TIGHT EMBEDDING OF SUBSPACES OF L_p IN ℓ_p^n FOR EVEN p

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(Communicated by Thomas Schlumprecht)

ABSTRACT. Using a recent result of Batson, Spielman and Srivastava, we obtain a tight estimate on the dimension of ℓ_p^n , p an even integer, needed to almost isometrically contain all k -dimensional subspaces of L_p .

In a recent paper [BSS] Batson, Spielman and Srivastava introduced a new method for sparsification of graphs which already proved to have functional analytic applications. Here we bring one more such application. Improving over a result of [BLM] (or see [JS] for a survey on this and related results), we show that for even p and for k of order $n^{2/p}$, any k -dimensional subspace of L_p nicely embeds into ℓ_p^n . This removes a log factor from the previously known estimate. The result in Theorem 2 is actually sharper than stated here and gives the best possible result in several respects, in particular in the dependence of k on n .

The theorem of [BSS] that we shall use is not specifically stated in [BSS], but it is stated as Theorem 1.6 in Srivastava's thesis [Sr]:

Theorem 1 ([BSS]). *Suppose $0 < \varepsilon < 1$ and $A = \sum_{i=1}^m v_i v_i^T$ are given with v_i column vectors in $\mathbb{R}^k$. Then there are nonnegative weights $\{s_i\}_{i=1}^m$, at most $\lceil k/\varepsilon^2 \rceil$ of which are nonzero, such that, putting $\tilde{A} = \sum_{i=1}^m s_i v_i v_i^T$,*

$$(1 - \varepsilon)^{-2} x^T A x \leq x^T \tilde{A} x \leq (1 + \varepsilon)^2 x^T A x$$

for all $x \in \mathbb{R}^k$.

Corollary 1. *Let X be a k -dimensional subspace of ℓ_2^m and let $0 < \varepsilon < 1$. Then there is a set $\sigma \subset \{1, 2, \dots, m\}$ of cardinality $n \leq C\varepsilon^{-2}k$ (C an absolute constant) and positive weights $\{s_i\}_{i \in \sigma}$ such that*

$$\|x\|_2 \leq \left(\sum_{i \in \sigma} s_i x^2(i) \right)^{1/2} \leq (1 + \varepsilon) \|x\|_2$$

for all $x = (x(1), x(2), \dots, x(m)) \in X$.

Proof. Let $u_1, u_2, \dots, u_k$ be an orthonormal basis for X ; $u_j = (u_{1,j}, u_{2,j}, \dots, u_{m,j})$, $j = 1, \dots, k$. Put $v_i^T = (u_{i,1}, u_{i,2}, \dots, u_{i,k})$, $i = 1, \dots, m$. Then $\sum_{i=1}^m v_i v_i^T = I_k$,

Received by the editors October 23, 2010.

2010 *Mathematics Subject Classification.* Primary 46B07.

The author was supported by the Israel Science Foundation and by the U.S.-Israel Binational Science Foundation.

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the $k \times k$ identity matrix. Let s_i be the weights given by Theorem 1 (and σ their support). Then, for all $x = \sum_{i=1}^k a_i u_i \in X$,

$$(1 - \varepsilon)^{-2} \|x\|_2 = a^T \sum_{i=1}^m v_i v_i^T a \leq a^T \sum_{i=1}^m s_i v_i v_i^T a \leq (1 + \varepsilon)^2 \|x\|_2.$$

Finally, notice that, for each $i = 1, \dots, m$, $a^T v_i v_i^T a = x(i)^2$, the square of the i -th coordinate of x . Thus,

$$a^T \sum_{i=1}^m s_i v_i v_i^T a = \sum_{i=1}^m s_i x(i)^2. \quad \square$$

We first prove a simpler version of the main result.

Proposition 1. *Let X be a k -dimensional subspace of L_p for some even p and let $0 < \varepsilon \leq 1/p$. Then X $(1 + \varepsilon)$ -embeds in ℓ_p^n for $n = O((\varepsilon p)^{-2} k^{p/2})$.*

Proof. Assume as we may that X is a k -dimensional subspace of ℓ_p^m for some finite m . Consider the set of all vectors which are coordinatewise products of $p/2$ vectors from X ; i.e. of the form

$$(x_1(1)x_2(1) \dots x_{p/2}(1), x_1(2)x_2(2) \dots x_{p/2}(2), \dots, x_1(m)x_2(m) \dots x_{p/2}(m)),$$

where $x_j = (x_j(1), x_j(2), \dots, x_j(m))$, $j = 1, 2, \dots, p/2$, are elements of X . We shall denote the vector above as $x_1 \cdot x_2 \cdot \dots \cdot x_{p/2}$. The span of this set in $\mathbb{R}^m$, which we denote by $X^{p/2}$, is clearly a linear space of dimension at most $k^{p/2}$. Consequently, by Corollary 1, there is a set $\sigma \subset \{1, 2, \dots, m\}$ of cardinality at most $C(\varepsilon p)^{-2} k^{p/2}$ and weights $\{s_i\}_{i \in \sigma}$ such that

$$\|y\|_2 \leq \left(\sum_{i \in \sigma} s_i y^2(i) \right)^{1/2} \leq \left(1 + \frac{\varepsilon p}{4} \right) \|y\|_2$$

for all $y \in X^{p/2}$. Restricting to y -s of the form

$$y = (x(1)^{p/2}, x(2)^{p/2}, \dots, x(m)^{p/2})$$

with $x = (x(1), x(2), \dots, x(m)) \in X$, we get

$$\|x\|_p^{p/2} \leq \left(\sum_{i \in \sigma} s_i x^p(i) \right)^{1/2} \leq \left(1 + \frac{\varepsilon p}{4} \right) \|x\|_p^{p/2}.$$

Raising these inequalities to the power $2/p$ gives the result. $\square$

We now state and prove the main result.

Theorem 2. *Let X be a k -dimensional subspace of L_p for some even $p \leq k$ and let $0 < \varepsilon \leq 1/p$. Then X $(1 + \varepsilon)$ -embeds in ℓ_p^n for $n = O(\varepsilon^{-2} (\frac{10k}{p})^{p/2})$. Equivalently, for some universal $c > 0$, for any n and any $k \leq c\varepsilon^{4/p} p n^{2/p}$, any k -dimensional subspace of L_p $(1 + \varepsilon)$ -embeds in ℓ_p^n .*

Proof. The only change from the previous proof is a better estimate of the dimension of the auxiliary subspace involved. An examination of the proof above shows that it is enough to apply Corollary 1 to any subspace containing all the vectors $x^{p/2} = x \cdot x \cdot \dots \cdot x$ ($p/2$ times), $x = (x(1), \dots, x(m)) \in X$. The smallest such subspace is the space of degree $p/2$ homogeneous polynomials in elements of X . Its

basis is the set of monomials of the form $u_{j_1}^{p_1} \cdot u_{j_2}^{p_2} \cdots u_{j_l}^{p_l}$ with $p_1 + \cdots + p_l = p/2$, where $u_1, \dots, u_k$ is a basis for X . The dimension of this space, which is the number of such monomials, is $\binom{k+p/2-1}{p/2} \leq (\frac{10k}{p})^{p/2}$. $\square$

Next we remark on the estimate $k \leq c\varepsilon^{4/p}pn^{2/p}$. This estimate improves (unfortunately, only for even p) on the known estimates (the best of which is in [BLM]) by removing a $\log n$ factor that was present in the best estimate until now. Also, the p in front of the $n^{2/p}$ is a nice feature. It is known that the dependence of k on p and n in this estimate is best possible even if one restricts to subspaces of L_p isometric to ℓ_2^k (see [BDGJN]). Actually the result above indicates that ℓ_2^k are the “worst” subspaces.

As for the dependence on ε , the published proofs give at best quadratic dependence, while here we get a linear dependence for $p = 4$ and better ones as p grows. For the special case of $X = \ell_2^k$ and $p = 4$, a better result is known: One can embed ℓ_2^k isometrically into $\ell_4^{4n^2}$ ([Ko]). But for $p = 6, 8, \dots$ we get a better result here, even for this special case, than was previously known. (The best the author knew was a linear dependence on ε ; this is unpublished. Here we get $\varepsilon^{2/3}$ for $p = 6$ and better for larger p -s.) It is not clear that this is the best possible dependence on ε , but note that for some combinations of ε and p and in particular for every ε and $p \sim 1/\varepsilon$ (and $k \geq p$), the dependence on ε becomes a constant, and, up to universal constants, we then get the best possible result with respect to all parameters.

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