A NOTE ON COMPOSITION OPERATORS ACTING ON HOLOMORPHIC SOBOLEV SPACES

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ABSTRACT. A holomorphic self-map φ of the unit disk is constructed such that the composition operator induced by φ is bounded on the Hardy-Sobolev space H_2^1 of order 2 as well as on the ordinary holomorphic Lipschitz space Lip_1 but unbounded on the Zygmund class Λ_1 . Among these three function spaces we have embedding relations $H_2^1 \subset \text{Lip}_1 \subset \Lambda_1$. So, the main points here are that our construction provides a composition operator which is bounded on smaller spaces, but not on a larger space and that all the function spaces involved are standard ones.

1. INTRODUCTION

Let $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$ and φ be a holomorphic self-map of $\mathbf{D}$, i.e., $\varphi(\mathbf{D}) \subset \mathbf{D}$. The *composition operator* C_φ induced by such a φ is the linear operator on the space of all holomorphic functions on $\mathbf{D}$ defined by $C_\varphi(f) = f \circ \varphi$. Books of Cowen and MacCluer [5] and Shapiro [12] are good references for various aspects of the basic theory of composition operators.

It is typically (but not always; see below) the case that when a composition operator is bounded on a smaller space, it is also bounded on a larger space. To illustrate such a phenomenon for holomorphic Sobolev spaces, we recall the holomorphic version of the well-known Sobolev embedding theorem: For $\alpha \geq -1$, $s > 0$ and $0 < p < \infty$, we have

$$(1.1) \quad sp > \alpha + 2 \implies A_{\alpha,s}^p \subset \Lambda_{s-(\alpha+2)/p};$$

see [2, Section 5]. Here $A_{\alpha,s}^p$ is the α -weighted Bergman-Sobolev space on $\mathbf{D}$ consisting of all holomorphic functions with fractional derivatives of order s belonging to the well-known α -weighted Bergman space A_α^p , and Λ_a denotes the holomorphic Lipschitz space of order $a > 0$. Here, $A_{-1,s}^p$ is to be interpreted as the Hardy-Sobolev space, denoted by H_s^p . Explicit definitions will be given in Section 2.

It is known that if C_φ is bounded on $A_{\alpha,s}^p$ and $\alpha + 2 < sp < \alpha + 2 + p$, then C_φ is also bounded on $\Lambda_{s-(\alpha+2)/p}$; see [3, Theorem 3.2]. The same implication was later

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shown to hold for $sp > \alpha + 2 + p$, leaving only those parameters with $sp = \alpha + 2 + p$ missing; see [4, Theorems 1.3 and 1.4]. The following natural question concerning the missing critical case was posed in [4, Problem 3.9]:

Let $sp = \alpha + 2 + p$ and suppose that C_φ is bounded on $A_{\alpha,s}^p$. Does it follow that C_φ is bounded on Λ_1 ?

Here we present the unexpected result that the answer to this question is “No” in general.

Theorem 1.1. *There exists a holomorphic self-map φ of $\mathbf{D}$ such that the composition operator C_φ is bounded on H_2^1 and bounded on Lip_1 , but not bounded on Λ_1 .*

As mentioned above, other examples are known of composition operators bounded on a smaller space, but not bounded on a larger space. One can be found in [5, Example 3.4], with the Hardy space H^2 as the smaller space and a certain weighted sequence space as the larger space. While that example is simple, with $\varphi(z) = z^2$ being the inducing function for the composition operator, the example given here is the only one we know that involves only “standard” function spaces. The tradeoff is that the construction of the inducing map φ in our example is rather subtle.

In Section 2 we review the holomorphic Sobolev spaces and Lipschitz spaces and provide some background on the hyperbolic metric that will be used in our construction of the holomorphic self-map φ of $\mathbf{D}$ to prove Theorem 1.1. That construction will be carried out in Section 3.

Notation. In the rest of the paper we use the notation $A \lesssim B$ or $B \gtrsim A$ for nonnegative quantities A and B to mean $A \leq CB$ for some inessential constant $C > 0$. Similarly, we use the notation $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$ hold. We will also use the notation $A(x) \sim B(x)$ as $x \rightarrow x_0$ for positive quantities $A(x)$ and $B(x)$ to mean that $A(x)$ and $B(x)$ are asymptotic; i.e., $A(x)B(x)^{-1} \rightarrow 1$ as $x \rightarrow x_0$.

2. BACKGROUND

2.1. Holomorphic Lipschitz spaces. For $0 < \epsilon < 1$, we let Λ_ϵ denote the space of all functions f holomorphic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$ such that

$$(2.1) \quad \|f\|_{\Lambda_\epsilon} := |f(0)| + \sup \frac{|f(z) - f(w)|}{|z - w|^\epsilon} < \infty,$$

where the supremum is taken over all $z, w \in \mathbf{D}$, $z \neq w$. For $\epsilon = 1$, we use Λ_1 to denote the Zygmund class of all functions f holomorphic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$ such that

$$\|f\|_{\Lambda_1} := |f(0)| + |f'(0)| + \sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbf{D}$ and $h > 0$. Equipped with norms $\|\cdot\|_{\Lambda_\epsilon}$, these spaces Λ_ϵ are all Banach spaces.

We remark in passing that the classes Λ_ϵ , $0 < \epsilon \leq 1$, admit well-known characterizations in terms of the growth rate of the first derivative for $0 < \epsilon < 1$ and of the second derivative for $\epsilon = 1$; see, for example, [5] or [6]. Also, in conjunction with the embedding property (1.1), we remark (although not needed in this paper) that

the notion of holomorphic Lipschitz spaces extends to arbitrary order in a standard way. Namely, given a positive integer n and $0 < \epsilon \leq 1$, the space $\Lambda_{n+\epsilon}$ consists of all functions f holomorphic on $\mathbf{D}$ and n -times continuously differentiable on $\overline{\mathbf{D}}$ such that $f^{(n)} \in \Lambda_\epsilon$ (with norm appropriately adjusted).

A characterization of when a composition operator is bounded on the Zygmund class has been known by various authors; see [9, Theorem 1], [8, Theorem 2.1] or [4, Theorem 2.2].

Theorem 2.1 ([9], [8], [4]). *Let φ be a holomorphic self-map of $\mathbf{D}$. Then C_φ is bounded on Λ_1 if and only if*

$$\sup_{z \in \mathbf{D}} \frac{|\varphi'(z)|^2(1-|z|)}{1-|\varphi(z)|} < \infty$$

and

$$(2.2) \quad \sup_{z \in \mathbf{D}} |\varphi''(z)|(1-|z|) \log \left(\frac{2}{1-|\varphi(z)|} \right) < \infty.$$

Let Lip_1 be the space of all functions corresponding to $\epsilon = 1$ in (2.1). It is well known and elementary that $f \in \text{Lip}_1$ if and only if $f' \in H^\infty$, the space of all bounded holomorphic functions on $\mathbf{D}$, with $\|f\|_{\text{Lip}_1} \approx |f(0)| + \|f'\|_\infty$. Also elementary is that C_φ is bounded on Lip_1 if and only if $\varphi' \in H^\infty$. Of course, $\text{Lip}_1 \subset \Lambda_1$.

2.2. Holomorphic Sobolev spaces. For $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A_α^p is the space of all functions f holomorphic on $\mathbf{D}$ for which

$$\|f\|_{A_\alpha^p} := \left\{ \int_{\mathbf{D}} |f(z)|^p dA_\alpha(z) \right\}^{1/p} < \infty.$$

Here dA_α denotes the weighted measure $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z)$, where dA is the area measure on $\mathbf{D}$. Also, the Hardy space H^p is the space of all f holomorphic on $\mathbf{D}$ for which

$$\|f\|_{H^p} := \sup_{0 < r < 1} \left\{ \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty.$$

We will often use the notation $A_{-1}^p = H^p$, which allows unified statements.

For $s \in \mathbf{R}$ and $f(z) = \sum_{n=0}^\infty a_n z^n$ holomorphic on $\mathbf{D}$, the s -fractional derivative $\mathcal{R}^s f$ of f is defined by

$$\mathcal{R}^s f(z) = \sum_{n=0}^\infty (1+n)^s a_n z^n, \quad z \in \mathbf{D};$$

see [2]. For $\alpha \geq -1$, $s \geq 0$, and $0 < p < \infty$, the holomorphic Sobolev space $A_{\alpha,s}^p$ is defined to be the space of all functions f holomorphic on $\mathbf{D}$ such that $\mathcal{R}^s f \in A_\alpha^p$ with “norm”

$$\|f\|_{A_{\alpha,s}^p} := \|\mathcal{R}^s f\|_{A_\alpha^p}.$$

As mentioned above, we often write $H_s^p = A_{-1,s}^p$. In case $1 \leq p < \infty$, which is the case for the space H_2^1 to appear in our example, the spaces $A_{\alpha,s}^p$ are all Banach spaces. In conjunction with the question stated in the introduction, we clarify that the term “boundedness” even for $0 < p < 1$ refers to boundedness with respect to the above norm.

When s is an integer, the Sobolev norm can be computed using ordinary derivatives. That is, when s is a positive integer,

$$\|f\|_{A_{\alpha,s}^p} \approx \sum_{k=0}^{s-1} |f^{(k)}(0)| + \|f^{(s)}\|_{A_{\alpha}^p};$$

see, for example, [4]. Thus, for example, $f \in H_2^1$ if and only if $f' \in H_1^1$ if and only if $f'' \in H^1$. Meanwhile, a classical theorem of Privalov ([6, Theorem 3.11]) states that functions in H_1^1 continuously extend up to $\overline{\mathbf{D}}$ and their boundary functions are absolutely continuous on $\partial\mathbf{D}$. In particular, $H_2^1 \subset \text{Lip}_1$.

Composition operators acting on holomorphic Sobolev spaces were studied by the current authors in [3] and [4]. One result that will be used below is that C_{φ} is bounded on H_2^1 if and only if $\varphi \in H_2^1$; see [13] or [4, Theorem 1.4].

2.3. The hyperbolic metric. The hyperbolic metric will play an important role in our construction. On the unit disk the hyperbolic metric $\rho_{\mathbf{D}}$ is defined by (see [1, p. 2])

$$\rho_{\mathbf{D}}(z_1, z_2) = \inf \left\{ \int_{\gamma} \frac{2|dz|}{1-|z|^2} : \gamma \text{ is an arc in } \mathbf{D} \text{ from } z_1 \text{ to } z_2 \right\}.$$

For example, $\rho_{\mathbf{D}}(0, z) = \log \frac{1+|z|}{1-|z|}$ is computed by integrating along the radius from 0 to z . This metric is invariant under conformal self-maps of the disk and therefore transfers to a natural conformally invariant metric on any simply connected domain $\Omega \subsetneq \mathbf{C}$. If $\psi : \mathbf{D} \rightarrow \Omega$ is any conformal map, the *hyperbolic distance* on Ω is given by $\rho_{\Omega}(w_1, w_2) = \rho_{\mathbf{D}}(z_1, z_2)$, where $w_j = \psi(z_j)$ for $j = 1, 2$. Since

$$\frac{2|dz|}{1-|z|^2} = \frac{2|dw|}{|\psi'(z)|(1-|z|^2)}, \quad \text{where } w = \psi(z),$$

ρ_{Ω} can be computed by integrating the function

$$(2.3) \quad h_{\Omega}(w) = \frac{2}{|\psi'(z)|(1-|z|^2)}, \quad w = \psi(z),$$

over arcs in Ω . The following lemma is immediate from this observation.

Lemma 2.2. *If γ is a crosscut of a simply connected domain Ω that separates the points a and b in Ω , then*

$$\rho_{\Omega}(a, b) \geq \inf_{c \in \gamma} \rho_{\Omega}(c, b).$$

Let $\delta_{\Omega}(z)$ denote the distance from a point z in a region Ω to the boundary of Ω . It is an easy consequence of the Koebe Distortion Theorem that if ψ is a conformal map of $\mathbf{D}$ onto Ω , then

$$(2.4) \quad \frac{1}{4}(1-|z|^2)|\psi'(z)| \leq \delta_{\Omega}(\psi(z)) \leq (1-|z|^2)|\psi'(z)|;$$

see [11, Corollary 1.4]. Along with (2.3), this leads to the very useful Euclidean lower bound for the hyperbolic distance given in the following lemma.

Lemma 2.3 ([12, Distance Lemma, p. 157]). *If a and b are points of a simply connected domain Ω , then*

$$\rho_{\Omega}(a, b) \geq \frac{1}{2} \log \left(1 + \frac{|a-b|}{\min\{\delta_{\Omega}(a), \delta_{\Omega}(b)\}} \right).$$

3. PROOF

This section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1. For each integer $n \geq 1$ define

$$\Gamma_n := \{z \in \mathbf{D} : |1 - z| = 2^{-n} \text{ and } |\operatorname{Im} z| \geq \epsilon_n\},$$

where $\epsilon_n > 0$ is sufficiently small and will be chosen later so that ϵ_n tends to 0 as $n \rightarrow \infty$. Now set

$$\Omega := \mathbf{D} \setminus \bigcup_{n=1}^{\infty} \Gamma_n,$$

and let ψ be the Riemann map of $\mathbf{D}$ onto Ω such that $\psi(0) = 0$ and $\psi'(0) > 0$. Since Ω is symmetric about the real axis, the uniqueness of the Riemann map implies that ψ is real on and only on $(-1, 1)$. Moreover, ψ maps the upper (lower, resp.) half of $\mathbf{D}$ onto the upper (lower, resp.) half of Ω . Observe that the boundary of Ω has finite one-dimensional Hausdorff measure, and so $\psi' \in H^1$ by [10, Theorem 10.11] (see also [7, Exercise VI.1]) and thus that ψ extends to be continuous on the closed disk $\overline{\mathbf{D}}$ by a theorem of Privalov ([6, Theorem 3.11]) mentioned before. Moreover, we have $\psi(1) = 1$ (and $\psi(-1) = -1$), because $\psi'(0) > 0$.

For each integer $n \geq 1$, set

$$x_n = 1 - \frac{1}{2} \left(\frac{1}{2^n} + \frac{1}{2^{n+1}} \right)$$

so that x_n is midway between Γ_n and Γ_{n+1} , and $\delta_n := \delta_{\Omega}(x_n) \approx 2^{-n}$ for all n . Then, with $r_n \in (-1, 1)$ defined so that $\psi(r_n) = x_n$, from (2.4) we have

$$(3.1) \quad |\psi'(r_n)|(1 - r_n^2) \approx \delta_n \approx 2^{-n}$$

for all n . It is important to note that this estimate is independent of the choice of $\{\epsilon_n\}$. Next, we show below that this sequence can be chosen to make the distances $\rho_n := \rho_{\mathbf{D}}(0, r_n) = \rho_{\Omega}(0, x_n)$ as large as needed.

For each integer $n \geq 1$, let

$$\gamma_n := \{z \in \mathbf{D} : |1 - z| = 2^{-n} \text{ and } |\operatorname{Im} z| < \epsilon_n\} \subset \Omega.$$

Note that γ_n is a crosscut of Ω that separates the points 0 and x_n . To begin with, choose $\epsilon_n > 0$ such that

$$(3.2) \quad \sup_{w \in \gamma_n} \delta_{\Omega}(w) \leq 2\epsilon_n \leq \delta_{\Omega}(x_n),$$

which holds for all ϵ_n sufficiently small (when n is fixed). Accordingly, applying Lemmas 2.2 and 2.3, we see from (3.2) that

$$\begin{aligned} \rho_n = \rho_{\Omega}(0, x_n) &\geq \inf_{w \in \gamma_n} \rho_{\Omega}(w, x_n) \\ &\geq \frac{1}{2} \inf_{w \in \gamma_n} \log \left(1 + \frac{|w - x_n|}{2\epsilon_n} \right). \end{aligned}$$

Thus, since

$$\inf_{w \in \gamma_n} |w - x_n| = x_n - \left(1 - \frac{1}{2^n} \right) = \frac{1}{2^{n+2}},$$

we obtain

$$\rho_n \geq \frac{1}{2} \log \left(1 + \frac{1}{\epsilon_n 2^{n+3}} \right)$$

for each $n \geq 1$. This shows that we can make ρ_n as large as we want by choosing $\epsilon_n > 0$ sufficiently small. We now make this choice to insure that $2^{-n}\rho_n \geq n$, in addition to (3.2), for each $n \geq 1$. Then, from (3.1),

$$(3.3) \quad |\psi'(r_n)|(1-r_n^2) \log \frac{1+r_n}{1-r_n} = |\psi'(r_n)|(1-r_n^2)\rho_n \gtrsim n$$

for all n .

Let g be the holomorphic function on $\mathbf{D}$ determined by the conditions that $g(0) = 0$ and $g' = \psi + 1$. Note that g is continuously differentiable on $\overline{\mathbf{D}}$. Since $g' > 0$ on $(-1, 1)$, we have

$$(3.4) \quad g(x_1) < g(x_2), \quad -1 < x_1 < x_2 < 1.$$

Also, since ψ maps the upper half of $\mathbf{D}$ onto the upper half of Ω , we have $\operatorname{Im} \psi(x+it) > 0$ for $x+it \in \mathbf{D}$ (of course x and t denote real numbers) with $t > 0$ and thus

$$(3.5) \quad \operatorname{Re} [g(x+iy) - g(x)] = \operatorname{Re} \int_0^y \psi(x+it)i \, dt < 0$$

for all $x+iy \in \mathbf{D}$ with $y > 0$. The same inequality remains valid for $y < 0$, because ψ maps the lower half of $\mathbf{D}$ onto the lower half of Ω . It follows from (3.4) and (3.5) that

$$(3.6) \quad \operatorname{Re} g(z) < g(1) = \int_0^1 \psi(x) \, dx + 1, \quad z \in \mathbf{D}.$$

Setting $s := g(1)$, we thus see that g maps $\mathbf{D}$ into the half-plane $\Pi_s := \{z \in \mathbf{C} : \operatorname{Re} z < s\}$. Note $s > 0$.

Let σ denote the Möbius map from the half-plane Π_s onto $\mathbf{D}$ determined by the conditions $\sigma(0) = 0$, $\sigma(s) = 1$ and $\sigma(\infty) = -1$, i.e.,

$$\sigma(z) = \frac{z}{2s-z}, \quad z \in \Pi_s.$$

Now let φ be the holomorphic self-map of $\mathbf{D}$ defined by $\varphi = \sigma \circ g$, and note that $\varphi(0) = 0$ and $\varphi(1) = 1$. Also, note that φ is real on $(-1, 1)$.

Note that $|\psi'(r_n)| \rightarrow \infty$ as $n \rightarrow \infty$ by (3.3). Also, $r_n \rightarrow 1$ as $n \rightarrow \infty$. Thus, since $g' = \psi + 1$, and σ is holomorphic at s with $\sigma'(s) = 2/s$, we compute

$$(3.7) \quad \begin{aligned} |\varphi''(r_n)|(1-r_n^2) &= \left| \psi'(r_n)\sigma'(g(r_n)) + (\psi(r_n)+1)^2\sigma''(g(r_n)) \right| (1-r_n^2) \\ &\sim \frac{2}{s} |\psi'(r_n)|(1-r_n^2) \end{aligned}$$

as $n \rightarrow \infty$.

Since $\varphi(r_n)$ is real and converges to 1 as $n \rightarrow \infty$, we see that $\varphi(r_n)$ is positive for all large n . Thus, to each of such large n corresponds $t_n \in (r_n, 1)$ such that

$$1 - |\varphi(r_n)| = \varphi(1) - \varphi(r_n) = \varphi'(t_n)(1-r_n)$$

by the Mean Value Theorem. Note that $g'(1) = \psi(1) + 1 = 2 > 0$; one may also use (3.6) and the Hopf Lemma to see the positivity of $g'(1)$, which we need below. Now, since

$$\varphi'(t) = \sigma'(g(t))g'(t) \rightarrow \sigma'(s)g'(1) = \frac{4}{s} > 0$$

as $t \rightarrow 1$, we see that

$$1 - |\varphi(r_n)| \sim \frac{4}{s}(1-r_n)$$

as $n \rightarrow \infty$ and thus

$$\log \left(\frac{1+r_n}{1-r_n} \right) \approx \log \left(\frac{2}{1-|\varphi(r_n)|} \right)$$

for all large n . Combining this estimate with (3.7) and (3.3), we obtain

$$|\varphi''(r_n)|(1-r_n^2) \log \frac{2}{1-|\varphi(r_n)|} \gtrsim n$$

for all large n , which shows that Condition (2.2) is violated for φ . So, Theorem 2.1 shows that C_φ is not bounded on Λ_1 , as required.

It remains to check that C_φ is bounded on H_2^1 and on Lip_1 . First note that $\sigma'(z) = 2s(2s-z)^{-2}$ and $\sigma''(z) = 4s(2s-z)^{-3}$ are both bounded on Π_s , which contains the range of g . So, $\sigma'(g)$ and $\sigma''(g)$ are both bounded on $\mathbf{D}$. Hence $\varphi' = (\psi+1)\sigma'(g) \in H^\infty$, and so C_φ is bounded on Lip_1 ; see the comment after Theorem 2.1. Also, since $\psi' \in H^1$, we have $\varphi'' = (\psi+1)^2\sigma''(g) + \psi'\sigma'(g) \in H^1$, which tells us that C_φ is bounded on H_2^1 ; see the remark at the end of subsection 2.2. This completes the proof. $\square$

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