SPECTRAL PROPERTIES OF COMMUTING OPERATIONS FOR n -TUPLES

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ABSTRACT. Let $\mathbf{R}$ and $\mathbf{S}$ be commuting n -tuples. We give some spectral and local spectral relations between $\mathbf{RS}$ and $\mathbf{SR}$. In particular, we show that $\mathbf{RS}$ has the single valued extension property or satisfies Bishop's property (β) if and only if $\mathbf{SR}$ has the corresponding property.

1. INTRODUCTION

Let X be a Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X . Given two bounded operators R and S defined on X , Jacobson's lemma says

$$\sigma(RS) \setminus \{0\} = \sigma(SR) \setminus \{0\}.$$

Several works have been devoted to the previous equation, which has been extended as follows:

$$(1) \quad \Sigma(RS) \setminus \{0\} = \Sigma(SR) \setminus \{0\},$$

where Σ runs over a large class of spectra. See [2, 3] for example.

We showed in [3] that RS and SR also share most of their local spectral properties. In particular, If RS has the single valued extension property (*SVEP*) (resp. Bishop's property (β)), then SR has (*SVEP*) (resp. Bishop's property (β)).

Jacobson's lemma does not extend to pairs of systems of two or more variables: for example if $UV = I \neq VU$, then it fails for the pair $(\mathbf{R}, \mathbf{S}) = (\frac{1}{2}I, \frac{1}{2}V), (I - UV, U)$. With enough mutual commutativity however, analogues of (1) can be extended to a product system $\mathbf{RS}$.

Recall that an operator $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{L}(X)^n$ is called a commuting n -tuple if $T_i T_j = T_j T_i$ for every $1 \leq i, j \leq n$. The notion of spectrum for commuting n -tuples was introduced by J. L. Taylor in [13]; see also [17]. Various spectra have been adapted to this situation using Taylor's formalism.

The main goal of this paper is to develop analogues of the preceding results in the setting of n -tuples where all spectra are considered in the sense of Taylor. To this aim, let $\mathbf{R} = (R_1, \dots, R_n)$ and $\mathbf{S} = (S_1, \dots, S_n)$ be commuting n -tuples and consider $\mathbf{RS} = (R_1 S_1, \dots, R_n S_n)$. To ensure that $\mathbf{RS}$ remains a commuting

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n -tuple, suitable extra conditions are needed. The first condition is the so-called “criss-cross commutativity” and has been intensively studied in the literature.

Recall that the commuting n -tuples $\mathbf{R} = (R_1, \dots, R_n)$ and $\mathbf{S} = (S_1, \dots, S_n)$ are said to be criss-cross commuting provided that

$$(2) \quad R_i S_j R_k = R_k S_j R_i \text{ and } S_i R_j S_k = S_k R_j S_i \text{ for every } 1 \leq i, j, k \leq n.$$

It follows that $\mathbf{SR}$ and $\mathbf{RS}$ are commuting n -tuples [9, 10], and then equation (1) is obtained for several Taylor spectra. See [5] and [11].

The second commutativity condition is given in [4]. It is assumed that $\mathbf{R}$ and $\mathbf{S}$ are commuting n -tuples. We will say that $\mathbf{R}$ and $\mathbf{S}$ are nearly commuting provided that

$$(3) \quad R_i S_j = S_j R_i \quad \text{for every } i \neq j.$$

Again, under near commutativity, $\mathbf{SR}$ and $\mathbf{RS}$ are commuting n -tuples, and they share their spectral properties except in

$$[0] := \{(z_1, \dots, z_n) : \prod_{i=1}^n z_i = 0\}.$$

More precisely,

$$(4) \quad \Sigma(RS) \setminus [0] = \Sigma(SR) \setminus [0],$$

where Σ runs over a large class of Taylor spectra.

Remark 1. Before proceeding further, we mention as observed in [3] that near commutativity and criss-cross commutativity are distinct. In [3], examples of n -tuples satisfying only near commutativity and criss-cross commutativity are given.

We pursue in this paper our investigations into common spectral properties of the n -tuples $\mathbf{RS}$ and $\mathbf{SR}$ under any of the previous commutativity conditions. In section 2 we introduce basic definitions of Taylor spectrum and give the corresponding local spectral properties. In section 3 we retrieve many results on common spectral properties of $\mathbf{RS}$ and $\mathbf{SR}$. The given proofs unify most of the earliest results in this context. Section 4 is devoted to local spectral theory for commuting n -tuples. We show that if $\mathbf{RS}$ has *SVEP* or Bishop’s property (β) , then so has $\mathbf{SR}$. We end in section 5 with some results involving local spectra of $\mathbf{RS}$ and $\mathbf{SR}$.

2. BASIC DEFINITIONS

Let $\mathbf{e} = \{e_1, e_2, \dots, e_n\}$ be indeterminate and define $\Lambda_n[\mathbf{e}]$ to be the *exterior algebra* on the generators $e_1, e_2, \dots, e_n$, that is, the linear space over the complex plane $\mathbb{C}$ endowed with an anti-commutative exterior product $e_i \wedge e_j = -e_j \wedge e_i$ ($1 \leq i, j \leq n$). For $F = (i_1, \dots, i_p)$ $\{F\} = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ with $i_1 < \dots < i_p$, we write $e_F = e_{i_1} \wedge \dots \wedge e_{i_p}$. The exterior algebra over $\mathbb{C}$ is then given by

$$\Lambda_n[\mathbf{e}] = \left\{ \sum_F \alpha_F e_F : e_F = e_{i_1} \wedge \dots \wedge e_{i_p} \text{ and } \alpha_F \in \mathbb{C} \right\}.$$

The identity element for the exterior product will be denoted by $e_\emptyset$. Consider $\Lambda_n^k[\mathbf{e}] =: \left\{ \sum_{|F|=k} \alpha_F e_F : \alpha_F \in \mathbb{C} \right\}$, where $|F|$ is the *cardinal* of $\{F\}$. Then clearly $\dim \Lambda_n^k[\mathbf{e}] = C_n^k$ for every $k \leq n$, $\Lambda_n^k[\mathbf{e}] \wedge \Lambda_n^l[\mathbf{e}] = \Lambda_n^{k+l}[\mathbf{e}]$ and $\Lambda_n[\mathbf{e}] = \bigoplus_{k=0}^n \Lambda_n^k$.

Given a Banach space X , the exterior algebra over X is defined to be

$$\Lambda_n[\mathbf{e}, X] = \left\{ \sum_F x_F e_F : e_F = e_{i_1} \wedge \cdots \wedge e_{i_p} \text{ and } x_F \in X \right\}.$$

The subspaces $\Lambda_n^p[\mathbf{e}, X] = \left\{ \sum_{|F|=p} x_F e_F : x_F \in X \right\}$ are given for $p \leq n$ in a similar way. Naturally $\Lambda_n^0[\mathbf{e}, X]$, $\Lambda_n^1[\mathbf{e}, X]$ and $\Lambda_n^n[\mathbf{e}, X]$ can be identified with X , X^n and X respectively.

Since no confusion is possible we will omit any reference to the indeterminate $e_1, e_2, \dots, e_n$, and for short we will write $\Lambda_n^k[X]$ and $\Lambda_n[X]$ for $\Lambda_n^k[\mathbf{e}, X]$ and $\Lambda_n[\mathbf{e}, X]$ respectively.

For $S \in \mathcal{L}(X)$, the same symbol S is usually kept to denote the extended operator defined on $\Lambda_n[X]$ by

$$\Lambda_n(S) : S\left(\sum_F x_F e_F\right) = \sum_F Sx_F e_F.$$

Given $i \in \{1, 2, \dots, n\}$, let $E_i : \Lambda_n[X] \rightarrow \Lambda_n[X]$, $E_i(e_F) = e_i \wedge e_F$ be the left multiplication operator induced by e_i . With any commuting n -tuple $\mathbf{T} = (T_1, \dots, T_n)$ we associate the linear mapping defined over $\Lambda_n[X]$ by

$$\alpha_{\mathbf{T}} = \sum_{i=1}^n T_i \otimes E_i : \sum_F x_F e_F \rightarrow \sum_F \sum_{i=1}^n T_i x_F e_i \wedge e_F.$$

Clearly,

$$(\alpha_{\mathbf{T}})^2 x_F e_F = \sum_{i,j=1}^n T_j T_i x_F e_j \wedge e_i \wedge e_F = \sum_{i < j} T_i T_j x_F (e_i \wedge e_j + e_j \wedge e_i) \wedge e_F = 0.$$

It follows that $(\alpha_{\mathbf{T}})^2 = 0$ and hence that $\text{Im}(\alpha_{\mathbf{T}}) \subset \ker(\alpha_{\mathbf{T}})$. Since $\alpha_{\mathbf{T}}$ takes $\Lambda_n^{k-1}[X]$ to $\Lambda_n^k[X]$, the last statement is equivalent to: $\text{Im}(\alpha_{\mathbf{T}}|_{\Lambda_n^{k-1}[X]}) \subset \ker(\alpha_{\mathbf{T}}|_{\Lambda_n^k[X]})$ for every $k \leq n$.

Set $\alpha_{\mathbf{T}}^k := \alpha_{\mathbf{T}}|_{\Lambda_n^k[X]}$. We construct a co-chain complex $K(\mathbf{T}, X)$, called the *Koszul complex* associated with $\mathbf{T}$ on X , as follows:

$$K(\mathbf{T}, X) : \quad \mathbf{0} \xrightarrow{\alpha_{\mathbf{T}}^{-1}} \Lambda_n^0[X] \xrightarrow{\alpha_{\mathbf{T}}^0} \Lambda_n^1[X] \xrightarrow{\alpha_{\mathbf{T}}^1} \cdots \xrightarrow{\alpha_{\mathbf{T}}^{n-1}} \Lambda_n^n[X] \xrightarrow{\alpha_{\mathbf{T}}^n} \mathbf{0}.$$

The operator $\mathbf{T}$ is said to be non-singular, or Taylor invertible, if $\ker(\alpha_{\mathbf{T}}^k) = \text{Im}(\alpha_{\mathbf{T}}^{k-1})$ for $k = 0, \dots, n$. The associated Koszul complex is said to be exact in this case. The Taylor spectrum of $\mathbf{T}$ on X^n is then the set

$$\sigma_T(\mathbf{T}) = \{\lambda \in \mathbb{C}^n : K(\mathbf{T} - \lambda, X) \text{ is not exact}\}.$$

The Taylor spectrum coincides with the usual spectrum in the case of single operators. We refer to [13, 15, 16, 17] for a detailed study of these spectra.

These notions are taken to sheaf theory canonically to define the corresponding local spectral properties. For $\mathcal{U}$ an open set in $\mathbb{C}^n$, denote by $\mathcal{O}(\mathcal{U}, X)$ the Fréchet algebra of analytic functions on $\mathcal{U}$. We define

$$K(\mathbf{T} - \mathbf{z}) : \mathbf{0} \xrightarrow{\alpha_{\mathbf{T}-\mathbf{z}}^{-1}} \Lambda_n^0[\mathcal{O}(\mathcal{U}, X)] \xrightarrow{\alpha_{\mathbf{T}-\mathbf{z}}^0} \Lambda_n^1[\mathcal{O}(\mathcal{U}, X)] \xrightarrow{\alpha_{\mathbf{T}-\mathbf{z}}^1} \cdots \xrightarrow{\alpha_{\mathbf{T}-\mathbf{z}}^{n-1}} \Lambda_n^n[\mathcal{O}(\mathcal{U}, X)] \xrightarrow{\alpha_{\mathbf{T}-\mathbf{z}}^n} \mathbf{0},$$

and we consider the associated cohomology groups

$$H^k(\mathbf{T} - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) = \ker(\alpha_{\mathbf{T}-\mathbf{z}}^k) / \text{Im}(\alpha_{\mathbf{T}-\mathbf{z}}^{k-1}), \quad k = 1, \dots, n.$$

Let $\mathbf{T}$ be a commuting n -tuple. We will say that $\mathbf{T}$ has the *SVEP* at λ if there exists an open polydisc D_λ centered at λ such that $H^k(\mathcal{O}(D_\lambda, X), \mathbf{T}) = 0$ for $k = 1, \dots, n-1$.

Following the definition given by J. Eschmeier [6], a commuting n -tuple $\mathbf{T}$ has *SVEP* provided that the cohomology groups $H^k(\mathcal{O}(\mathcal{U}(z), X)) = \{0\}$ for $k = 1, \dots, n-1$ and for every $z \in \mathbb{C}$. As usual $\mathcal{O}(\mathcal{U}(z), X)$ stands for the space of all X -valued analytic functions near $\{z\}$. It is then not difficult to show that $\mathbf{T}$ has *SVEP* in the sense of [6] if and only if $\mathbf{T}$ has *SVEP*, at every $\lambda \in \mathbb{C}$.

A detailed study of the *SVEP* for commuting n -tuples is developed in [6]. It is in particular shown that if $(T_1, \dots, T_n)$ has *SVEP*, then $(T_1, \dots, T_{n-1})$ has *SVEP*, and the same for $(T_{\sigma(1)}, \dots, T_{\sigma(n)})$ for every permutation σ of $\{1, \dots, n\}$.

Other notions including the (β) condition and spectral decomposition are defined in a similar way. We refer to [7, 16] for more details.

3. COMMON SPECTRAL PROPERTIES BETWEEN $\mathbf{RS}$ AND $\mathbf{SR}$

Using a nice lemma from [7], we retrieve in a simple way several results given in [4, 9, 10, 11]. We thank J. Eschmeier for fruitful discussions and for pointing out to us this next lemma.

Lemma 1 ([7, Lemma 2.2.4]). *Let $a = (a_1, \dots, a_n)$ be a commuting n -tuple of morphisms of module X . Let $b_1, \dots, b_n$ belong to the commutant of a and consider $c = a_1 b_1 + \dots + a_n b_n$. Then $M_c : H^p(a, X) \rightarrow H^p(a, X)$, induced by the componentwise action of c , is the zero operator for each p .*

3.1. Common spectral properties under near commutativity. Let $\mathbf{R} = (R_1, \dots, R_n) \in \mathcal{L}(X)$ and $\mathbf{S} = (S_1, \dots, S_n) \in \mathcal{L}(X)$ be commuting n -tuples that are nearly commuting. We fix $i \in \{1, \dots, n\}$ and we set $\mathbf{SR} = (S_1 R_1, \dots, S_i R_i, \dots, S_n R_n)$ and $(\mathbf{SR})_i = (S_1 R_1, \dots, R_i S_i, \dots, S_n R_n)$.

Let $z \in \mathbb{C}^n$ be such that $z_i \neq 0$. The componentwise actions

$$\phi : u \in \Lambda^p[X] \mapsto R_i u \in \Lambda^p[X] \quad \text{and} \quad \psi : u \in \Lambda^p[X] \mapsto S_i u \in \Lambda^p[X]$$

define morphisms of complexes

$$\begin{aligned} K(\mathbf{SR} - \mathbf{z}, X) &\xrightarrow{\phi} K((\mathbf{SR})_i - \mathbf{z}, X), \\ K((\mathbf{SR})_i - \mathbf{z}, X) &\xrightarrow{\psi} K(\mathbf{SR} - \mathbf{z}, X). \end{aligned}$$

The mapping $z_i I - \psi\phi$ is then given by the componentwise action and satisfies $z_i I - S_i R_i = (z_i - S_i R_i) = \sum_{v=1}^n (z_v - S_v R_v) C_v$, where $C_v = \delta_{iv} I \in \{\mathbf{SR}\}'$ for $v = 1, \dots, n$ (here $\{\mathbf{SR}\}'$ denotes the commutant of $\mathbf{SR}$). It follows from Lemma 1 that $\psi\phi = z_i I$ on $H^p(\mathbf{SR} - \mathbf{z}, X)$ for $p \in \{0, \dots, n\}$. Similarly, from the identity $z_i I - R_i S_i = (z_i - R_i S_i) = \sum_{v=1}^n (z_v - R_v S_v) C'_v$, where $C'_v = \delta_{iv} I \in \{(\mathbf{SR})_i\}'$ for $v = 1, \dots, n$, we get $\phi\psi = z_i I$ on $H^p((\mathbf{SR})_i - \mathbf{z}, X)$ for $p \in \{0, \dots, n\}$.

Thus,

$$\begin{aligned} \phi : H^p(\mathbf{SR} - \mathbf{z}, X) &\longrightarrow H^p((\mathbf{SR})_i - \mathbf{z}, X), \\ \psi : H^p((\mathbf{SR})_i - \mathbf{z}, X) &\longrightarrow H^p(\mathbf{SR} - \mathbf{z}, X) \end{aligned}$$

are vector-space bounded isomorphisms, for every $p = 0, \dots, n$, which are inverse to each other (because $z_i \neq 0$).

As an immediate consequence from our preceding discussion, we observe that $H^p(\mathbf{SR} - \mathbf{z}, X) = \{0\}$ (resp. is closed) if and only if $H^p((\mathbf{SR})_{\mathbf{i}} - \mathbf{z}, X) = \{0\}$ (resp. is closed). Using the definitions and notation from [7, chapter 2.6], we get

Proposition 1. *Let $\mathbf{SR}$ and $(\mathbf{SR})_{\mathbf{i}}$ be as above, $z \in \mathbb{C}^n$ be such that $z_i \neq 0$ and $k \in \{0, \dots, n\}$. Then:*

- (1) $z \in \sigma(\mathbf{SR}) \iff z \in \sigma((\mathbf{SR})_{\mathbf{i}}),$
- (2) $z \in \sigma_e(\mathbf{SR}) \iff z \in \sigma_e((\mathbf{SR})_{\mathbf{i}}),$
- (3) $z \in \sigma^{\delta,k}(\mathbf{SR}) \iff z \in \sigma^{\delta,k}((\mathbf{SR})_{\mathbf{i}}),$
- (4) $z \in \sigma_e^{\delta,k}(\mathbf{SR}) \iff z \in \sigma_e^{\delta,k}((\mathbf{SR})_{\mathbf{i}}),$
- (5) $z \in \sigma^{\pi,k}(\mathbf{SR}) \iff z \in \sigma^{\pi,k}((\mathbf{SR})_{\mathbf{i}}),$
- (6) $z \in \sigma_e^{\pi,k}(\mathbf{SR}) \iff z \in \sigma_e^{\pi,k}((\mathbf{SR})_{\mathbf{i}}).$

Let $\mathcal{I}$ be a subset of $\{1, \dots, n\}$ and denote $(\mathbf{RS})_{\mathcal{I}} = ((R_1 S_1)_{\mathcal{I}}, \dots, (R_n S_n)_{\mathcal{I}})$, with $(R_i S_i)_{\mathcal{I}} = S_i R_i$ if $i \in \mathcal{I}$ and $(R_i S_i)_{\mathcal{I}} = R_i S_i$ otherwise. Clearly $((\mathbf{RS})_{\mathcal{I}})_{\mathcal{I}} = \mathbf{RS}$ and $\mathbf{SR} = (\mathbf{RS})_{\{1, \dots, n\}}$. We also write

$$[0]^{\mathcal{I}} = \{(\lambda_1, \dots, \lambda_n) \in [0] : \prod_{i \in \mathcal{I}} \lambda_i \neq 0\}.$$

Now inductive reasoning in the previous proposition leads obviously to the next theorem.

Theorem 1. *Let $\mathbf{R} = (R_1, \dots, R_n) \in \mathcal{L}(X)$ and $\mathbf{S} = (S_1, \dots, S_n) \in \mathcal{L}(X)$ be commuting n -tuples nearly commuting.*

- For $\mathcal{I}, J \subset \{1, \dots, n\}$, we have

$$\Sigma((\mathbf{RS})_{\mathcal{I}}) \setminus [0]^{T \cup \mathcal{J}} = \Sigma((\mathbf{SR})_{\mathcal{J}}) \setminus [0]^{T \cup \mathcal{J}}.$$

- In particular,

$$\Sigma(\mathbf{RS}) \setminus [0] = \Sigma(\mathbf{SR}) \setminus [0]$$

$$\text{for } \Sigma \in \{\sigma, \sigma_e, \sigma^{\pi,k}, \sigma_e^{\pi,k}, \sigma^{\delta,k}, \sigma_e^{\delta,k}\}.$$

3.2. Common spectral properties under the criss-cross commutativity.

Let $\mathbf{R}$ and $\mathbf{S}$ be criss-cross commuting n -tuples. It is not difficult to check that $S_i \alpha_{\mathbf{RS}-\mathbf{z}} = \alpha_{\mathbf{SR}-\mathbf{z}} S_i$ for every $i = 1, \dots, n$. It will follow as in the first part of this section that for every $\mathbf{z} \neq (0, \dots, 0)$, the Koszul complexes $K(\mathbf{RS} - \mathbf{z}, X)$ and $K(\mathbf{SR} - \mathbf{z}, X)$ are isomorphic. Precisely, we write the same arguments with i such that $z_i \neq 0$. This yields to the next criss-cross version of Theorem 1.

Theorem 2. *Let $\mathbf{R} = (R_1, \dots, R_n) \in \mathcal{L}(X)$ and $\mathbf{S} = (S_1, \dots, S_n) \in \mathcal{L}(X)$ be commuting n -tuples satisfying the criss-cross commutativity condition. We have*

$$\Sigma(\mathbf{RS}) \setminus \{0\} = \Sigma(\mathbf{SR}) \setminus \{0\}$$

for any $\Sigma \in \{\sigma, \sigma_e, \sigma^{\pi,k}, \sigma_e^{\pi,k}, \sigma^{\delta,k}, \sigma_e^{\delta,k}\}$ and $0 \leq k \leq n$.

Theorem 2 provides a simple proof for various joint spectra and generalizes the results in [9, 10]. See also [11].

4. SINGLE VALUED EXTENSION PROPERTY AND BISHOP'S PROPERTY (β)
FOR **SR** AND **RS**

Recall that a single operator T satisfies the *SVEP* at $\lambda \in \mathbb{C}$ if there exists a neighborhood $\mathcal{U}$ of λ such that the zero function is the only analytic function f defined on $\mathcal{U}$ satisfying $(T - \mu)f(\mu) = 0$ for every $\mu \in \mathcal{U}$. The definition above can be viewed as a particular case of the n -tuples one (see [1]).

A result of Finch in [8] asserts that for an operator $T \in \mathcal{L}(X)$ with *SVEP* at zero we have: T is invertible if and only if $T(X) = X$. The following extension of Finch's result for n -tuples has been observed in [17].

Proposition 2. *Let $\mathbf{T}$ be a commuting n -tuple with *SVEP* at zero. Then, $\mathbf{T}$ is Taylor invertible if and only if $T_1(X) + T_2(X) + \cdots + T_n(X) = X$.*

Remark that the condition $T_1(X) + T_2(X) + \cdots + T_n(X) = X$ implies that $H^n(T) = \{0\}$, and from the proof of lemma 2.1 in [13], it follows that $H^n(\mathbf{T} - \lambda) = \{0\}$ for $\lambda \in D$, with D a polydisc containing zero. We deduce that $H^n(\mathcal{O}(\mathcal{U}(z), X)) = 0$ and hence that $\mathbf{T}$ is Taylor invertible.

The commuting n -tuple $\mathbf{T}$ is said to possess *Bishop's property* (β) if the Koszul complex $K.(T - z, \mathcal{O}(U, X))$ has a separated cohomology in degree n and it is exact in degree less than n for every open set $\mathcal{U} \subset \mathbb{C}^n$. Equivalently, $\mathbf{T}$ possesses (β) if and only if it has *SVEP*, and $\alpha_{\mathbf{T}-\mathbf{z}}^{n-1}(\mathcal{O}(\mathcal{U}, X))$ is closed for every open set $\mathcal{U} \subset \mathbb{C}^n$. The last assertion can be rephrased as follows: the canonical mapping

$$(5) \quad \Lambda^{n-1}[\mathcal{O}(\mathcal{U}, X)] / \ker(\alpha_{\mathbf{T}-\mathbf{z}}^{n-1}) \longrightarrow \text{Im}(\alpha_{\mathbf{T}-\mathbf{z}}^{n-1})$$

is bounded below for every open set $\mathcal{U} \subset \mathbb{C}^n$.

Moreover, outlining the same proof for *SVEP*, one may deduce that if $(T_1, \dots, T_n)$ possesses *Bishop's property* (β) , then $(T_1, \dots, T_{n-1})$ possesses *Bishop's property* (β) and also $(T_{\sigma(1)}, \dots, T_{\sigma(n)})$ for every permutation σ .

Of course, this also generalizes the definition of Bishop's property (β) for a single operator, which asserts that the map

$$T - z : \mathcal{O}(\mathcal{U}, X) \longrightarrow \mathcal{O}(\mathcal{U}, X)$$

is one to one with closed range for every open set $\mathcal{U} \subset \mathbb{C}$.

4.1. The case of near commutativity.

Theorem 3. *Let $\mathbf{R}$ and $\mathbf{S}$ be commuting n -tuples nearly commuting. Then $\mathbf{T} = (R_1 S_1, \dots, R_i S_i, \dots, R_n S_n)$ has *SVEP* if and only if $\mathbf{T}'_i = (R_1 S_1, \dots, S_i R_i, \dots, R_n S_n)$ has *SVEP* for every $i \in \{1, \dots, n\}$.*

*In particular, **RS** has *SVEP* if and only if **SR** has *SVEP*.*

The next lemma is crucial in our proof and is of independent interest. We will assume for convenience that $i = n$.

Lemma 2. Let $\mathcal{U}$ be an open neighborhood of zero in $\mathbb{C}^n$, $\mathbf{T} = (T_1, \dots, T_n)$ be a bounded commuting n -tuple such that $\hat{\mathbf{T}} = (T_1, \dots, T_{n-1})$ has *SVEP* at zero (in $\mathbb{C}^{n-1}$) and M_{z_n} be the pointwise multiplication by the independent variable z_n . We have

$$M_{z_n} H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) = \{0\} \implies H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) = \{0\}.$$

In particular, if $(T_1, \dots, T_{n-1})$ has *SVEP*, then the claim $M_{z_n} H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) = \{0\}$ yields to $H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) = \{0\}$ for every open set $\mathcal{U}$.

Proof of Lemma 2. We first fix some notation. For $g = \sum_I g_I e_I \in \Lambda[\mathcal{O}(\mathcal{U}, X)]$, we denote

$$\rho(g(z_1, \dots, z_n)) = g(z_1, \dots, z_{n-1}, 0) \text{ and } \pi(g) = \rho\left(\sum_{n \notin I} g_I e_I\right).$$

For simplicity, we shall use the notation $\delta_{\mathbf{T}}$ for the operator $\alpha_{\mathbf{T}-\mathbf{z}}$. Let $f \in \ker(\delta_{\mathbf{T}}^p)$ be such that $z_n f = \delta_{\mathbf{T}}(g)$ for some $g \in \Lambda^{p-1}[\mathcal{O}(\mathcal{U}, X)]$. We have

$$\begin{aligned} 0 = \rho(z_n f) &= \rho\left(\sum_{n \notin I} \sum_{i=1}^{n-1} (T_i - z_i) g_I e_i \wedge e_I\right) \\ &\quad + \rho\left(\sum_{n \notin I} (T_n - z_n) g_I e_n \wedge e_I + \sum_{n \in I} \sum_{i=1}^{n-1} (T_i - z_i) g_I e_i \wedge e_I\right). \end{aligned}$$

Hence we have

$$(6) \quad \rho\left(\sum_{n \notin I} \sum_{i=1}^{n-1} (T_i - z_i) g_I e_i \wedge e_I\right) = 0$$

and

$$(7) \quad \rho\left(\sum_{n \notin I} (T_n - z_n) g_I e_n \wedge e_I + \sum_{n \in I} \sum_{i=1}^{n-1} (T_i - z_i) g_I e_i \wedge e_I\right) = 0.$$

Equation (6), together with the *SVEP* of $\hat{\mathbf{T}} = (T_1, \dots, T_{n-1})$, yields to the existence of $h = \sum_{J \subset \{1, \dots, n-1\}} h_J e_J$ such that $\pi(g) = \delta_{\hat{\mathbf{T}}}(h)$. That is,

$$\rho\left(\sum_{n \in I} \sum_{i=1}^{n-1} (T_i - z_i) g_I e_i \wedge e_I\right) = \sum_{i=1}^{n-1} (T_i - z_i) h_J e_i \wedge e_J.$$

Now, equation (7) gives

$$\begin{aligned} 0 &= \rho\left(\sum_{n \notin I} (T_n - z_n) g_I e_n \wedge e_I + \sum_{n \notin I} \sum_{i=1}^{n-1} (T_i - z_i) g_I e_i \wedge e_I\right) \\ &= e_n \wedge [T_n \rho\left(\sum_{n \notin I} g_I e_I\right) - (-1)^p \rho\left(\sum_{n \in I} \sum_{i=1}^{n-1} (T_i - z_i) g_I e_i \wedge e_{I \setminus \{n\}}\right)], \end{aligned}$$

and again using *SVEP* of the operator $\hat{\mathbf{T}}$, we deduce that $T_n h - (-1)^p \tilde{\pi}(g) = \delta_{\hat{\mathbf{T}}}(k)$ for some $k \in \Lambda[\mathcal{O}(\mathcal{U}, X)]$. Here, we set $\tilde{\pi}(g) = \rho\left(\sum_{n \in I} g_I e_{I \setminus \{n\}}\right)$. Combining this last expression with the formula $\rho(g) = \pi(g) + \tilde{\pi}(g) \wedge e_n$, we claim that

$G = g - \rho(g) + (-1)^p z_n h \wedge e_n$ satisfies $z_n f = \delta_{\mathbf{T}}(G)$. Indeed,

$$\begin{aligned}
\delta_{\mathbf{T}}(G) &= \delta_{\mathbf{T}}(g) - \delta_{\mathbf{T}}(\rho(g)) + \delta_{\mathbf{T}}((-1)^p z_n h \wedge e_n) \\
&= z_n f - \delta_{\mathbf{T}}[\pi(g) + \tilde{\pi}(g) \wedge e_n] + (-1)^p z_n \delta_{\hat{\mathbf{T}}} h \wedge e_n \\
&= z_n f - \delta_{\mathbf{T}}\pi(g) - \delta_{\hat{\mathbf{T}}}\tilde{\pi}(g) \wedge e_n + (-1)^p z_n \delta_{\hat{\mathbf{T}}} h \wedge e_n \\
&= z_n f - [\delta_{\hat{\mathbf{T}}} + (T_n - z_n) \otimes E_n] \delta_{\hat{\mathbf{T}}}(h) - \delta_{\hat{\mathbf{T}}}\tilde{\pi}(g) \wedge e_n + (-1)^p z_n \delta_{\hat{\mathbf{T}}} h \wedge e_n \\
&= z_n f - (-1)^{p-1} (T_n - z_n) \delta_{\hat{\mathbf{T}}}(h) \wedge e_n - \delta_{\hat{\mathbf{T}}}\tilde{\pi}(g) \wedge e_n + (-1)^p z_n \delta_{\hat{\mathbf{T}}} h \wedge e_n \\
&= z_n f - (-1)^{p-1} [T_n \delta_{\hat{\mathbf{T}}}(h) - (-1)^p \delta_{\hat{\mathbf{T}}}\tilde{\pi}(g)] \wedge e_n \\
&= z_n f + (-1)^p \delta_{\hat{\mathbf{T}}} [T_n h - (-1)^p \tilde{\pi}(g)] \wedge e_n \\
&= z_n f + (-1)^p \delta_{\hat{\mathbf{T}}}(\delta_{\hat{\mathbf{T}}} k) \wedge e_n \\
&= z_n f.
\end{aligned}$$

To end the proof of the lemma, we notice that $H(z) = \frac{G(z)}{z_n}$ is analytic and satisfies $\delta_{\mathbf{T}}(H) = f$.

Proof of Theorem 3. Let $z \in \mathbb{C}^n$ be arbitrary. The componentwise actions

$$\begin{cases} \Phi : \Lambda^p[\mathcal{O}(\mathcal{U}, X)] \longrightarrow \Lambda^p[\mathcal{O}(\mathcal{U}, X)], & u \mapsto R_n u, \\ \Psi : \Lambda^p[\mathcal{O}(\mathcal{U}, X)] \longrightarrow \Lambda^p[\mathcal{O}(\mathcal{U}, X)], & u \mapsto S_n u \end{cases}$$

induce the next morphisms of complexes

$$\begin{aligned}
K.(\mathbf{T} - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) &\xrightarrow{\Phi} K.(\mathbf{T}' - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)), \\
K.(\mathbf{T}' - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) &\xrightarrow{\Psi} K.(\mathbf{T} - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)),
\end{aligned}$$

since $I - \psi\phi$ is given by componentwise action with

$$z_n I - S_n R_n = \sum_{v=1}^n (z_v - T_v) C_v,$$

where $C_v = \delta_{n,v} I \in \{\mathbf{T}'\}'$ for $v = 1, \dots, n$. It follows (Lemma 2.2.4 in [7]) that $\Psi\Phi = M_{z_n}$ is the pointwise multiplication by the variable z_n on $H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(D, X))$.

On the other hand, from the identity $z_n I - R_n S_n = \sum_{v=1}^n (z_v - T'_v) C'_v$, where $C'_v = \delta_{n,v} I \in \{\mathbf{T}'\}'$ for $v = 1, \dots, n$, it follows exactly in the same way that $\Phi\Psi = M_{z_n}$ on $H^p(\mathbf{T}' - \mathbf{z}, \mathcal{O}(D, X))$ for $p = 0, \dots, n$.

Thus, we have seen that

$$(8) \quad \begin{cases} H^p(\mathbf{T}' - \mathbf{z}, \mathcal{O}(D, X)) \xrightarrow{\Phi} H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(D, X)), \\ H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(D, X)) \xrightarrow{\Psi} H^p(\mathbf{T}' - \mathbf{z}, \mathcal{O}(D, X)) \end{cases}$$

are continuous vector-space morphisms.

Now if we assume that $H^p(\mathbf{T} - \mathbf{z}, \mathcal{O}(D, X)) = \{0\}$, we will derive by using the previous discussion that $M_{z_n} H^p(\mathbf{T}' - \mathbf{z}, \mathcal{O}(D, X)) = \{0\}$, and hence, from Lemma 2 we get $H^p(\mathbf{T}' - \mathbf{z}, \mathcal{O}(\mathcal{U}, X)) = \{0\}$. In particular, $\mathbf{T}$ has *SVEP* if and only if $\mathbf{T}'$ has *SVEP*.

Combining equation (5) with some minor modifications of the previous proof, we obtain the next (β) -version of Lemma 2. This lemma is to be used in the proof of the corresponding theorem for Bishop's property (β) .

Lemma 3. Let $\mathcal{U}$ be an open neighborhood of zero in $\mathbb{C}^n$ and let $\mathbf{T} = (T_1, \dots, T_n)$ be a bounded commuting n -tuple such that $(T_1, \dots, T_{n-1})$ satisfies Bishop's property (β) . Then the pointwise multiplication by the independent variable z_n on $\Lambda^{n-1}[\mathcal{O}(\mathcal{U}, X)]/\text{Im}(\alpha_{\mathbf{T}-\mathbf{z}}^{n-2})$ is bounded below.

Theorem 4. Let $\mathbf{R}$ and $\mathbf{S}$ be commuting n -tuples nearly commuting. Then $\mathbf{T} = (R_1S_1, \dots, R_iS_i, \dots, R_nS_n)$ has (β) if and only if $\mathbf{T}'_i = (R_1S_1, \dots, S_iR_i, \dots, R_nS_n)$ has (β) for every $i \in \{1, \dots, n\}$.

In particular, $\mathbf{RS}$ has (β) if and only if $\mathbf{SR}$ has (β) .

Proof. As for *SVEP*, we will take $i = n$, and since (β) implies *SVEP*, we only need to show the last step. That is, the pointwise multiplication by the independent variable z_n on $\Lambda^{n-1}[\mathcal{O}(\mathcal{U}, X)]/\text{Im}(\alpha_{\mathbf{T}}^{n-2})$ is bounded below if and only if the corresponding mapping on $\Lambda^{n-1}[\mathcal{O}(\mathcal{U}, X)]/\text{Im}(\alpha_{\mathbf{T}'-\mathbf{z}}^{n-2})$ is bounded below. Assume that $\alpha_{\mathbf{T}'-\mathbf{z}}(f_p)$ converges to zero. It will be immediate that $\lim_{n \rightarrow \infty} \alpha_{\mathbf{T}-\mathbf{z}}(S_nf_p) = 0$. Now the fact that $\Lambda^{n-1}[\mathcal{O}(\mathcal{U}, X)]/\text{Im}(\alpha_{\mathbf{T}}^{n-2})$ is bounded below yields to the existence of g_p such that $S_nf_p - \alpha_{\mathbf{T}-\mathbf{z}}g_p$ tends to zero. We apply R_n , and we get

$$\lim_{n \rightarrow \infty} (R_nS_nf_p - \alpha_{\mathbf{T}'-\mathbf{z}}(R_ng_p)) = 0.$$

Equivalently,

$$\lim_{n \rightarrow \infty} (z_nf_p + (R_nS_n - z_n)f_p - \alpha_{\mathbf{T}'-\mathbf{z}}(R_ng_p)) = 0.$$

From the Kronecker formula $(R_nS_n - z_n) = E_n^* \alpha_{\mathbf{T}'-\mathbf{z}} + \alpha_{\mathbf{T}'-\mathbf{z}} E_n^*$, we deduce that

$$\lim_{n \rightarrow \infty} (z_nf_p + (E_n^* \alpha_{\mathbf{T}'-\mathbf{z}} + \alpha_{\mathbf{T}'-\mathbf{z}} E_n^*)f_p - \alpha_{\mathbf{T}'-\mathbf{z}}(R_ng_p)) = 0,$$

and then

$$\lim_{n \rightarrow \infty} (z_nf_p + \alpha_{\mathbf{T}'-\mathbf{z}}[E_n^*f_p - R_ng_p]) = 0.$$

We end the proof of our theorem by using Lemma 3.

4.2. The case of criss-cross commutativity. In the setting of criss-cross commutativity, we have

Theorem 5. Let $\mathbf{R}$ and $\mathbf{S}$ be commuting n -tuples satisfying the criss-cross commuting. Then,

$\mathbf{RS}$ has *SVEP* (resp. has (β)) if and only if $\mathbf{SR}$ has *SVEP* (resp. has (β)).

Proof. For $n = 1$, the result can be found in [3, Proposition 2.1].

For $n = 2$, the *SVEP* of (R_1S_1, R_2S_2) reads as follows:

$$(9) \quad \begin{cases} \ker(R_1S_1 - z_1) \cap \ker(R_2S_2 - z_2) = \{0\}, \\ \{(f_1, f_2) : (R_1S_1 - z_1)f_2 = (R_2S_2 - z_2)f_1\} \\ \quad = \{((R_1S_1 - z_1)g, (R_2S_2 - z_2)g) : g \in \mathcal{O}(D, X)\}. \end{cases}$$

Assume that (R_1S_1, R_2S_2) has *SVEP* and let us show that

$$(10) \quad \begin{cases} \ker(S_1R_1 - z_1) \cap \ker(S_2R_2 - z_2) = \{0\}, \\ \{(f_1, f_2) : (S_1R_1 - z_1)f_2 = (S_2R_2 - z_2)f_1\} \\ \quad = \{((S_1R_1 - z_1)g, (S_2R_2 - z_2)g) : g \in \mathcal{O}(D, X)\}. \end{cases}$$

Consider f satisfying

$$(11) \quad (S_1R_1 - z_1)f = (S_2R_2 - z_2)f = 0$$

and apply R_1 to the latter equality. We will get

$$(12) \quad (R_1 S_1 - z_1) R_1 f = (R_2 S_2 - z_2) R_1 f = 0.$$

From equation (9), it follows that $R_1 f \equiv 0$. Now from equation (11), we have $z_1 f \equiv 0$ and hence $f \equiv 0$.

For the second affirmation, let (f_1, f_2) be such that $(S_1 R_1 - z_1) f_2 = (S_2 R_2 - z_2) f_1$. Applying R_1 gives $(R_1 S_1 - z_1) R_1 f_2 = (R_2 S_2 - z_2) R_1 f_1$. The *SVEP* of $(R_1 S_1, R_2 S_2)$ implies that there exists $g \in \mathcal{O}(D, X)$ satisfying $R_1 f_1 = (R_1 S_1 - z_1) g$ and $R_1 f_2 = (R_2 S_2 - z_2) g$. We derive that $S_1 R_1 f_1 = (S_1 R_1 - z_1) S_1 g$ and $S_1 R_1 f_2 = (S_2 R_2 - z_2) S_1 g$. The latter expression gives

$$z_1 f_1 = (S_1 R_1 - z_1)(S_1 g - f_1) \text{ and } z_1 f_2 = (S_2 R_2 - z_2)(S_1 g - f_1).$$

It now suffices to establish that $\frac{1}{z_1}(S_1 g - f_1) \in \mathcal{O}(D, X)$. To this aim, if we take $z_1 = 0$ we obtain $0 = (S_2 R_2 - z_2)(S_1 g(0, z_2) - f_1(0, z_2))$. Using the *SVEP* of $S_2 R_2$, we deduce that $S_1 g(0, z_2) - f_1(0, z_2) \equiv 0$. Finally, $\frac{1}{z_1}(S_1 g - f_1) \in \mathcal{O}(D, X)$.

In an inductive proof, assume now that we have showed our claim for $n-1$ and let $\mathbf{T} = (R_1 S_1, \dots, R_n S_n)$ be with *SVEP*. In particular, $(R_1 S_1, \dots, R_{n-1} S_{n-1})$ will satisfy *SVEP* and hence the same is the case for $(S_1 R_1, \dots, S_{n-1} R_{n-1})$. Now, if $H^p((R_1 S_1, \dots, R_n S_n)) = 0$, using the same morphisms as in Lemma 2 we get $M_{z_n} H^p((S_1 R_1, \dots, S_n R_n)) = \{0\}$. Then $(S_1 R_1, \dots, S_n R_n)$ has *SVEP*.

The proof for Bishop's property (β) is handled in the same way as for the near commutativity condition.

5. LOCAL SPECTRA FOR $\mathbf{RS}$ AND $\mathbf{SR}$

We first recall the definition of local spectrum for n -tuples. Let $\mathbf{T} = (T_1, \dots, T_n)$ denote a commuting n -tuple of bounded linear operators acting on the Banach space X . For $x \in X$, $\rho(\mathbf{T}, x)$ is the set of those points $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ for which there exist an open set $U \ni \lambda$ and X -valued analytic functions on U : $f_1, \dots, f_n$ such that

$$(13) \quad \sum_{j=1}^n (T_j - z_j) f_j(z) = x, \quad z \in U.$$

The set $\sigma(\mathbf{T}, x) =: \mathbb{C}^n \setminus \rho(\mathbf{T}, x)$ is called *the analytic local spectrum of $\mathbf{T}$ at x* .

We have the following result, which generalizes the one in the single operator case.

Proposition 3. *Let $\mathbf{R}$ and $\mathbf{S}$ be commuting n -tuples and let $x \in X$. We have:*

(1) *If $\mathbf{R}$ and $\mathbf{S}$ are nearly commuting (3), then*

$$\sigma(\mathbf{SR}, S_i x) \subset \sigma(\mathbf{SR}_{\{i\}}, x) \subset \sigma(\mathbf{SR}, S_i x) \cup [0]^{\{i\}}.$$

(2) *If $\mathbf{R}$ and $\mathbf{S}$ are criss-cross commuting, then*

$$\sigma(\mathbf{SR}, S_i x) \subset \sigma(\mathbf{RS}, x) \subset \bigcap_{j=1}^n \sigma(\mathbf{SR}, S_j x) \cup [0]^{\{j\}}.$$

Proof. (1) For simplicity, we will assume that $i = 1$. Let $\lambda \notin \sigma(\mathbf{SR}_{\{1\}}, x)$ and $f_1, \dots, f_n$ be X -valued analytic functions in a neighborhood $\mathcal{U}$ of λ such that

$x = (R_1 S_1 - z_1)f_1(z) + \sum_{j=2}^n (S_j R_j - z_j)f_j(z)$ for every $z \in \mathcal{U}$. Then $S_1 x = (S_1 R_1 - z_1)S_1 f_1(z) + \sum_{j=2}^n (S_j R_j - z_j)S_1 f_j(z)$. Since $S_1 f_j$ are X -valued analytic functions in a neighborhood U of λ , we derive that $\lambda \notin \sigma(\mathbf{SR}, S_1 x)$.

For the other inclusion, let $\lambda \notin \sigma(\mathbf{SR}_{\{1\}}, S_1 x) \cup [0]^{\{1\}}$ and $f_1, \dots, f_n$ be X -valued analytic functions in a neighborhood $\mathcal{U}$ of λ such that $S_1 x = (S_1 R_1 - z_1)f_1(z) + \sum_{j=2}^n (S_j R_j - z_j)f_j(z)$ for every $z \in \mathcal{U}$. Then by applying R_1 , $S_1[-x + R_1 f_1] + \sum_{j=2}^n (S_j R_j - z_j)f_j = z_1 f_1$. Since $\lambda \notin [0]^{\{1\}}$, we may choose $\mathcal{U}$ such that $z_1 \neq 0$ for every $z \in \mathcal{U}$. Now consider $g_1 = \frac{1}{z_1}[-x + R_1 f_1]$ and $g_j = \frac{1}{z_1}R_1 f_j$ for $j = 2, \dots, n$. Then a simple and direct calculation gives

$$\begin{aligned} (R_1 S_1 - z_1)g_1 + \sum_{j=2}^n (S_j R_j - z_j)g_j &= \frac{1}{z_1}[R_1(S_1 R_1 - z_1)f_1 - R_1 S_1 x + z_1 x] \\ &\quad + \frac{1}{z_1}R_1\left[\sum_{j=2}^n (S_j R_j - z_j)f_j\right] \\ &= \frac{1}{z_1}\left[R_1\left(\sum_{j=1}^n (S_j R_j - z_j)f_j\right) - R_1 S_1 x + z_1 x\right] \\ &= x. \end{aligned}$$

Thus, $\lambda \notin \sigma(\mathbf{SR}, x)$.

(2) The second statement is obtained similarly.

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