PLANAR LOOPS WITH PRESCRIBED CURVATURE: EXISTENCE, MULTIPLICITY AND UNIQUENESS RESULTS

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ABSTRACT. Let $k : \mathbb{C} \rightarrow \mathbb{R}$ be a smooth given function. A k -loop is a closed curve u in $\mathbb{C}$ having prescribed curvature $k(p)$ at every point $p \in u$. We use variational methods to provide sufficient conditions for the existence of k -loops. Then we show that a breaking symmetry phenomenon may produce multiple k -loops, in particular when k is radially symmetric and somewhere increasing. If $k > 0$ is radially symmetric and non-increasing, we prove that any embedded k -loop is a circle; that is, round circles are the only convex loops in $\mathbb{C}$ whose curvature is a non-increasing function of the Euclidean distance from a fixed point. The result is sharp, as there exist radially increasing curvatures $k > 0$ which have embedded k -loops that are not circles.

INTRODUCTION

In this paper we prove existence, multiplicity and uniqueness results for the following *k-loop problem*: given a non constant function $k : \mathbb{C} \rightarrow \mathbb{R}$, find a k -loop, that is, a closed curve u in $\mathbb{C}$ having prescribed curvature $k(p)$ at every point $p \in u$.

Several papers deal with loops with prescribed curvature and with related questions; see for example [4], [5], [8], [9], [12], [13]. For similar problems in higher dimensions we quote [3], [6], [7], [13] and references therein.

In Section 2 we provide global sufficient conditions for the existence of k -loops by studying the ordinary differential system

$$(0.1) \quad \begin{cases} u'' = \left(\int_0^1 |u'|^2 \right)^{1/2} k(u)(iu') & \text{on } (0, 1), \\ u \in H_{per}, \end{cases}$$

where H_{per} is the Sobolev space of 1-periodic functions in $H_{loc}^1(\mathbb{R}, \mathbb{C})$. Any non constant weak solution u to (0.1) is smooth by regularity theory. As u'' is orthogonal to u' , we have that $|u'|$ is a constant. Thus $u'' \cdot (iu') = |u'|^3 k(u)$; that is, u is a k -loop.

If the prescribed curvature $k \in C^2(\mathbb{C})$ satisfies

$$(k_1) \quad M_k := \sup_{z \in \mathbb{C}} |(\nabla k(z) \cdot z)| < 1,$$

$$(k_2) \quad \text{there exists a constant } k_\infty > 0 \text{ such that } k(z) = k_\infty + o(|z|^{-1}) \text{ as } |z| \rightarrow \infty,$$

then problem (0.1) can be plainly studied via variational methods. Indeed solutions to (0.1) turn out to be critical points of an energy functional $\mathcal{E}_k : H_{per} \rightarrow \mathbb{R}$ that

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enjoys good regularity and geometrical properties thanks to assumptions (k_1) and (k_2) (see Section 1 for details). In particular the Nehari manifold

$$\Sigma = \{u \in H_{per} \mid u \text{ is non-constant, } \mathcal{E}'_k(u) \cdot u = 0\}$$

is a smooth submanifold of H_{per} and the functional $\mathcal{E}_k$ is positively bounded from below on Σ . Notice that every k -loop admits a parametrization $u \in \Sigma$. In spite of the non-homogeneous nature of problem (0.1), it turns out that Σ is a natural constraint for $\mathcal{E}_k$; that is, any stationary point for $\mathcal{E}_k(u)$ on Σ is a k -loop. In particular, if $u \in \Sigma$ achieves the infimum

$$(0.2) \quad \underline{c} := \inf_{u \in \Sigma} \mathcal{E}_k(u),$$

then u is a *minimal k -loop*, that means that u has the minimal energy among all k -loops.

The infimum $\underline{c}$ might be not achieved, due to the lack of compactness produced by the group of translations $u \mapsto u + p$. On the other hand, in Lemma 1.4 we show that $\underline{c} \leq \pi/k_\infty$. Then we prove in Theorem 2.1 that a minimal k -loop $u \in \Sigma$ exists, provided that $\underline{c} < \pi/k_\infty$. In particular, it turns out that $\underline{c} < \pi/k_\infty$ if $k(z) > k_\infty$ for $|z|$ large; see Theorem 2.5.

The main results in the present paper highlight the connection between the monotonicity properties of the prescribed curvature k along radial directions and the existence of geometrically distinct k -loops.

For the sake of clarity in this introduction we restrict ourselves to radially symmetric curvatures. It is straightforward to notice that if $k(z) = k(|z|)$, then $\mathcal{R} \circ u$ is a k -loop for any k -loop u and for any rotation $\mathcal{R}$ of the complex plane. Thus, any non-round k -loop generates a 1-dimensional manifold of distinct k -loops.

Multiple k -loops may be produced by a breaking symmetry phenomenon for problem (0.2). For instance, one can construct a curvature $k(z) = k(|z|)$ satisfying (k_1) , (k_2) and such that there exist a k -loop which is a circle C_R about the origin and a minimal k -loop which is not a circle (see Corollary 3.2). Here we have parametrized C_R by the map $t \mapsto Re^{2\pi it}$, so that C_R can be regarded as a k -loop in H_{per} . More precisely, it turns out that C_R is not a minimal k -loop if k increases along the normal directions to C_R .

Theorem 0.1. *Assume that $k \in C^2(\mathbb{C})$ is a radially symmetric curvature satisfying (k_1) and (k_2) . If there is a radius $R > 0$ with $Rk(R) = 1$ and $k'(R) > 0$, then the circle C_R is not a minimal k -loop.*

Theorem 0.1 is a consequence of a more general result that will be stated in Section 3 (see Theorem 3.1).

Roughly speaking, if the prescribed curvature is somewhere increasing, then one can expect breaking symmetry phenomena and multiple k -loops. This might happen also for curvatures that do not satisfy the assumptions (k_1) and (k_2) . In Example 4.2 we notice that the curvature of any ellipse increases with the distance from the center. Conversely, non-increasing curvatures may have only round k -loops.

Theorem 0.2. *Let $k \in C^0(\mathbb{C})$ be a positive and radially symmetric function. Assume that k is non-increasing as a function of the distance from the origin. Then any embedded k -loop is a circle.*

The result is sharp, in view of Example 4.2 in Section 4.

Our proof of Theorem 0.2 is based on Osserman's construction for the four-vertex theorem. One of the main tools is the *Touching Lemma* for the mean curvature operator, that follows from Hopf's maximum principle.

Several uniqueness (up to homothety) results are available for similar geometrical problems. A complete list of references would lead us far from our purpose. We quote the pioneering papers [2] by Alexandrov and [1] by Aeppli, where the prescribed curvature is assumed to be homogeneous of degree -1 . Treibergs and Wei [13] proved the uniqueness of embedded radial graphs over the unit sphere $\mathbb{S}^n$ having positive prescribed mean curvature $H \in C^1(\mathbb{R}^{n+1})$, such that $H(p) + \nabla H(p) \cdot p \leq 0$ for any $p \in \mathbb{R}^{n+1}$.

A result similar to Theorem 0.2 is missing for the H -bubble problem. It would be of interest to know if spheres are the only convex surfaces in $\mathbb{R}^3$ whose mean curvature H is a non-increasing function of the distance from the origin.

Notation. Let z_1, z_2 be two points in the complex plane $\mathbb{C}$. We denote by $z_1 z_2 \in \mathbb{C}$ and by $z_1 \cdot z_2 \in \mathbb{R}$ their complex and scalar products, respectively. Let $R > 0$ and $z \in \mathbb{C}$. The open disk of center z and radius R is denoted by $D_R(z)$. If $z = 0$ we simply write D_R instead of $D_R(0)$. The unit circle $\mathbb{S}^1 = \partial D_1$ is parametrized by the function

$$(0.3) \quad \omega(t) := e^{2\pi i t}, \quad \omega : (0, 1) \rightarrow \mathbb{C}.$$

A *loop* is a closed curve parametrized by a periodic function $g : \mathbb{R} \rightarrow \mathbb{C}$ of class C^2 and such that $g'(t) \neq 0$ for any $t \in I$. We will often identify the curve g with its image. It is well known that the only loops having constant curvature are circles. An *embedded loop* is a closed curve without self-intersections. Any compact, connected, 1-dimensional submanifold of $\mathbb{C}$ without boundary is an embedded loop.

We let

$$H_{per} = \{\bar{u} \circ \omega \mid \bar{u} \in H^1(\mathbb{S}^1, \mathbb{C})\},$$

where ω is defined in (0.3). We define in H_{per} the Hilbertian scalar product

$$\langle u, v \rangle = \int_0^1 u' v' + \left(\int_0^1 u \right) \cdot \left(\int_0^1 v \right).$$

Notice that H_{per} contains $\mathbb{C}$ as a closed subspace. We denote by H_{per}^{-1} the topological dual space to H_{per} .

1. PRELIMINARIES

We start by recalling the main features of the variational approach to problem (0.1). For details we refer to [5] and to [9]. Let $k \in C^0(\mathbb{C})$ be a given bounded function, and set

$$m(z) = \int_0^1 k(sz) s \, ds, \quad m : \mathbb{C} \rightarrow \mathbb{R}.$$

For $u \in H_{per}$ we put

$$\mathcal{L}(u) = \left(\int_0^1 |u'|^2 \right)^{1/2}, \quad \mathcal{A}_k(u) = \int_0^1 m(u) u \cdot (iu').$$

The functional $\mathcal{A}_k(u)$ is well defined on H_{per} as $u \in L^\infty$ for any $u \in H_{per}$, by the Sobolev embedding theorem. The real number $\mathcal{A}_k(u)$ measures the algebraic area

enclosed by the curve u with respect to the weight k . Moreover, the following isoperimetric inequality holds:

$$(1.1) \quad 4\pi|\mathcal{A}_k(u)| \leq \|k\|_\infty \mathcal{L}(u)^2 \quad \text{for any } u \in H_{per}.$$

For any constant $k_\infty \neq 0$ it turns out that

$$(1.2) \quad \mathcal{A}_{k_\infty}(u) = \frac{k_\infty}{2} \int_0^1 u \cdot (iu') \, ds.$$

In the next lemma we state some remarks on the area functional. We omit the simple proofs.

Lemma 1.1. *Let $k \in C^1(\mathbb{C})$. Assume that $M_k = \sup_{z \in \mathbb{C}} |(\nabla k(z) \cdot z)|$ is finite and that there exists $\lim_{|z| \rightarrow \infty} k(z) = k_\infty \in \mathbb{R}$. Then*

$$(1.3) \quad |(k(z) - k_\infty)z| \leq M_k,$$

$$(1.4) \quad |(2m(z) - k(z))z| \leq M_k,$$

$$(1.5) \quad 2|\mathcal{A}_{k-k_\infty}(u)| \leq M_k \mathcal{L}(u) \quad \text{for any } u \in H_{per}.$$

The energy $\mathcal{E}_k : H_{per} \rightarrow \mathbb{R}$ is defined by

$$\mathcal{E}_k(u) = \mathcal{L}(u) + \mathcal{A}_k(u) = \left(\int_0^1 |u'|^2 \, ds \right)^{1/2} + \int_0^1 m(u) u \cdot (iu') \, ds.$$

If k is of class C^1 , then the functional $\mathcal{E}_k$ is Frechét differentiable on $H_{per} \setminus \mathbb{C}$ and

$$(1.6) \quad \mathcal{E}'_k(u) \cdot \varphi = \frac{1}{\mathcal{L}(u)} \int_0^1 u' \cdot \varphi' + \int_0^1 k(u) \varphi \cdot (iu') \, ds \quad \text{for any } u \in H_{per} \setminus \mathbb{C}, \varphi \in H_{per}$$

(see for example [5]). In particular, any critical point for $\mathcal{E}_k$ on $H_{per} \setminus \mathbb{C}$ parametrizes a smooth k -loop.

Remark 1.2. Let k be as in Lemma 1.1, and assume in addition that $k_\infty \neq 0$. Then the energy $\mathcal{E}_k$ is unbounded from below. Fix any map $u \in H_{per}$ such that $\mathcal{A}_{k_\infty}(u) < 0$. Then, using (1.5), we get

$$\begin{aligned} \mathcal{E}_k(su) &= \mathcal{L}(su) + \mathcal{A}_{k-k_\infty}(su) + \mathcal{A}_{k_\infty}(u) \\ &\leq (1 + M_k)s\mathcal{L}(u) + s^2\mathcal{A}_{k_\infty}(u) \rightarrow -\infty \quad \text{as } s \rightarrow \infty. \end{aligned}$$

Next we introduce the curvature function

$$f(z) := 2k(z) + (\nabla k(z) \cdot z), \quad f : \mathbb{C} \rightarrow \mathbb{R}.$$

Since $\int_0^1 f(sz) s \, ds = k(z)$, then from (1.6) we infer

$$(1.7) \quad \mathcal{E}'_k(u) \cdot u = \mathcal{L}(u) + \int_0^1 k(u) u \cdot (iu') \, ds = \mathcal{E}_f(u).$$

Thus, $\mathcal{E}'_k(u) \cdot u$ equals the energy of u with respect to the curvature f . If k is of class C^2 , then, coherently with (1.6), the functional $\mathcal{E}_f$ is Frechét differentiable on $H_{per} \setminus \mathbb{C}$, and

$$(1.8) \quad \mathcal{E}'_f(u) \cdot \varphi = \frac{1}{\mathcal{L}(u)} \int_0^1 u' \cdot \varphi' + \int_0^1 (2k(u) + (\nabla k(u) \cdot u)) \varphi \cdot (iu') \, ds$$

for any $u \in H_{per} \setminus \mathbb{C}$, $\varphi \in H_{per}$. Lastly we introduce the Nehari manifold

$$\Sigma = \{u \in H_{per} \mid \mathcal{L}(u) > 0, \mathcal{E}'_k(u) \cdot u = \mathcal{E}_f(u) = 0\}$$

and the infimum

$$(1.9) \quad \underline{c} := \inf_{\Sigma} \mathcal{E}_k.$$

Remark 1.3. Assume that $k \in C^2(\mathbb{C})$ satisfies $M_k < 1$ and $k(z) \rightarrow k_{\infty} = 0$ as $|z| \rightarrow \infty$. Then from (1.5), (1.7) and (1.3) it easily follows that the energy is coercive with respect to the seminorm $\mathcal{L}(u)$ and that the manifold Σ is empty.

From now on we assume that $k \in C^2(\mathbb{C})$ satisfies the assumptions (k_1) and (k_2) in the introduction. In the next lemma we show that the set Σ is not empty and smooth, and it is a natural constraint for $\mathcal{E}_k$.

Lemma 1.4. *The following facts hold.*

- (1) Let $u \in H_{per} \setminus \mathbb{C}$. Then $u \in \Sigma$ if and only if $\mathcal{E}_k(u) = \sup_{s>0} \mathcal{E}_k(su)$.
- (2) Σ is a non-empty submanifold of H_{per} of class C^1 .
- (3) $0 < \underline{c} \leq \pi/k_{\infty}$.
- (4) Every critical point u for $\mathcal{E}_k$ on Σ solves $\mathcal{E}'_k(u) = 0$.

Proof. We start by noticing that

$$(1.10) \quad \mathcal{E}'_f(u) \cdot u = -\mathcal{L}(u) + \int_0^1 (\nabla k(u) \cdot u) u \cdot (iu') \quad \text{for any } u \in \Sigma$$

by (1.8). Thus from (k_1) we readily get

$$(1.11) \quad -(1 + M_k)\mathcal{L}(u) \leq \mathcal{E}'_f(u) \cdot u \leq (-1 + M_k)\mathcal{L}(u) < 0 \quad \text{for any } u \in \Sigma.$$

To prove (1) we fix any function u in $H_{per} \setminus \mathbb{C}$ such that $\mathcal{E}_k(u) = \sup_{s>0} \mathcal{E}_k(su)$. Then the derivative of the function $s \mapsto \mathcal{E}_k(su)$ vanishes at $s = 1$; that is, $u \in \Sigma$. Conversely, if $u \in \Sigma$ we put $F(s) := \mathcal{E}_k(su)$ for $s > 0$. We use (1.7), (1.10) and (1.11) to get

$$\begin{aligned} sF'(s) &= \mathcal{E}_f(su), \\ sF'(s) + s^2F''(s) &= \mathcal{E}'_f(su) \cdot (su) \leq (-1 + M_k)s^2\mathcal{L}(u). \end{aligned}$$

In particular, $s = 1$ is a critical point for F on $\{s > 0\}$ and every critical point for F is a local maximum for F . Thus the function F achieves its absolute maximum at $s = 1$; that is, $\mathcal{E}_k(u) = \sup_{s>0} \mathcal{E}_k(su)$. This completes the proof of (1).

We point out that from (1) and from Remark 1.2 it follows that Σ is non-empty and

$$(1.12) \quad \underline{c} = \inf_{\substack{u \in H_{per} \\ \mathcal{A}_{k_{\infty}}(u) < 0}} \sup_{s>0} \mathcal{E}_k(su).$$

Since $k \in C^2(\mathbb{C})$ we have that $\mathcal{E}_f$ is of class C^1 on $H_{per} \setminus \mathbb{C}$. In addition, from (1.11) it follows that $\mathcal{E}'_f(u) \neq 0$ for any $u \in \Sigma$. Thus the constraint Σ has a normal direction at every point, and claim (2) is proved.

To check that $\underline{c}$ is positive we use the isoperimetric inequality (1.1). For any $u \in \Sigma$ we get $0 = \mathcal{E}_f(u) = \mathcal{L}(u) + \mathcal{A}_f(u) \geq \mathcal{L}(u) - \|f\|_{\infty}(4\pi)^{-1}\mathcal{L}(u)^2$. In particular,

$$(1.13) \quad \|f\|_{\infty}\mathcal{L}(u) \geq 4\pi \quad \text{for any } u \in \Sigma.$$

As $\mathcal{E}_f(u) = 0$ for any $u \in \Sigma$, then from (1.7) we get

$$(1.14) \quad 2\mathcal{E}_k(u) = \mathcal{L}(u) + \int_0^1 (2m(u) - k(u)) u \cdot (iu') \quad \text{for any } u \in \Sigma.$$

Thus from (1.4) we infer

$$(1.15) \quad 2\mathcal{E}_k(u) \geq (1 - M_k)\mathcal{L}(u) \quad \text{for any } u \in \Sigma,$$

that compared with (1.13) gives $\underline{c} > 0$, since $M_k < 1$. To prove that $\underline{c} \leq \pi/k_\infty$ we parametrize the unit circle around 0 with the curve $\omega(t) = e^{2\pi it}$ as in (0.3). By (1.12) it turns out that

$$\underline{c} \leq \sup_{s>0} \mathcal{E}_k(s(\omega + p)) = \sup_{s>0} [\mathcal{E}_{k_\infty}(s(\omega + p)) + \mathcal{A}_{k-k_\infty}(s(\omega + p))]$$

for any $p \in \mathbb{C}$. It is easy to compute

$$\mathcal{E}_{k_\infty}(s(\omega + p)) = 2\pi s - \pi k_\infty s^2.$$

Since $s(\omega + p)$ parametrizes the circle of radius s around sp , then

$$|\mathcal{A}_{k-k_\infty}(s(\omega + p))| = \left| \int_{D_s(sp)} (k(z) - k_\infty) dz \right| \leq M_k \int_{D_s(sp)} \frac{1}{|z|} dz$$

by the divergence theorem and by (1.3). Thus for $|p|$ large it holds that

$$|\mathcal{A}_{k-k_\infty}(s(\omega + p))| \leq M_k \int_{D_s(sp)} \frac{1}{s(|p| - 1)} dz = M_k \frac{\pi s}{|p| - 1},$$

and therefore

$$(1.16) \quad \sup_{s>0} \mathcal{E}_k(s(\omega + p)) \leq \sup_{s>0} \pi s \left(2 + \frac{M_k}{|p| - 1} - k_\infty s \right) = \frac{\pi}{k_\infty} + O(|p|^{-1}).$$

Hence $\underline{c} \leq \pi/k_\infty$, as desired.

It remains to prove (4). If $u \in \Sigma$ is a critical point for $\mathcal{E}_k$ on Σ , then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $\mathcal{E}'_k(u) \cdot \varphi = \lambda \mathcal{E}'_f(u) \cdot \varphi$ for any $\varphi \in H_{per}$. In particular $\lambda \mathcal{E}'_f(u) \cdot u = \mathcal{E}'_k(u) \cdot u = 0$ as $u \in \Sigma$. Thus $\lambda = 0$ by (1.11), and $\mathcal{E}'_k(u) = 0$. $\square$

The next lemma will be crucial later on.

Lemma 1.5. *For any $p \in \mathbb{C}$ there exists a unique $s_p > 0$ such that $s_p(\omega + p) \in \Sigma$. Moreover, the function $p \mapsto s_p$ is of class C^1 on $\mathbb{C}$.*

Proof. Argue as in the proof of claim (1) to check that for any $p \in \mathbb{C}$ the map $s \mapsto \mathcal{E}_k(s(\omega + p))$ achieves its maximum on $(0, \infty)$ at its unique critical point $s_p \in (0, \infty)$. Thus $s_p(\omega + p) \in \Sigma$. As s_p is uniquely and implicitly defined by the equation $\mathcal{E}_f(s_p(\omega + p)) = 0$, and since the functional $\mathcal{E}_f$ is continuously differentiable on $H_{per} \setminus \mathbb{C}$, then the function $p \mapsto s_p$ is of class C^1 on $\mathbb{C}$ by (1.11). $\square$

Remark 1.6. Lemma 1.4 holds whenever $M_k < 1$ and $k(z) \rightarrow k_\infty > 0$ as $|z| \rightarrow \infty$.

2. EXISTENCE

We are now in a position to prove our existence result.

Theorem 2.1. *Assume that $k \in C^2(\mathbb{C})$ satisfies (k_1) , (k_2) and let $\underline{c}$ be the infimum in (1.9). If $\underline{c} < \pi/k_\infty$, then there exists a minimal k -loop.*

Proof. Let $u_n \in \Sigma$ be a minimizing sequence for $\mathcal{E}_k$ on Σ , that is,

$$(2.1) \quad \mathcal{E}_k(u_n) \rightarrow \underline{c} < \frac{\pi}{k_\infty}.$$

By Ekeland's variational principle we can assume that u_n is a sequence of approximate critical points for $\mathcal{E}_k$ on the constraint Σ . This means that

$$(2.2) \quad \mathcal{E}'_k(u_n) - \lambda_n \mathcal{E}'_f(u_n) \rightarrow 0 \quad \text{in } H_{per}^{-1}$$

for some $\lambda_n \in \mathbb{R}$. The seminorms $\mathcal{L}(u_n)$ are uniformly bounded by (1.15). Since the L^∞ norm of $f(u) = 2k(u) + (\nabla k(u) \cdot u)$ is bounded by a constant that depends only on k , then the sequence $\mathcal{E}'_f(u_n)$ is bounded in H_{per}^{-1} by (1.8). Finally, we notice that the sequence $\mathcal{E}'_f(u_n) \cdot u_n$ is bounded and bounded away from zero by (1.11).

To conclude, we only have to prove that the sequence

$$p_n := \int_0^1 u_n$$

is bounded in $\mathbb{C}$. Indeed, in this case we can assume that $u_n \rightharpoonup u$ weakly in H_{per} , for some $u \in H_{per}$. In particular, $u_n \rightarrow u$ uniformly on $[0, 1]$ by the Sobolev embedding theorem. Test (2.2) with u_n to get

$$0 = \mathcal{E}'_k(u_n) \cdot u_n = \lambda_n \mathcal{E}'_f(u_n) \cdot u_n + o(1).$$

Since the sequence $\mathcal{E}'_f(u_n) \cdot u_n$ is bounded away from zero, then $\lambda_n \rightarrow 0$, which implies $\mathcal{E}'_k(u_n) \rightarrow 0$ by (2.2). Since $|k(u_n)(u_n - u) \cdot (iu'_n)| \leq \|k\|_\infty \|u_n - u\|_\infty |u'_n| \rightarrow 0$ in L^1 , then, in particular,

$$o(1) = \mathcal{E}'_k(u_n) \cdot (u_n - u) = \frac{1}{\mathcal{L}(u_n)} \int_0^1 u'_n \cdot (u'_n - u') + o(1) = \frac{1}{\mathcal{L}(u_n)} \int_0^1 |u'_n - u'|^2 + o(1).$$

Taking (1.13) into account we infer that $u'_n \rightarrow u'$ strongly in L^2 . Therefore $u_n \rightarrow u$ in the H_{per} -norm. Then $\mathcal{E}'(u) = 0$ and $\mathcal{E}_k(u) = \underline{c}$ easily follow by continuity.

Assume by contradiction that the averages p_n are not bounded. Then, for a subsequence, it turns out that $u_n - p_n \rightarrow U$ weakly in H_{per} and uniformly on $[0, 1]$, where $U \in H_{per}$ has zero mean value on $(0, 1)$. We can also assume that there exists

$$\alpha := \lim_{n \rightarrow \infty} \mathcal{L}(u_n) \in \mathbb{R}.$$

Notice that $\alpha > 0$ by (1.13). Since $u_n - p_n \rightarrow U$ and $|u_n| \rightarrow \infty$ uniformly, then from (k₁) and (k₂) it follows that

$$(2.3) \quad (k(u_n) - k_\infty)u_n \rightarrow 0 \quad \text{uniformly,}$$

$$(2.4) \quad k(u_n)(u_n - p_n) \rightarrow k_\infty U \quad \text{uniformly,}$$

$$(2.5) \quad \nabla k(u_n) \cdot u_n \rightarrow 0 \quad \text{uniformly.}$$

Using (2.3) we get

$$\int_0^1 k(u_n)u_n \cdot (iu'_n) = k_\infty \int_0^1 u_n \cdot (iu'_n) + o(1) = k_\infty \int_0^1 U \cdot (iU') + o(1),$$

as $\int_0^1 (iu'_n) = 0$. In particular, from $\mathcal{E}'_k(u_n) \cdot u_n = 0$ and from (1.7) we infer that $U \neq 0$ and

$$(2.6) \quad \alpha = -k_\infty \int_0^1 U \cdot (iU').$$

Next we notice that $(k(u_n) - k_\infty)p_n = (k(u_n) - k_\infty)(p_n - u_n) + (k(u_n) - k_\infty)u_n \rightarrow 0$ uniformly, since $k(u_n) \rightarrow k_\infty$ uniformly, $\sup_n \|u_n - p_n\| < \infty$, and by (2.3). This implies

$$\int_0^1 k(u_n)p_n \cdot (iu'_n) = k_\infty p_n \cdot \int_0^1 (iu'_n) + o(1) = o(1).$$

Using (1.6) once again we infer

$$\mathcal{E}'_k(u_n) \cdot (u_n - p_n) = -\mathcal{E}'_k(u_n) \cdot p_n = \int_0^1 k(u_n)p_n \cdot (iu'_n) = o(1),$$

as $u_n \in \Sigma$. By similar arguments, from (1.8) we also obtain

$$(2.7) \quad \mathcal{E}'_f(u_n) \cdot (u_n - p_n) = \alpha + 2k_\infty \int_0^1 U \cdot (iU') + o(1) = -\alpha + o(1)$$

by (2.4), (2.5) and (2.6). Consequently, since by (2.2) we have that

$$\mathcal{E}'_k(u_n) \cdot (u_n - p_n) = \lambda_n \mathcal{E}'_f(u_n) \cdot (u_n - p_n) + o(1),$$

then $o(1) = \lambda_n(-\alpha + o(1))$. Since $\alpha > 0$, then $\lambda_n \rightarrow 0$. Thus $\mathcal{E}'_k(u_n) \cdot \varphi \rightarrow 0$ for any $\varphi \in H_{per}$ by (2.2), as the sequence $\mathcal{E}'_f(u_n)$ is bounded in H_{per}^{-1} . Therefore

$$o(1) = \mathcal{E}'_k(u_n) \cdot \varphi = \frac{1}{\alpha} \int_0^1 U' \varphi' + 2k_\infty \int_0^1 \varphi \cdot (iU') + o(1) \quad \text{for any } \varphi \in H_{per}.$$

Thus, U solves $U'' = \alpha k_\infty (iU')$, and, in particular, $\mathcal{L}(U) = \alpha = \lim_n \mathcal{L}(u_n)$ by (2.6). This is sufficient to conclude that $u_n - p_n \rightarrow U$ strongly in H_{per} . In addition, U is a non-constant solution to the linear ordinary differential system $U'' = \mathcal{L}(U)k_\infty (iU')$. Thus U parametrizes a circle of radius k_∞^{-1} , and

$$(2.8) \quad \mathcal{L}(U) = -k_\infty \int_0^1 U \cdot (iU') \geq \frac{2\pi}{k_\infty}, \quad \mathcal{E}_{k_\infty}(U) = \frac{L(U)}{2}.$$

Next we claim that $\mathcal{E}_k(u_n) = \mathcal{E}_{k_\infty}(U) + o(1)$. Notice that

$$(2m(u_n) - k_\infty)(u_n) = 2 \int_0^1 (k(su_n) - k_\infty)(su_n) ds \rightarrow 0$$

pointwise in $(0, 1)$ by (k_2) , (1.3) and Lebesgue's Theorem. Since, in addition, $iu'_n \rightarrow iU'$ in L^2 , then

$$2\mathcal{A}_{k-k_\infty}(u_n) = \int_0^1 (2m(u_n) - k_\infty)(u_n) \cdot (iu'_n) = o(1)$$

by (1.4) and again by Lebesgue's Theorem. In conclusion, we get

$$\mathcal{E}_k(u_n) = \mathcal{L}(u_n) + \frac{k_\infty}{2} \int_0^1 (u_n - p_n) \cdot (iu'_n) + \mathcal{A}_{k-k_\infty}(u_n) = \mathcal{E}_{k_\infty}(U) + o(1) = \frac{\pi}{k_\infty} + o(1).$$

Taking (2.1) and (2.8) into account we get

$$\frac{\pi}{k_\infty} > \underline{c} = \mathcal{E}_k(u_n) + o(1) = \mathcal{E}_{k_\infty}(U) + o(1) \geq \frac{\pi}{k_\infty} + o(1),$$

a contradiction for n sufficiently large. This proves that the sequence p_n is bounded in $\mathbb{C}$ and completes the proof of the theorem. $\square$

Remark 2.2. We notice that the assumption on the sign of k_∞ is not restrictive. Indeed, $t \mapsto u(t)$ is a k -loop if and only if $t \mapsto u(1-t)$ has curvature $-k$.

Remark 2.3. By suitably modifying the compactness argument in the proof of Theorem 2.1 one can completely describe the behavior of any sequence of approximate solutions. More precisely, assume that $u_n \in \Sigma$ satisfies $\mathcal{E}_k(u_n) \rightarrow c$ and $\mathcal{E}'_k(u_n) - \lambda_n \mathcal{E}'_f(u_n) \rightarrow 0$ for some $c \in \mathbb{R}$ and for a sequence $\lambda_n \in \mathbb{R}$. Then, up to a subsequence, either u_n converges in H_{per} to a k -loop u such that $\mathcal{E}_k(u) = c$, or $\mu := \pi c k_\infty^{-1}$ is a positive integer, and

$$\left| \int_0^1 u_n \right| \rightarrow \infty, \quad u_n - \int_0^1 u_n \rightarrow U \quad \text{in } H_{per},$$

where $U(t) = k_\infty^{-1} \sigma e^{2\pi i \mu t}$ for some $\sigma \in \mathbb{S}^1$.

Remark 2.4. Assume that $\underline{c} = \pi/k_\infty$ and that no minimal k -loop exists. Then using Remark 2.3 and arguing as in [7] one can find sufficient conditions for the existence of highly unstable k -loops.

Some conditions to ensure that $\underline{c} < \pi/k_\infty$ can be easily given. For instance, notice that

$$\underline{c} \leq \sup_{s>0} \mathcal{E}_k(s(\omega + p))$$

for any $p \in \mathbb{C}$, by (1.12), where ω is defined in (0.3). Since $s(\omega + p)$ parametrizes the circle of radius s around sp , then $\mathcal{A}_k(s(\omega + p)) = -\int_{D_s(sp)} k(z) dz$ by the divergence theorem. Thus

$$(2.9) \quad \mathcal{E}_k(s(\omega + p)) = 2\pi s - \int_{D_s(sp)} k(z) dz.$$

In particular, $\underline{c} < \pi/k_\infty$ if there exists a point $p \in \mathbb{C}$ such that

$$\sup_{s>0} \left(2\pi s - \int_{D_s(sp)} k(z) dz \right) < \frac{\pi}{k_\infty}.$$

This happens, for instance, if $k(z) > k_\infty$ on $\mathbb{C}$ (take $p = 0$). Actually, weaker sufficient conditions can be given.

Theorem 2.5. *Assume that $k \in C^2(\mathbb{C})$ satisfies (k_1) , (k_2) . If $k(z) > k_\infty$ for $|z|$ large, then there exists a minimal k -loop.*

Proof. We only have to show that $\underline{c} < \pi/k_\infty$. Put as before $\omega(t) = e^{2\pi i t}$. For any $p \in \mathbb{C}$ let $s_p > 0$ be the unique positive number defined in Lemma 1.5. Then $\underline{c} \leq \mathcal{E}_k(s_p(\omega + p))$, since $s_p(\omega + p) \in \Sigma$.

Recall from (1.7) that $\mathcal{E}_f(u) = \mathcal{L}(u) + k_\infty \int_0^1 u \cdot (iu') + \int_0^1 (k(u) - k_\infty) u \cdot (iu')$ for any $u \in H_{per}$, and therefore

$$0 = \mathcal{E}_f(s_p(\omega + p)) = 2\pi s_p - 2\pi k_\infty s_p^2 + s_p \int_0^1 (k(s_p(\omega + p)) - k_\infty) s_p(\omega + p) \cdot (i\omega').$$

In particular we infer

$$(2.10) \quad k_\infty s_p = 1 - \int_0^1 (k(s_p(\omega + p)) - k_\infty) s_p(\omega + p) \cdot \omega$$

since $i\omega' = -2\pi\omega$. From (1.3) and (2.10) we get

$$(2.11) \quad 1 - M_k \leq k_\infty s_p \leq 1 + M_k \quad \text{for any } p \in \mathbb{C}.$$

Next we take a sequence of points $p_n \in \mathbb{C}$ such that $|p_n| \rightarrow \infty$. By (2.9), (2.11), and since $k(z) > k_\infty$ for $|z|$ large enough, we get

$$\underline{c} \leq \mathcal{E}_k(s_{p_n}(\omega + p_n)) = 2\pi s_{p_n} - \int_{D_{s_{p_n}}(s_{p_n} p_n)} k(z) dz < 2\pi s_{p_n} - \pi k_\infty s_{p_n}^2.$$

Thus $\underline{c} < \sup_{s>0} (2\pi s - \pi k_\infty s^2) = \pi/k_\infty$, and the theorem is completely proved. $\square$

Remark 2.6. A slightly different proof can be obtained by following the arguments in [6], proof of Corollary 2.13. As in [6] the assumption on the sign of $k(z) - k_\infty$ can be weakened. It is sufficient to ask that there exist $\sigma \in \mathbb{S}^1$, $R, \delta > 0$ such that $k(z) > k_\infty$ for any $z \in \mathbb{C} \setminus D_R$ with $|\sigma - z|^{-1} < \delta$.

3. BREAKING SYMMETRY AND MULTIPLICITY

In this section we identify the circle $C_R = \{|z| = R\}$ with its parametrization $t \mapsto Re^{2\pi it}$. With this notation we have that C_R is a k -loop if and only if $Rk(\cdot) \equiv 1$ on C_R . Theorem 0.1 is an immediate consequence of the next result.

Theorem 3.1. *Let $k \in C^2(\mathbb{C})$ be a curvature satisfying (k_1) and (k_2) . Assume that there exists $R > 0$ such that $Rk(z) \equiv 1$ and $\nabla k(z) \cdot z \geq 0$ for any $z \in C_R$. If the circle C_R is a minimal k -loop, then $\nabla k(z) \cdot z \equiv 0$ on C_R .*

Proof. We start by noticing that $u \in H_{per}$ is a k -loop if and only if $u_R := Ru$ is a k_R -loop, where $k_R(z) = R^{-1}k(R^{-1}z)$. Therefore we are allowed to assume $R = 1$ and $C_R = \mathbb{S}^1$. Hence $k \equiv 1$ on the unit sphere, $\mathbb{S}^1$ is a minimal k -loop and

$$(3.1) \quad A(z) := \nabla k(z) \cdot z \geq 0 \quad \text{on } \mathbb{S}^1.$$

Since $k \in C^2(\mathbb{C})$, then the energy $\mathcal{E}_k$ is twice differentiable on $H_{per} \setminus \mathbb{C}$, and

$$\begin{aligned} \mathcal{E}_k''(u)[\varphi, \psi] &= \frac{1}{\mathcal{L}(u)} \int_0^1 \varphi' \cdot \psi' - \frac{1}{\mathcal{L}(u)^3} \left(\int_0^1 u' \cdot \varphi' \right) \left(\int_0^1 u' \cdot \psi' \right) \\ &\quad + \int_0^1 \varphi \cdot [k(u)(i\psi') + (\nabla k(u) \cdot \psi)(iu')] \end{aligned}$$

for all $u \in H_{per} \setminus \mathbb{C}$, and for any $\varphi, \psi \in H_{per}$ (see [5]). To compute $\mathcal{E}_k''(\omega)$ we notice that $\mathcal{L}(\omega) = 2\pi$ and $\omega'' = 2\pi(i\omega') = -(2\pi)^2\omega$. In particular $\int \omega' \cdot \varphi' = (2\pi)^2 \int \omega \cdot \varphi$ for any $\varphi \in H_{per}$. Since k is constant on $\mathbb{S}^1$, then $\nabla k(\omega) = A(\omega)\omega$, where A is defined in (3.1). Thus we get

$$\begin{aligned} \mathcal{E}_k''(\omega)[\varphi, \psi] &= \frac{1}{2\pi} \int_0^1 \varphi' \cdot \psi' - 2\pi \left(\int_0^1 \omega \cdot \varphi \right) \left(\int_0^1 \omega \cdot \psi \right) \\ (3.2) \quad &\quad + \int_0^1 \varphi \cdot [(i\psi') - 2\pi A(\omega)(\omega \cdot \psi)\omega] \end{aligned}$$

for any $\varphi, \psi \in H_{per}$. From (3.2) we first get $\mathcal{E}_k''(\omega)[\varphi, \omega] = -2\pi \int_0^1 (1 + A(\omega)) \omega \cdot \varphi$ for any $\varphi \in H_{per}$, and in particular

$$(3.3) \quad \mathcal{E}_k''(\omega)[\omega, \omega] = -2\pi \left(1 + \int_0^1 A(\omega) \right) < 0.$$

Taking instead $\varphi \equiv p$ to be a constant function we get

$$(3.4) \quad \mathcal{E}_k''(\omega)[p, \omega] = -2\pi \int_0^1 A(\omega) \omega \cdot p,$$

since ω has zero mean value on $(0, 1)$. From (3.2) we infer also

$$(3.5) \quad \mathcal{E}_k''(\omega)[p, p] = -2\pi \int_0^1 A(\omega)(\omega \cdot p)^2.$$

For any point $p \in \mathbb{C}$ we put $s(t) := s_{tp}$, where $s_{tp} > 0$ is defined by the condition

$$s_{tp}(\omega + tp) = s(t)(\omega + tp) \in \Sigma,$$

as in Lemma 1.5. Since ω is a minimal k -loop, the function $g(t) := \mathcal{E}_k(s(t)(\omega + tp))$ attains its minimum at $t = 0$. Notice that $s(0) = 1$, since ω is a k -loop. Moreover the map $s(t)$ is of class C^1 on $(0, \infty)$ by Lemma 1.5 and

$$(3.6) \quad \mathcal{E}_k'(s(t)(\omega + tp)) \cdot (\omega + tp) = 0 \quad \text{for any } t \in \mathbb{R}.$$

We compute

$$g'(t) = \mathcal{E}_k'(s(t)(\omega + tp)) \cdot (s'(t)(\omega + tp) + s(t)p) = s(t) \mathcal{E}_k'(s(t)(\omega + tp)) \cdot p,$$

by (3.6). Thus g is twice differentiable, and

$$g''(t) = s'(t) \mathcal{E}_k'(s(t)(\omega + tp)) \cdot p + s(t) \mathcal{E}_k''(s(t)(\omega + tp)) [p, s'(t)(\omega + tp) + p].$$

From $s(0) = 1$ and $\mathcal{E}_k'(\omega) = 0$ we get

$$(3.7) \quad g''(0) = \beta \mathcal{E}_k''(\omega)[p, \omega] + \mathcal{E}_k''(\omega)[p, p],$$

where we have set $\beta = s'(0)$. To compute β we differentiate (3.6) with respect to t :

$$\mathcal{E}_k'(s(t)(\omega + tp)) \cdot p + \mathcal{E}_k''(s(t)(\omega + tp))[\omega + tp, s'(t)(\omega + tp) + s(t)p] = 0.$$

Thus at $t = 0$ it holds that $\mathcal{E}_k''(\omega)[\omega, \beta\omega + p] = 0$, compared with (3.7), gives

$$\begin{aligned} g''(0) &= - \frac{(\mathcal{E}_k''(\omega)[p, \omega])^2}{\mathcal{E}_k''(\omega)[\omega, \omega]} + \mathcal{E}_k''(\omega)[p, p] \\ &= 2\pi \left[\frac{\left(\int_0^1 A(\omega)(\omega \cdot p) \right)^2}{1 + \int_0^1 A(\omega)} - \int_0^1 A(\omega)(\omega \cdot p)^2 \right] \end{aligned}$$

by (3.3), (3.4) and (3.5). Since 0 is a minimum point for g , then $g''(0) \geq 0$; that is,

$$\begin{aligned} \left(\int_0^1 A(\omega)(\omega \cdot p)^2 \right) \left(1 + \int_0^1 A(\omega) \right) &\leq \left(\int_0^1 A(\omega)(\omega \cdot p) \right)^2 \\ &\leq \left(\int_0^1 A(\omega)(\omega \cdot p)^2 \right) \left(\int_0^1 A(\omega) \right) \end{aligned}$$

by the Hölder inequality. We infer that $\int A(\omega)(\omega \cdot p)^2 = 0$ for any $p \in \mathbb{C}$, since $A(\omega) \geq 0$ by assumption. Take $p = 1 \in \mathbb{C}$ and then $p = i$ to get

$$0 = \int_0^1 A(\omega)[(\omega \cdot 1)^2 + (\omega \cdot i)^2] = \int_0^1 A(\omega).$$

Therefore $A(\omega) \equiv 0$, and the theorem is proved. $\square$

From now on we restrict our attention to radially symmetric curvatures $k(P) \equiv k(|P|)$. Quite trivially, if k is any continuous radially symmetric map on $\mathbb{C}$ such that $k(z) \rightarrow k_\infty > 0$ as $|z| \rightarrow \infty$, then there exists $R > 0$ such that $Rk(R) = 1$. In this case the circle C_R is a k -loop. In addition, if u is a k -loop, then $\mathcal{R} \circ u$ is

a k -loop for any rotation $\mathcal{R}$ of the complex plane. Therefore, the next multiplicity result immediately follows from Theorem 0.1.

Corollary 3.2. *Let $k \in C^2(\mathbb{C})$ be a radially symmetric function satisfying (k_1) , (k_2) . Assume that $k'(R) > 0$ for any R such that $Rk(R) = 1$. If $\underline{c}$ is achieved, then no minimal k -loop is a circle around the origin. Hence there exist at least one round k -loop and a rotationally-invariant family of non-round k -loops.*

It has to be noticed that the class of curvatures described in Corollary 3.2 is non-empty. For instance, let $\lambda > 0$ and put

$$k(r) = 1 + \lambda \frac{r^2 - 1}{r^4 + 1}.$$

If λ is sufficiently close to 0, then k satisfies (k_1) and (k_2) . In addition $Rk(R) = 1$ if and only if $R = 1$, and $k'(1) > 0$. The existence of a minimal k -loop is given by Theorem 2.5.

Remark 3.3. Let k be any continuous radially symmetric curvature. Then the existence of a round k -loop centered at 0 is a necessary condition for the existence of k -loops. Assume that $g : \mathbb{R} \rightarrow \mathbb{C}$ is a periodic parametrization by arclength of a k -loop. Then g solves

$$(3.8) \quad \begin{cases} g'' = k(|g|)(ig'), \\ |g'| = 1. \end{cases}$$

Let $P_0 = g(t_0)$ be a point on g such that $|P_0| = \bar{R} := \max\{|P| \mid P \in g\}$. Set $v(t) := |g(t)|^2$ and compute $v'' = 2(|g'|^2 + g \cdot g'') \geq 2(1 - |k(|g|)g|)$. Since v takes its maximum value at t_0 , then $0 \geq v''(t_0) \geq 2(1 - |k(\bar{R})\bar{R}|)$. Hence $\bar{R}|k(\bar{R})| \geq 1$. By the continuity of the function $r \mapsto r|k(r)|$, there exists a radius $R \in (0, \bar{R}]$ such that $R|k(R)| = 1$. Hence the circle C_R is a k -loop.

Remark 3.4. As an immediate consequence of Remark 3.3 we notice that no k -loops may exist if $k : \mathbb{C} \rightarrow \mathbb{R}$ is a continuous radially symmetric function such that $R|k(R)| < 1$ for any $R > 0$.

4. UNIQUENESS UP TO SIMILARITY

In this section we prove Theorem 0.2. We start by fixing some notation. Let g be an embedded loop in $\mathbb{C}$, and let $k : g \rightarrow \mathbb{R}$ be its curvature. We assume that g is positively oriented, so that the interior is to the left. Let P_1, P_2 be two distinct points in g . We denote by $A_g(P_1, P_2)$ the closed arc of g having endpoints P_1 and P_2 , oriented accordingly to the orientation of g . The open arc will be denoted by $\overset{\circ}{A}_g(P_1, P_2)$.

Following Osserman's definition in [10], we let $C_R^g(X) = \{P \in \mathbb{C} \mid |P - X| = R\}$ be the circumscribed circle about g . Thus, R is the minimum positive number r such that there exists a circle of radius r including g .

To prove Theorem 0.2 we need a preliminary result, which is based on Hopf's maximum principle. More precisely, the next lemma is a consequence of the *Touching Lemma* for the mean curvature operator. Although it is essentially well known, we state it here for the sake of completeness.

Lemma 4.1. *Let $P \in g \cap C_R^g(X)$. If $k(M) \leq 1/R$ for every $M \in g$ close to P , then $g \cap C_R^g(X)$ contains an arc around P .*

Proof. Up to a rotation and a translation we can assume that $X = 0$ and $P = iR$. An arc of $C_R(0)$ around P is the graph of the function $\gamma(t) = \sqrt{R^2 - t^2}$. Since g is an immersion, then g is locally the graph of a function $f : I = (-\delta, \delta) \rightarrow \mathbb{R}$ such that $f(0) = R$. By assumption $k \leq 1/R$ in a neighborhood of P , and therefore

$$\left(\frac{\gamma'}{\sqrt{1 + |\gamma'|^2}} \right)' + \frac{1}{R} = 0 \leq \left(\frac{f'}{\sqrt{1 + |f'|^2}} \right)' + \frac{1}{R}$$

for any t close to 0. In addition we have that $f \leq \gamma$ and $f(0) = \gamma(0) = R$. Thus $f \equiv \gamma$ in a neighborhood of 0 by Theorem 2.3 in [11]. $\square$

Proof of Theorem 0.2. Let k and g be as in the statement of the theorem. Then g is convex, since its curvature is positive. Let $C_R^g(X)$ be the circumscribed circle about g . By Lemma 3 in [10] it turns out that $k(P) \geq 1/R$ for any $P \in g \cap C_R^g(X)$.

We claim that g coincides with its circumscribed circle. By contradiction, we assume that $g \cap C_R^g(X)$ is strictly contained in $C_R^g(X)$. We distinguish two cases, depending on the number of connected components in $C_R^g(X) \setminus g$.

Case 1. The set $C_R^g(X) \setminus g$ is not connected. Then we can fix two distinct points P_1 and P_2 on $g \cap C_R^g(X)$ such that the arcs $A_g(P_1, P_2)$ and $A_g(P_2, P_1)$ are not contained in the circle $C_R^g(X)$. By Lemma 4 in [10], there exist two points, $Q_1 \in \overset{\circ}{A}_g(P_1, P_2)$ and $Q_2 \in \overset{\circ}{A}_g(P_2, P_1)$, such that

$$|Q_i - X| < R = |P_j - X|, \quad k(Q_i) < 1/R \leq k(P_j),$$

for $i, j = 1, 2$. Since k is a non-increasing function of the distance from the origin, then $|Q_i| > |P_j|$. Thus $X \neq 0$ and

$$0 < |Q_i|^2 - |P_j|^2 = |Q_i - X|^2 - |P_j - X|^2 + 2X \cdot (Q_i - P_j) < 2X \cdot (Q_i - P_j);$$

that is,

$$(4.1) \quad \min\{X \cdot Q_1, X \cdot Q_2\} > \max\{X \cdot P_1, X \cdot P_2\}.$$

Denote by $[Q_1, Q_2]$ the segment joining Q_1 and Q_2 , and by $[P_1, P_2]$ the segment joining P_1 and P_2 . By the convexity of the curve g the segments $[Q_1, Q_2]$ and $[P_1, P_2]$ intersect at a point Q . Since $Q \in [Q_1, Q_2]$, then $X \cdot Q > \max\{X \cdot P_1, X \cdot P_2\}$ by (4.1). On the other hand, $X \cdot Q \leq \max\{X \cdot P_1, X \cdot P_2\}$ as $Q \in [P_1, P_2]$, a contradiction. Thus Case 1 is excluded.

Case 2. The set $C_R^g(X) \setminus g$ is connected. We find $P_1, P_2 \in g$ such that

$$C_R^g(X) \setminus g = \overset{\circ}{A}_g(P_2, P_1)$$

and such that the arc $A_g(P_1, P_2)$ is contained in $C_R^g(X)$. In particular $k(P) = 1/R$ for any $P \in A_g(P_1, P_2)$. Since $C_R^g(X)$ is the smallest circle circumscribing g , then $A_g(P_1, P_2)$ is larger than a semicircle ([10], Lemma 2). In particular the antipodal $\tilde{P}_1 = 2X - P_1$ to P_1 belongs to $A_g(P_1, P_2)$. Up to a rotation and up to a change of indexes we can assume that $P_1, \tilde{P}_1$ lie on the same vertical line $\mathcal{V}$, with P_1 above $\tilde{P}_1$ and with $A_g(P_1, \tilde{P}_1)$ on the left of $\mathcal{V}$.

By Lemma 4 in [10], there exists a point $Q \in \overset{\circ}{A}_g(P_2, P_1)$ such that $k(Q) < 1/R$. Since g is convex, then Q lies on the right of the vertical line $\mathcal{V}$. By Lemma 4.1 there exists $M \in A_g(Q, P_1)$ on the right of $\mathcal{V}$ such that $k(M) > 1/R$. Since $k \equiv 1/R$

on $A_g(P_1, P_2)$ and since k is radially symmetric and non-increasing by assumption, then

$$(4.2) \quad |M| < |P| < |Q| \quad \text{for any } P \in A_g(P_1, P_2) \subset C_R^g(X).$$

In particular, $X \neq 0$, as $|X - Q| < R$. We put $P_0 := X(1 - R|X|^{-1}) \in C_R^g(X)$ and $\tilde{P}_0 := 2X - P_0 \in C_R^g(X)$.

We claim that 0 cannot belong to $\mathbb{C} \setminus \overline{D_R(X)}$. Indeed, if $|X| \geq R$, then the point P_0 is the minimal distance projection of 0 on $\overline{D_R(X)}$. Since $|M| < |P|$ for any $P \in A_g(P_1, P_2)$ and since M is in the interior of $D_R(X)$, then

$$P_0 \in \mathring{A}_g(P_2, P_1) \subset \mathring{A}_g(\tilde{P}_1, P_1).$$

But then, $\tilde{P}_0 \in A_g(P_1, \tilde{P}_1)$. Thus $|\tilde{P}_0| < |Q|$ by (4.2). However, this is impossible, as $|\tilde{P}_0| = |X| + R$ and $|Q| \leq |Q - X| + |X| < R + |X|$.

Thus $0 \in D_R(X)$ and $|P_0| = R - |X| > 0$. Since $|P_1|, |\tilde{P}_1| < |Q|$ and since Q lies on the right of the axes $\mathcal{V}$ joining $P_1, \tilde{P}_1$, then 0 is on the left of $\mathcal{V}$. Therefore $P_0 \in A_g(P_1, \tilde{P}_1) \subset C_R^g(X)$ and $k(P_0) = 1/R$. Thus $k \equiv 1/R$ on $C_{|P_0|}(0)$ and $k \leq 1/R$ outside $D_{|P_0|}(0)$, as k is radially symmetric and non increasing. Finally, we notice that the circle $C_{|P_0|}(0)$ is tangent to $C_R^g(X)$ at P_0 from the interior. Thus $k \leq 1/R$ in a neighborhood of P_1 , and therefore P_1 is in the interior of $g \cap C_R^g(X)$ by Lemma 4.1, a contradiction. The theorem is completely proved. $\square$

Theorem 0.2 is sharp in view of the next example.

Example 4.2. There exist a positive, radially symmetric and increasing curvature $k : \mathbb{C} \rightarrow \mathbb{R}$ that has a round k -loop and a non-round embedded k -loop.

To exhibit such a curvature we fix a pair of positive numbers $a < b$. Let $k_{a,b}$ be any smooth function such that

$$(4.3) \quad k_{a,b}(z) := \frac{ab}{(a^2 + b^2 - |z|^2)^{3/2}}$$

on the disk $\{|z| \leq b\}$. Notice that $k_{a,b}$ can be taken to be radially symmetric and strictly increasing as a function of the distance from the origin.

The ellipse $E_{a,b}$ of equation $a^{-2}x^2 + b^{-2}y^2 = 1$ has curvature k at any point. For the proof, it is convenient to parametrize $E_{a,b}$ by $g(t) = (a \cos t, b \sin t)$.

By elementary continuity arguments there exists a radius $R \in (a, b)$ such that $Rk_{a,b}(z) \equiv 1$ on the circle C_R , coherently with Remark 3.3. Thus the circle C_R and the ellipse $E_{a,b}$ are distinct k -loops.

Theorem 0.2 provides a new characterization of circles.

Corollary 4.3. *Round circles are the only convex loops in $\mathbb{C}$ whose curvature is a non increasing function of the Euclidean distance from a fixed point.*

We conclude the paper by pointing out the following uniqueness (up to homothety) result, which is an immediate consequence of Theorem 0.2.

Corollary 4.4. *Let k be a continuous, positive and radially symmetric function. If k is radially decreasing, then any embedded k -loop is a circle around the origin.*

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