ON STABLY FREE MODULES OVER LAURENT POLYNOMIAL RINGS

ABED ABEDOLFATAH

(Communicated by Harm Derksen)

ABSTRACT. We prove constructively that for any finite-dimensional commutative ring R and $n \geq \dim(R) + 2$, the group $E_n(R[X, X^{-1}])$ acts transitively on $\text{Um}_n(R[X, X^{-1}])$. In particular, we obtain that for any finite-dimensional ring R , every finitely generated stably free module over $R[X, X^{-1}]$ of rank $> \dim R$ is free; i.e., $R[X, X^{-1}]$ is $(\dim R)$ -Hermite.

1. INTRODUCTION

We denote by R a commutative ring with unity and $\mathbb{N}$ the set of non-negative integers. $\text{Um}_n(R)$ is the set of unimodular rows of length n over R , that is, all $(x_0, \dots, x_{n-1}) \in R^n$ such that $x_0R + \dots + x_{n-1}R = R$. If $u, v \in \text{Um}_n(R)$ and G is a subgroup of $\text{GL}_n(R)$, we write $u \sim_G v$ if there exists g in G such that $v = ug$. Recall that $E_n(R)$ denotes the subgroup of $\text{GL}_n(R)$, generated by all $E_{ij}(a) := I_n + ae_{ij}$ (where $i \neq j$, $a \in R$ and e_{ij} denotes the $(n \times n)$ -matrix whose only non-zero entry is 1 on the (i, j) entry). We abbreviate the notation $u \sim_{E_n(R)} v$ to $u \sim_E v$. We say that a ring R is Hermite (resp. d -Hermite) if any finitely generated stably free R -module (resp., any finitely generated stably free R -module of rank $> d$) is free.

In [6], A. A. Suslin proved:

Theorem 1.1 (A. A. Suslin). *If R is a Noetherian ring and*

$$A = R[X_1^{\pm 1}, \dots, X_k^{\pm 1}, X_{k+1}, \dots, X_n],$$

then for $n \geq \max(3, \dim(R) + 2)$ the group $E_n(A)$ acts transitively on $\text{Um}_n(A)$.

In particular we obtain that $E_n(R[X, X^{-1}])$ acts transitively on $\text{Um}_n(R[X, X^{-1}])$ for any Noetherian ring R , where $n \geq \max(3, \dim(R) + 2)$. In [7], I. Yengui proved:

Theorem 1.2 (I. Yengui). *Let R be a ring of dimension d , $n \geq d + 1$, and let $f \in \text{Um}_{n+1}(R[X])$. Then there exists $E \in E_{n+1}(R[X])$ such that $f \cdot E = e_1$.*

In this article we generalize by proving:

Received by the editors May 29, 2010 and, in revised form, September 25, 2010 and October 19, 2010.

2010 *Mathematics Subject Classification.* Primary 13Axx; Secondary 13A50.

Key words and phrases. Stably free modules, Hermite rings, unimodular rows, Laurent polynomial rings, constructive mathematics.

©2011 American Mathematical Society
 Reverts to public domain 28 years from publication

Theorem 1.3. *For any finite-dimensional ring R , $E_n(R[X, X^{-1}])$ acts transitively on $\text{Um}_n(R[X, X^{-1}])$, where $n \geq \dim(R) + 2$.*

This gives a positive answer to Yengui's question (Question 9 of [7]). The proof we give is a close adaptation of Yengui's proof of the Laurent case.

2. PRELIMINARY RESULTS ON UNIMODULAR ROWS

A. A. Suslin proved in [6], that if $f = (f_0, \dots, f_n) \in \text{Um}_{n+1}(R[X])$, where f_1 is unitary and $n \geq 1$, then there exists $w \in \text{SL}_2(R[X]) \cdot E_{n+1}(R[X])$ such that $f \cdot w = e_1$. In fact, this theorem is a crucial point in his proof of Serre's conjecture. R. A. Rao generalized in [4], Corollary 2. 5] by proving:

Theorem 2.1 (R. A. Rao [4]). *Let $f = (f_0, \dots, f_n) \in \text{Um}_{n+1}(R[X])$, where $n \geq 2$. If some f_i is unitary, then f is completable to a matrix in $E_n(R[X])$.*

Recall that the boundary ideal of an element a of a ring R is the ideal $\mathcal{I}(a)$ of R generated by a and all $y \in R$ such that ay is nilpotent. Moreover, $\dim R \leq d \Leftrightarrow \dim(R/\mathcal{I}(a)) \leq d - 1$ for all $a \in R$ [3].

Theorem 2.2 ([2], Theorem 2.4). *Let R be a ring of dimension $\leq d$ and $a = (a_0, \dots, a_n) \in \text{Um}_{n+1}(R)$ where $n \geq d + 1$. Then there exist $b_1, \dots, b_n \in R$ such that*

$$\langle a_1 + b_1 a_0, \dots, a_n + b_n a_0 \rangle = R.$$

In fact, we can obtain a stronger result if $f \in \text{Um}_{n+1}(R_S)$, where S is a multiplicative subset of R :

Proposition 2.3. *Let S be a multiplicative subset of R such that $S^{-1}R$ has dimension d . Let $(a_0, \dots, a_n) \in M_{n+1}(R)$ be a row such that $(\frac{a_0}{1}, \dots, \frac{a_n}{1}) \in \text{Um}_{n+1}(S^{-1}R)$, where $n > d$. Then there exist $b_1, \dots, b_n \in R$ and $s \in S$ such that*

$$s \in (a_1 + b_1 a_0)R + \dots + (a_n + b_n a_0)R.$$

Proof. By induction on d , if $d = 0$, then $R_S/\mathcal{I}(\frac{a_n}{1}) \cong (R/J)_{\bar{S}}$ is trivial, where $\bar{S} = \{s + J \mid s \in S\}$, $J = i^{-1}(\mathcal{I}(\frac{a_n}{1}))$, and $i : R \rightarrow R_S$ is the natural homomorphism. So $1 \in \langle \frac{a_n}{1}, \frac{b_n}{1} \rangle$ in R_S , where $b_n \in R$ and $\frac{a_n b_n}{1}$ is nilpotent. Since $1 \in \langle \frac{a_1}{1}, \dots, \frac{a_{n-1}}{1}, \frac{a_n}{1}, \frac{b_n a_0}{1} \rangle$, so by [[2], Lemma 2.3], $1 \in \langle \frac{a_1}{1}, \dots, \frac{a_{n-1}}{1}, \frac{a_n + b_n a_0}{1} \rangle$; i.e., there exist $s \in S$ such that $s \in a_1 R + \dots + a_{n-1} R + (a_n + b_n a_0)R$.

Assume now that $d > 0$. By the induction assumption with respect to the ring $R_S/\mathcal{I}(\frac{a_n}{1}) \cong (R/J)_{\bar{S}}$ we can find $\bar{b}_1, \dots, \bar{b}_{n-1} \in R/J$ such that

$$\langle \frac{\bar{a}_1 + \bar{b}_1 \bar{a}_0}{\bar{1}}, \dots, \frac{\bar{a}_{n-1} + \bar{b}_{n-1} \bar{a}_0}{\bar{1}} \rangle = (R/J)_{\bar{S}}.$$

So $\langle \frac{a_1 + b_1 a_0}{1}, \dots, \frac{a_{n-1} + b_{n-1} a_0}{1} \rangle = R_S/\mathcal{I}(\frac{a_n}{1})$. This means that

$$\langle \frac{a_1 + b_1 a_0}{1}, \dots, \frac{a_{n-1} + b_{n-1} a_0}{1}, \frac{a_n}{1}, \frac{b_n}{1} \rangle = R_S$$

where $\frac{a_n b_n}{1}$ is nilpotent. So by [[2], Lemma 2.3],

$$\langle \frac{a_1 + b_1 a_0}{1}, \dots, \frac{a_{n-1} + b_{n-1} a_0}{1}, \frac{a_n + b_n a_0}{1} \rangle = R_S. \quad \square$$

Let $f \in \text{Um}_{n+1}(R[X])$, where $n \geq \frac{d}{2} + 1$, with R a local ring of dimension d . M. Roitman's argument in [[5], Theorem 5], shows how one could decrease the degree of all but one (special) co-ordinate of f . In the absence of a monic polynomial as a co-ordinate of f he uses a Euclid's algorithm and this is achieved via the following:

Lemma 2.4 (M. Roitman, [5], Lemma 1). *Let $(x_0, \dots, x_n) \in \text{Um}_{n+1}(R)$, $n \geq 2$, and let t be an element of R which is invertible mod $(Rx_0 + \dots + Rx_{n-2})$. Then*

$$(x_0, \dots, x_n) \sim_{\text{E}_{n+1}(R)} (x_0, \dots, t^2 x_n) \sim_{\text{E}_{n+1}(R)} (x_0, \dots, tx_{n-1}, tx_n).$$

3. THE MAIN RESULTS

Definitions 3.1. Let $f \in R[X, X^{-1}]$ be a nonzero Laurent polynomial. We denote $\deg(f) = \text{hdeg}(f) - \text{ldeg}(f)$, where $\text{hdeg}(f)$ and $\text{ldeg}(f)$ denote respectively the highest and the lowest degree of f .

Let $\text{hc}(f)$ and $\text{lc}(f)$ denote respectively the coefficients of the highest and the lowest degree term of f . An element $f \in R[X, X^{-1}]$ is called doubly unitary if $\text{hc}(f), \text{lc}(f) \in U(R)$.

For example, $\deg(X^{-3} + X^2) = 5$.

Lemma 3.2. *Let $f_1, \dots, f_n \in R[X, X^{-1}]$ such that $\text{hdeg}(f_i) \leq k-1$, $\text{ldeg}(f_i) \geq -m$ for all $1 \leq i \leq n$. Let $f \in R[X, X^{-1}]$ with $\text{hdeg}(f) = k$, $\text{ldeg}(f) \geq -m$, where $k, m \in \mathbb{N}$. Assume that $\text{hc}(f) \in U(R)$ and the coefficients of $f_1, \dots, f_n$ generate the ideal (1) of R . Then $I = \langle f_1, \dots, f_n, f \rangle$ contains a polynomial h of $\text{hdeg}(h) = k-1$, $\text{ldeg}(h) \geq -m$ and $\text{hc}(h) \in U(R)$.*

Proof. Since $X^m f_1, \dots, X^m f_n, X^m f \in R[X]$, by [[1], §4, Lemma 1(b)], I contains a polynomial $h_1 \in R[X]$ of degree $m + k - 1$ which is unitary. So $h = X^{-m} h_1 \in I$ of $\text{hdeg}(h) = k - 1$, $\text{ldeg}(h) \geq -m$ and $\text{hc}(h) \in U(R)$. $\square$

Proposition 3.3. *Let $I \trianglelefteq R[X, X^{-1}]$ be an ideal, $J \trianglelefteq R$, such that I contains a doubly unitary polynomial. If $I + J[X, X^{-1}] = R[X, X^{-1}]$, then $(I \cap R) + J = R$.*

Proof. Let us denote by h_1 a doubly unitary polynomial in I . Since $I + J[X, X^{-1}] = R[X, X^{-1}]$, there exist $h_2 \in I$ and $h_3 \in J[X, X^{-1}]$ such that $h_2 + h_3 = 1$. Let $g_i = X^{-\text{ldeg}(h_i)} h_i$, for $i = 1, 2, 3$. Since $X^l \in \sum_{i=1}^3 g_i R[X]$, for some $l \geq 0$, and $g_1 \equiv u \pmod{XR[X]}$, where $u \in U(R)$, we obtain that $\langle g_1, g_2, g_3 \rangle = \langle 1 \rangle$ in $R[X]$. By [[8], Lemma 2], we obtain $(\langle g_1, g_2 \rangle \cap R) + J = R$. So $(I \cap R) + J = R$. $\square$

Theorem 3.4. *Let $f = (f_0, \dots, f_n) \in \text{Um}_{n+1}(R[X, X^{-1}])$, where $n \geq 2$. Assume that f_0 is a doubly unitary polynomial; then*

$$f \sim_{\text{E}_{n+1}(R[X, X^{-1}])} (1, 0, \dots, 0).$$

Proof. By (2.4),

$$\begin{aligned} f &\sim_{\text{E}} (X^{-\text{ldeg}(f_0)} f_0, X^{-\text{ldeg}(f_0)} f_1, f_2, \dots, f_n) \\ &\sim_{\text{E}} (X^{-\text{ldeg}(f_0)} f_0, X^{-\text{ldeg}(f_0)+2k} f_1, X^{2k} f_2, \dots, X^{2k} f_n) = (g_0, \dots, g_n) \end{aligned}$$

where $k \in \mathbb{N}$. For sufficiently large k , we obtain that $g_0, \dots, g_n \in R[X]$. Clearly, $X^l \in \sum_{i=0}^n g_i R[X]$ for some $l \geq 0$. But $g_0 \equiv u \pmod{XR[X]}$, where $u \in U(R)$. Then $X^l R[X] + g_0 R[X] = R[X]$, so $g \in \text{Um}_n(R[X])$. By (2.1), $g \sim_{\text{E}} e_1$. $\square$

Remark 3.5. Let $a = (a_1, \dots, a_n) \in \text{Um}_{n+1}(R)$, where $n \geq 2$. If

$$a \sim_{\text{E}_n(R/\text{Nil}(R))} e_1,$$

then $a \sim_{\text{E}_n(R)} e_1$.

Proposition 3.6. If R is a zero-dimensional ring and $f = (f_0, \dots, f_n) \in \text{Um}_{n+1}(R[X, X^{-1}])$, where $n \geq 1$, then

$$f \sim_{\text{E}} e_1.$$

Proof. We prove by induction on $\deg f_0 + \deg f_1$. We may assume that R is a reduced ring. Let $a = \text{hc}(f_0)$ and $b = \text{lc}(f_0)$. Assume that $ab \in U(R)$. Then by elementary transformations of the form

$$f_1 - X^{\text{ldeg}(f_1) - \text{ldeg}(f_0)} b^{-1} \text{lc}(f_1) f_0,$$

we obtain that $f \sim_E (f_0, h_1, f_2, \dots, f_n)$, where $\text{ldeg}(h_1) > \text{ldeg}(f_0)$. By elementary transformations of the form

$$f_1 - X^{\text{hdeg}(f_1) - \text{hdeg}(f_0)} a^{-1} \text{hc}(f_1) f_0,$$

we obtain that $f \sim_E (f_0, g_1, f_2, \dots, f_n)$, where $\text{ldeg}(g_1) \geq \text{ldeg}(f_0)$ and $\text{hdeg}(g_1) < \text{hdeg}(f_0)$. So we may assume that $\deg f_0 \leq \deg f_1$ and $ab \notin U(R)$. Assume that $a \notin U(R)$. We have $Ra = Re$ for some idempotent e . Let $c = \text{hc}(f_1)$. Since $e \in Ra$, we may assume that $c \neq 0$ and that $c \in R(1 - e)$. Note that

$$(1 - e)f = (f_0(1 - e), \dots, f_n(1 - e)) \in \text{Um}_{n+1}(R(1 - e)[X, X^{-1}]) \text{ and}$$

$$ef = (f_0e, \dots, f_ne) \in \text{Um}_{n+1}(Re[X, X^{-1}]).$$

By the inductive assumption, there are matrices

$$A \in \text{E}_{n+1}(R(1 - e)[X, X^{-1}]), \quad B \in \text{E}_{n+1}(Re[X, X^{-1}])$$

so that $(1 - e)fA = (1 - e, 0, \dots, 0)$ and $efB = (e, 0, \dots, 0)$. Let

$$A = \prod_{s=1}^k \text{E}_{ij}(h_s), \quad B = \prod_{s=1}^t \text{E}_{ij}(g_s)$$

where

$$\text{E}_{ij}(h_s) = (1 - e)I_{n+1} + h_se_{ij}, \quad \text{E}_{ij}(g_s) = eI_{n+1} + g_se_{ij}$$

and $i \neq j \in \{1, \dots, n + 1\}$, $h_s \in R(1 - e)[X, X^{-1}]$, $g_s \in Re[X, X^{-1}]$. Let

$$A' = \prod_{s=1}^k (I_{n+1} + h_se_{ij}), \quad B' = \prod_{s=1}^t (I_{n+1} + g_se_{ij}).$$

Clearly, $(1 - e)A' = A$, $eB' = B$ and $A', B' \in \text{E}_{n+1}(R[X, X^{-1}])$. Let $C = A'B'$; then $C \in \text{E}_{n+1}(R[X, X^{-1}])$ and

$$(1 - e)C = (1 - e)A'(1 - e)B' = A(1 - e)I_{n+1} = (1 - e)A' = A.$$

Similarly, we have $eC = B$. Let $fC = (g_0, \dots, g_n) = g$. Thus

$$g_0(1 - e) = 1 - e \text{ and } g_1e = e.$$

So

$$\begin{aligned} f &\sim_{\text{E}_{n+1}(R[X, X^{-1}])} (g_0, \dots, g_n) \sim_{\text{E}_{n+1}(R[X, X^{-1}])} (g_0 + e, \dots, g_n) \\ &= (1 + g_0e, \dots, g_n) \sim_{\text{E}_{n+1}(R[X, X^{-1}])} (1 + g_0e, -g_0e, \dots, g_n) \sim_{\text{E}_{n+1}(R[X, X^{-1}])} e_1. \end{aligned}$$

Similarly, if $b \notin U(R)$, then $f \sim_{\text{E}} e_1$. $\square$

Proposition 3.7. *If R is a zero-dimensional ring, then*

$$\mathrm{SL}_n(R[X, X^{-1}]) = \mathrm{E}_n(R[X, X^{-1}])$$

for all $n \geq 2$.

Proof. Clearly, $\mathrm{E}_n(R[X, X^{-1}]) \subseteq \mathrm{SL}_n(R[X, X^{-1}])$. Let $M \in \mathrm{SL}_n(R[X, X^{-1}])$. By (3.6), we can perform suitable elementary transformations to bring M to M_1 with first row $(1, 0, \dots, 0)$. Now a sequence of row transformations brings M_1 to

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & M' \end{pmatrix}$$

where $M' \in \mathrm{SL}_{n-1}(R[X, X^{-1}])$. The proof now proceeds by induction on n . $\square$

Lemma 3.8. *Let $(f_0, \dots, f_n) \in \mathrm{Um}_{n+1}(R[X, X^{-1}])$, where $n \geq 2$. Assume that $\mathrm{hc}(f_0)$ is invertible modulo f_0 . Then*

$$f \sim_{\mathrm{E}} (f_0, g_1, \dots, g_n)$$

where $\mathrm{hdeg}(g_i) < \mathrm{hdeg}(f_0)$, $\mathrm{ldeg}(g_i) \geq \mathrm{ldeg}(f_0)$, for all $1 \leq i \leq n$.

Proof. By (2.4), $f \sim_{\mathrm{E}} (f_0, X^{2k}f_1, \dots, X^{2k}f_n)$ for all $k \in \mathbb{Z}$. So we may assume that $\mathrm{ldeg}(f_i) > \mathrm{ldeg}(f_0)$. Let $a = \mathrm{hc}(f_0)$. By (2.4) we have

$$f \sim_{\mathrm{E}} (f_0, a^2f_1, \dots, a^2f_n).$$

Using elementary transformations of the form

$$a^2f_i - aX^{\mathrm{hdeg}(f_i) - \mathrm{hdeg}(f_0)} \mathrm{hc}(f_i)f_0$$

we lower the degrees of f_i , for all $1 \leq i \leq n$, and obtain the required row. $\square$

Lemma 3.9. *Let R be a ring of dimension $d > 0$ and*

$$f = (r, f_1, \dots, f_n) \in \mathrm{Um}_{n+1}(R[X, X^{-1}])$$

where $r \in R$, $n \geq d + 1$. Assume that for every ring T of dimension $< d$ and $n \geq \dim(T) + 1$, the group $\mathrm{E}_{n+1}(T[X, X^{-1}])$ acts transitively on $\mathrm{Um}_{n+1}(T[X, X^{-1}])$. Then $f \sim_{\mathrm{E}(R[X, X^{-1}])} e_1$.

Proof. Since $\dim(R/\mathcal{I}(r)) < \dim(R)$ over $R/\mathcal{I}(r)$, we can complete $(f_1, \dots, f_n)$ to a matrix in $\mathrm{E}_n(R/\mathcal{I}(r)[X, X^{-1}])$. If we lift this matrix, we obtain that

$$(r, f_1, \dots, f_n) \sim_{\mathrm{E}_{n+1}(R[X, X^{-1}])} (r, 1 + rw_1 + h_1, \dots, rw_n + h_n) \sim_{\mathrm{E}_{n+1}(R[X, X^{-1}])} (r, 1 + h_1, \dots, h_n)$$

where $h_i, w_i \in R[X, X^{-1}]$ and $rh_i = 0$ for all $1 \leq i \leq n$. Then

$$f \sim_{\mathrm{E}_{n+1}(R[X, X^{-1}])} (r - r(1 + h_1), 1 + h_1, \dots, h_n) \sim_{\mathrm{E}_{n+1}(R[X, X^{-1}])} e_1. \quad \square$$

Lemma 3.10. *Let R be a ring of dimension $d > 0$ and*

$$f = (f_0, \dots, f_n) \in \mathrm{Um}_{n+1}(R[X, X^{-1}])$$

such that $n \geq d + 1$, $f_0 = ag$ and $a^t = \mathrm{hc}(f_0)$, where $a \in R \setminus U(R)$, $0 \neq t \in \mathbb{N}$. Assume that for every ring T of dimension $< d$ and $n \geq \dim(T) + 1$, the group $\mathrm{E}_{n+1}(T[X, X^{-1}])$ acts transitively on $\mathrm{Um}_{n+1}(T[X, X^{-1}])$. Then $f \sim_{\mathrm{E}(R[X, X^{-1}])} e_1$.

Proof. We prove by induction on the number M of non-zero coefficients of the polynomial f_0 that $f \sim_E e_1$. If $M = 1$, then $f_0 = rX^m$ where $r \in R, m \in \mathbb{Z}$. By (2.4), $f \sim_E (r, X^{-m}f_1, f_2, \dots, f_n)$. So by (3.9), we obtain that $f \sim_E e_1$. Assume now that $M > 1$. Let S be the multiplicative subset of R generated by a, b , where $b = \text{lc}(g)$; i.e., $S = \{a^{k_1}b^{k_2} \mid k_1, k_2 \in \mathbb{N}\}$. By the inductive step, with respect to the ring R/abR , we obtain from f a row $\equiv (1, 0, \dots, 0) \pmod{abR[X, X^{-1}]}$. We can perform such transformations so that at every stage the row contains a doubly unitary polynomial in $R_S[X, X^{-1}]$. Indeed, if we have to perform, e.g., the elementary transformation

$$(g_0, \dots, g_n) \rightarrow (g_0, g_1 + hg_0, \dots, g_n)$$

and g_1 is a doubly unitary polynomial in $R_S[X, X^{-1}]$, then we replace this elementary transformation by the two transformations

$$\begin{aligned} (g_0, \dots, g_n) &\rightarrow (g_0 + abX^m g_1 + abX^k g_1, g_1, \dots, g_n) \\ &\rightarrow (g_0 + abX^m g_1 + abX^k g_1, g_1 + h(g_0 + abX^m g_1 + abX^{-k} g_1), \dots, g_n) \end{aligned}$$

where $m > \text{hdeg}(g_0)$, $k < \text{ldeg}(g_0)$. So we may assume that

$$(f_0, \dots, f_n) \equiv (1, 0, \dots, 0) \pmod{abR[X, X^{-1}]}$$

and f_0 is a doubly unitary polynomial in $R_S[X, X^{-1}]$. By (3.8), we may assume that $\text{hdeg}(f_i) < \text{hdeg}(f_0)$, $\text{ldeg}(f_i) \geq \text{ldeg}(f_0)$.

We prove that f can be transformed by elementary transformation into a row with one constant entry. We use an argument similar to that in the proof of [[5], Theorem 5].

Assume that the number of coefficients of $f_2, \dots, f_n$ is $\geq 2(n-1)$. Since $d > 0$, we obtain that $2(n-1) \geq d+1$. Let $a_1, \dots, a_t$ be the coefficients of $f_2, \dots, f_n$ and $J = \frac{a_1}{1}R_S + \dots + \frac{a_t}{1}R_S$. Let $I = R_S[X, X^{-1}]f_0 + R_S[X, X^{-1}]f_1$. Since $I + J[X, X^{-1}] = R_S[X, X^{-1}]$ and f_0 is doubly unitary in $R_S[X, X^{-1}]$, by (3.3), we obtain that $(I \cap R_S) + J = R_S$. So $(\frac{f_0 h_0 + f_1 h_1}{s} + \frac{r_1 a_1}{s_1 1} + \dots + \frac{r_t a_t}{s_t 1}) = \frac{1}{1}$, where $h_0, h_1 \in R[X, X^{-1}]$ and $r_i \in R$, $s, s_i \in S$ for all $1 \leq i \leq t$. This means that $(\frac{f_0 h_0 + f_1 h_1}{1}, \frac{a_1}{1}, \dots, \frac{a_t}{1}) \in \text{Um}_{t+1}(R_S)$. By (2.3), there exist $s \in S$ and $b_1, \dots, b_t \in R$, such that

$$s \in (a_1 + b_1(f_0 h_0 + f_1 h_1))R + \dots + (a_t + b_t(f_0 h_0 + f_1 h_1))R.$$

Using elementary transformations, we may assume that $J = R_S$. By (3.2), the ideal $\langle f_0, f_2, \dots, f_n \rangle$ contains a polynomial h such that $a^{k_1}b^{k_2} = \text{hc}(h)$ and $\text{hdeg}(h) = \text{hdeg}(f_0) - 1$, $\text{ldeg}(h) \geq \text{ldeg}(f_0)$ where $k_1, k_2 \in \mathbb{N}$. Let $r = \text{hc}(f_1)$, so

$$f \sim_E (f_0, a^{2k_1}b^{2k_2}f_1, f_2, \dots, f_n) \sim_E (f_0, a^{2k_1}b^{2k_2}f_1 + (1 - a^{k_1}b^{k_2}r)h, f_2, \dots, f_n).$$

Then we may assume that $a^{k_1}b^{k_2} = \text{hc}(f_1)$. By the proof of Lemma (3.8), we can decrease the $\text{hdeg}(f_i)$ for all $2 \leq i \leq n$.

Repeating the argument above, we obtain that

$$f \sim_E (rX^m, g_1, \dots, g_n) \sim_E (r, g_1X^{-m}, g_2, \dots, g_n)$$

where $r \in R, m \in \mathbb{Z}, g_1, \dots, g_n \in R[X, X^{-1}]$. By (3.9), $f \sim_E e_1$. □

Lemma 3.11. *Let R be a ring of dimension $d > 0$ and*

$$f = (f_0, \dots, f_n) \in \text{Um}_{n+1}(R[X, X^{-1}])$$

such that $n \geq d + 1$, $f_0 = cg$ and $c^t = \text{lc}(f_0)$, where $c \in R \setminus U(R)$, $0 \neq t \in \mathbb{N}$. Assume that for every ring T of dimension $< d$ and $n \geq \dim(T) + 1$, the group $E_{n+1}(T[X, X^{-1}])$ acts transitively on $\text{Um}_{n+1}(T[X, X^{-1}])$. Then $f \sim_{E(R[X, X^{-1}])} e_1$.

Proof. By making the change of variable: $X \rightarrow X^{-1}$ and Proposition (3.10), we obtain that $f \sim_{E(R[X, X^{-1}])} e_1$. $\square$

Theorem 3.12. *Let R be a ring of dimension d and $n \geq d + 1$. Then $E_{n+1}(R[X, X^{-1}])$ acts transitively on $\text{Um}_{n+1}(R[X, X^{-1}])$.*

Proof. Let $f = (f_0, \dots, f_n) \in \text{Um}_{n+1}(R[X, X^{-1}])$. We prove the theorem by induction on d ; we may assume that R is a reduced ring. If $d = 0$, by (3.6), we are done. Assume that the theorem is true for the dimensions $0, 1, \dots, d - 1$, where $d > 0$. We prove by induction on the number N of non-zero coefficients of the polynomials $f_0, \dots, f_n$, that $f \sim_E e_1$ if $\dim R = d$. Starting with $N = 1$, let $N > 1$. Let $a = \text{hc}(f_0)$ and $c = \text{lc}(f_0)$. If $ac \in U(R)$, then by (3.4), we are done. Otherwise, assume that $a \notin U(R)$. By the inductive step, with respect to the ring R/aR , we obtain from f a row $\equiv (1, 0, \dots, 0) \pmod{aR[X, X^{-1}]}$ using elementary transformations. We can perform such transformations so that at every stage the row contains a polynomial $g \in R[X, X^{-1}]$ such that $\text{hc}(g) = a^t$, where $t \in \mathbb{N}$. Indeed, if we have to perform, e.g., the elementary transformation

$$(g_0, \dots, g_n) \rightarrow (g_0, g_1 + hg_0, \dots, g_n)$$

and $\text{hc}(g_1) \in U(R_a)$, then we replace this elementary transformation by the two transformations:

$$(g_0, \dots, g_n) \rightarrow (g_0 + aX^m g_1, g_1, \dots, g_n) \rightarrow (g_0 + aX^m g_1, g_1 + h(g_0 + aX^m g_1), \dots, g_n)$$

where $m > \text{hdeg}(g_0)$.

So we have $f_0 = ag$, and $a^t = \text{hc}(f_0)$, where $0 \neq t \in \mathbb{N}$. By (3.10), $f \sim_E e_1$. Similarly, if $c \notin U(R)$, by (3.11) we obtain that $f \sim_E e_1$. $\square$

Corollary 3.13. *For any ring R with Krull dimension $\leq d$, all finitely generated stably free modules over $R[X, X^{-1}]$ of rank $> d$ are free.*

The following conjecture is the analogue of Conjecture 8 of [7] in the Laurent case:

Conjecture 3.14. *For any ring R with Krull dimension $\leq d$, all finitely generated stably free modules over $R[X_1^{\pm 1}, \dots, X_k^{\pm 1}, X_{k+1}, \dots, X_n]$ of rank $> d$ are free.*

ACKNOWLEDGMENT

The author would like to thank Professor Moshe Roitman, his M.Sc. thesis advisor, for his interest in this project.

REFERENCES

1. H. Bass, *Libération des modules projectifs sur certains anneaux de polynômes*, Sémin. Bourbaki 1973/74, exp. 448, Lecture Notes in Math., vol. 431, Springer-Verlag, Berlin and New York, 1975, 228-254. MR0472826 (57:12516)
2. T. Coquand, H. Lombardi, C. Quitté, *Generating non-noetherian modules constructively*, Manuscripta Math. 115 (2004) 513-520. MR2103665 (2005h:13014)
3. T. Coquand, H. Lombardi, M. F. Roy, *An elementary characterization of Krull dimension*, From sets and types to analysis and topology: towards practicable foundations for constructive mathematics, Oxford University Press, 2005. MR2188647 (2007a:54023)

4. R. A. Rao, *An elementary transformation of a special unimodular vector to its top coefficient vector*, Proc. Amer. Math. Soc. 93 (1985), no. 1, 21-24. MR766519 (86a:13011)
5. M. Roitman, *On stably extended projective modules over polynomial rings*, Proc. Amer. Math. Soc. 97 (1986), no. 4, 585-589. MR845969 (87f:13007)
6. A. A. Suslin, *On the structure of the special linear group over polynomial rings*, Izv. Math. 11 (1977), no. 2, 221-238.
7. I. Yengui, *Stably free modules over $\mathbf{R}[X]$ of rank $> \dim \mathbf{R}$ are free*, Math. Comp. 80 (2011), 1093-1098.
8. I. Yengui, *The Hermite ring conjecture in dimension one*, J. Algebra, 320 (2008), no. 1, 437-441. MR2417998 (2009g:16019)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, MOUNT CARMEL, HAIFA 31905, ISRAEL
E-mail address: abedelfatah@gmail.com