx_1 dx_2.$$

In fact, the proof is similar to that of Proposition 3.3 with the minor modification that $\|\cdot\|_{\mathbb{B}(x_2)}$ is replaced by $\|\cdot\|_{\mathcal{B}}$.

Secondly, we claim that

$$(4.26) \quad \left\| \left(\int_0^\infty |F(x, t)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \leq C \left\| \left(\iint_{\Gamma(x)} |F(y, t)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)}.$$

We observe that the right part of the inequality above is the T_2^1 norm of the function $F(x, t)$. By Proposition 2.2, we obtain $F(x, t) = \sum_j \lambda_j a_j(x, t)$, where $a_j(x, t)$ are T_2^1 atoms.

To prove (4.26), it is enough to prove that for each T_2^1 atom $a(x, t)$ associated to some cube Q , there exists a constant C such that

$$\left\| \left(\int_0^\infty |a(x, t)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^1(\mathbb{R}^n)} \leq C.$$

In fact, by Hölder's inequality and the definition of the T_2^1 atom (see Section 2), we have

$$\int_{\mathbb{R}^n} \left(\int_0^\infty |a(y, t)|^2 \frac{dt}{t} \right)^{1/2} dy = \int_Q \left(\int_0^{\ell(Q)} |a(y, t)|^2 \frac{dt}{t} \right)^{1/2} dy \\ \leq \left(\iint_{\hat{Q}} |a(y, t)|^2 \frac{dy dt}{t} \right)^{1/2} |Q|^{1/2} \leq C.$$

Therefore, we get (4.26). Let $F = \left(\int_0^\infty |\psi_{t_1} \psi_{t_2} * f(x_1, x_2)|^2 \frac{dt_2}{t_2} \right)^{1/2}$. Substituting F back into (4.26), we have

$$\int_{\mathbb{R}^n} \left(\int_0^\infty \int_0^\infty |\psi_{t_1} \psi_{t_2} * f(x_1, x_2)|^2 \frac{dt_2}{t_2} \frac{dt_1}{t_1} \right)^{1/2} dx_1 \\ \leq C \int_{\mathbb{R}^n} \left(\iint_{\Gamma(x_1)} \int_0^\infty |\psi_{t_1} \psi_{t_2} * f(y_1, x_2)|^2 \frac{dt_2}{t_2} \frac{dy_1 dt_1}{t_1^{n+1}} \right)^{1/2} dx_1.$$

Integrating x_2 on both sides of the inequality above, we obtain

$$(4.27) \quad \iint \left(\int_0^\infty \int_0^\infty |\psi_{t_1} \psi_{t_2} * f(x_1, x_2)|^2 \frac{dt_2}{t_2} \frac{dt_1}{t_1} \right)^{1/2} dx_1 dx_2 \\ \leq C \iint \left(\iint_{\Gamma(x_1)} \|\psi_{t_1} \psi_{t_2} * f(y_1, x_2)\|_B^2 \frac{dy_1 dt_1}{t_1^{n+1}} \right)^{1/2} dx_1 dx_2.$$

(4.25), together with (4.27), implies (4.24).

4.2. $H^1(\mathbb{R}^n \times \mathbb{R}^m) \simeq H_{S, \mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m)$. It is easy to see that $\|f\|_{H_{\mathcal{N}, S}^1(\mathbb{R}^n \times \mathbb{R}^m)} \leq \|f\|_{H_{S, \mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m)}$, which, together with the Main Theorem, implies $H_{S, \mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m) \subset H^1(\mathbb{R}^n \times \mathbb{R}^m)$.

The proof of the inclusion $H^1(\mathbb{R}^n \times \mathbb{R}^m) \subset H_{S, \mathcal{N}}^1(\mathbb{R}^n \times \mathbb{R}^m)$ can be obtained by the similar argument to that of Method 1 of Proposition 3.2.

4.3. By Subsections 4.1 and 4.2, we can obtain all the equivalent characterizations of $H^1(\mathbb{R}^n \times \mathbb{R}^m)$ mentioned in Theorem 1.2.

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REFERENCES

- [1] S.-Y. A. Chang and R. Fefferman, The Calderón-Zygmund decomposition on product domains, *Amer. J. Math.*, 104 (1982), 445–468. MR658542 (84a:42028)
- [2] S.-Y. A. Chang and R. Fefferman, Some recent developments in Fourier analysis and H^p -theory on product domains, *Bull. Amer. Math. Soc. (N.S.)*, 12 (1985), 1–43. MR766959 (86g:42038)
- [3] R.R. Coifman, Y. Meyer and E.M. Stein, Some new function spaces and their applications to harmonic analysis, *J. Funct. Anal.*, 62 (1985), 304–335. MR791851 (86i:46029)
- [4] R. Fefferman, Calderón-Zygmund theory for product domains: H^p spaces, *Proc. Nat. Acad. Sci. USA*, 83 (1986), 840–843. MR828217 (87h:42032)
- [5] R. Fefferman, Multiparameter Fourier analysis, Beijing Lecture in Harmonic Analysis, edited by E.M. Stein, 47–130, Princeton Univ. Press, Princeton, NJ, 1986. MR864371 (89a:42001)
- [6] C. Fefferman and E.M. Stein, H^p spaces of several variables, *Acta Math.*, 129 (1972), 137–193. MR0447953 (56:6263)
- [7] R. Gundy and E.M. Stein, H^p theory for the poly-disc, *Proc. Nat. Acad. Sci.*, 76 (1979), 1026–1029. MR524328 (80j:32012)
- [8] Y.S. Han, M.Y. Lee, C.C. Lin and Y.C. Lin, Calderón-Zygmund operators on product Hardy spaces, *J. Funct. Anal.*, 258 (2010), 2834–2861. MR2593346
- [9] Y.S. Han, G.Z. Lu and K. Zhao, Discrete Calderón’s identity, atomic decomposition and boundedness criterion of operators on multiparameter Hardy spaces, *J. Geom. Anal.*, 20 (2010), 670–689. MR2610894
- [10] J.L. Journé, A covering lemma for product spaces, *Proc. Amer. Math. Soc.*, 96 (1986), 593–598. MR826486 (87g:42028)
- [11] J.L. Journé, Calderón-Zygmund operators on product spaces, *Rev. Mat. Iberoamericana*, 1 (1985), 55–91. MR836284 (88d:42028)
- [12] J.L. Journé, Two problems of Calderón-Zygmund theory on product spaces, *Ann. Inst. Fourier (Grenoble)*, 38 (1988), 111–132. MR949001 (90b:42031)
- [13] M. Lacey, S. Petermichl, J. Pipher and B. Wick, Multiparameter Riesz commutators, *Amer. J. Math.*, 131 (2009), 731–769. MR2530853 (2010d:42021)

- [14] K. Merryfield, On the area integral, Carleson measures and H^p in the polydisc, Indiana Univ. Math. J., 34 (1985), 663–685. MR794581 (87c:42023)
- [15] C. Muscalu, J. Pipher, T. Tao and C. Thiele, Bi-parameter paraproducts, Acta Math., 193 (2004), 269–296. MR2134868 (2005m:42028)
- [16] C. Muscalu, J. Pipher, T. Tao and C. Thiele, Multi-parameter paraproducts, Rev. Mat. Iberoamericana, 22 (2006), 963–976. MR2320408 (2008b:42037)
- [17] E.M. Stein, Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton, NJ, 1993. MR1232192 (95c:42002)
- [18] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, NJ, 1970. MR0290095 (44:7280)

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