FORMALITY OF PASCAL ARRANGEMENTS

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ABSTRACT. In this paper we construct a family of subspace arrangements whose intersection lattices have the shape of Pascal's triangle. We prove that even though the intersection lattices are not geometric, the complex complement of the arrangements are rationally formal.

1. INTRODUCTION

Let V be a complex vector space of dimension ℓ . In this paper, a subspace arrangement $\mathcal{A}$ is a finite collection of linear subspaces of V , and we assume that there are no inclusions between the elements of $\mathcal{A}$. Let $L(\mathcal{A})$ be the intersection lattice of $\mathcal{A}$ defined by all possible intersections of elements from $\mathcal{A}$ ordered by reverse inclusion. We call $M(\mathcal{A}) := V \setminus \bigcup_{X \in \mathcal{A}} X$ the complement of $\mathcal{A}$.

In this paper the main characters are the following class of arrangements, which we call *Pascal arrangements*.

Definition 1.1. Let n be a positive integer and let $\ell = 2n - 1$. For $1 \leq k \leq n$ let X_k be the subspace defined by

$$X_k = \{(v_1, \dots, v_\ell) \in \mathbb{C}^\ell \mid v_k = \dots = v_{k+n-1}\}.$$

Define the subspace arrangement $\mathcal{P}_n$ to be the collection $\{X_1, \dots, X_n\}$.

The arrangements $\mathcal{P}_n$ are a certain type of hypergraph arrangement as defined by Björner in [1] and further studied by Björner and Welker in [2], Kozlov in [8], and Peeva, Reiner and Welker in [10]. The authors came upon these arrangements through their recent work [9], in which we study a generalization of hypergraph arrangements. Also, the arrangements $\mathcal{P}_n$ were a main focus in a combinatorial study by Brockman and Sagan but were unfortunately not published (for reference see [3]).

An integral part of the study of the topology and combinatorics of subspace arrangements is that of the formality, or lack thereof, of the complex complement. The key work done by Yuzvinsky in [12] and by Feichtner and Yuzvinsky in [6] shows that if the intersection lattice of a subspace arrangement is geometric, then its complex complement is rationally formal. Both [12] and [6] rely on the wonderful models of De Concini and Procesi from [4]. Later, Denham and Suciu in [5], Grbić

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and Theriault in [7], and the authors in [9] all exhibited non-geometric subspace arrangements whose complex complements are not rationally formal.

The main result of this paper is that the complements of the $\mathcal{P}_n$ are, to the contrary, rationally formal in spite of their intersection lattices being far from geometric. In Section 2 we investigate the combinatorics of the arrangements $\mathcal{P}_n$. Then in Section 3 we prove that the complement is rationally formal.

2. COMBINATORICS OF $\mathcal{P}_n$

2.1. The intersection lattice. Fix a positive integer n ; then for each k with $1 \leq k \leq n$ the subspace X_k has dimension n . The X_k correspond to the atoms of the intersection lattice which we denote by k . Now we compute intersections of the subspaces which correspond to the join of atoms of the lattice. Let $1 \leq k_1 < \cdots < k_s \leq n$. Then since the dimension of the vector space is $2n - 1$ and the dimension of X_k is n , the intersection

$$X_{k_1} \cap \cdots \cap X_{k_s} = X_{k_1} \cap X_{k_s} = \{(v_1, \dots, v_\ell) \in \mathbb{C}^\ell \mid v_{k_1} = \cdots = v_{k_s+n-1}\}$$

has dimension $n + k_1 - k_s$ and the join is

$$k_1 \vee \cdots \vee k_s = k_1 \vee k_s.$$

We denote this subspace by $X_{(k_1, k_s)}$ and the corresponding element of the lattice by (k_1, k_s) . We rank the intersection lattice $L(\mathcal{P}_n)$ by codimension and note that this is not a ranking in the usual sense because the bottom element V has rank 0, but then the next level of the intersection lattice corresponding to all the atoms has rank $n - 1$. Each level above this increases in rank by one from the previous level, and the maximal element $(1, n)$ has rank $2n - 2$. The i^{th} level of $L(\mathcal{P}_n)$ for $n - 1 < i \leq 2n - 2$ consists of elements of the form (a, b) where $b - a = i - n + 1$. There are $2n - i - 1$ of these subspaces, and there are n atoms.

Now we examine the structure of the lattice by considering which elements are less than a fixed element (a, b) . In the case when $b = a + 1$ the element $(a, a + 1)$ is greater than the atoms a and $a + 1$. For arbitrary a and b the element (a, b) sits directly above $(a + 1, b)$ and $(a, b - 1)$. All other elements in the $(i - 1)^{\text{st}}$ level will not be comparable to (a, b) , and all other elements less than (a, b) will be less than either $(a + 1, b)$ or $(a, b - 1)$. Now the next proposition follows and justifies the name of these arrangements.

Proposition 2.1. *The intersection lattice of $\mathcal{P}_n$ with the element V removed has the shape of the top n rows of Pascal's triangle.*

Remark 2.2. Because of this shape, for $n \geq 3$ the intersection lattice $L(\mathcal{P}_n)$ is not geometric.

2.2. The characteristic polynomial. The characteristic polynomial has been a crucial invariant in the study of both hyperplane and subspace arrangements. In this subsection we compute the characteristic polynomials for $\mathcal{P}_n$. First we recall the definitions. For any subspace arrangement $\mathcal{A}$ the Möbius function $\mu : L(\mathcal{A}) \rightarrow \mathbb{Z}$ is defined recursively as $\mu(V) = 1$ and

$$\mu(X) = - \sum_{Y \subsetneq X} \mu(Y),$$

where $X \in L(\mathcal{A})$. Then the characteristic polynomial of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim(X)}.$$

Turning our attention to $\mathcal{P}_n$ we see that $\mu(X_i) = -1$, $\mu(X_{(a,a+1)}) = 1$, and $\mu(X_{(a,b)}) = 0$ whenever $b - a > 1$. Hence we have proved the following proposition.

Proposition 2.3. *The characteristic polynomial of $\mathcal{P}_n$ is*

$$\chi(\mathcal{P}_n, t) = (n-1)t^{n-1} - nt^n + t^{2n-1}.$$

We illustrate the ideas for $n = 4$ in the following example.

Example 2.4. Let $n = 4$. Then $\ell = 7$ and there are 4 subspaces X_1, X_2, X_3 , and X_4 . Figure 1 is a picture of the lattice with the corresponding lattice element at each vertex. The Möbius values of the atoms are all -1 , the Möbius values of the codimension 4 elements are 1, and the Möbius values of the higher codimension elements are all 0. Hence the characteristic polynomial is $\chi(\mathcal{P}_4, t) = 3t^3 - 4t^4 + t^7$.

FIGURE 1. The intersection lattice of $\mathcal{P}_4$

3. THE COMPLEX COMPLEMENT

In this section we compute the rational comology algebra of the complement of the arrangements $\mathcal{P}_n$ and show that the complements $M(\mathcal{P}_n)$ are rationally formal. The main tool is the relative atomic complex defined by Yuzvinsky in [11] and used by Feichtner and Yuzvinsky in [6] to prove that the complement of a subspace arrangement with a geometric intersection lattice is formal.

We now define the relative atomic complex. Let $\mathcal{A} = \{X_1, \dots, X_n\}$ be a subspace arrangement and fix an order on these elements, $X_1 < X_2 < \dots < X_n$. We associate the integer s with the subspace X_s . Then we will use $\sigma = \{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ to denote a subset of atoms in the intersection lattice of $\mathcal{A}$ such that $1 \leq i_1 < i_2 < \dots < i_k$. Let $D_{\mathcal{A}}$ be the differential graded algebra over $\mathbb{Q}$ with a generator a_σ in degree $2 \operatorname{codim} \bigvee \sigma - |\sigma|$ for each σ . The differential is defined by

$$(3.1) \quad da_\sigma = \sum_{j: \bigvee \sigma \setminus i_j = \bigvee \sigma} (-1)^j a_{\sigma \setminus i_j},$$

and the product structure is defined by

$$(3.2) \quad a_\sigma a_\tau = \begin{cases} (-1)^{\varepsilon(\sigma, \tau)} a_{\sigma \cup \tau} & \text{codim } \bigvee \sigma + \text{codim } \bigvee \tau = \text{codim } \bigvee (\sigma \cup \tau), \\ 0 & \text{otherwise,} \end{cases}$$

where $\varepsilon(\sigma, \tau)$ is the sign associated with the permutation that re-orders the linearly ordered $\sigma \cup \tau$ so that the elements of τ come after that of σ .

Feichtner and Yuzvinsky in [6] prove that the relative atomic complex, $D_{\mathcal{A}}$, is a rational model for the complex complement $M(\mathcal{A})$ by showing that it is quasi-isomorphic to the rational models of $M(\mathcal{A})$ developed by De Concini and Procesi in [4].

Returning to our examples we denote the relative atomic complex of $\mathcal{P}_n$ as P_n . Let $\sigma = \{k_1, \dots, k_s\}$ and notice that the join $\bigvee \sigma$ is the lattice element (k_1, k_s) and that the corresponding subspace has codimension $n - 1 + k_s - k_1$. Therefore, we have $\deg(a_\sigma) = 2(n - 1 + k_s - k_1) - s$. Now let $\tau = \{j_1, \dots, j_r\}$ with $k_1 \leq j_1$. The join $\sigma \vee \tau$ is $(k_1, \max\{k_s, j_r\})$, and that the corresponding subspace has codimension $\max\{k_s, j_r\} - k_1 + n - 1$.

Lemma 3.1. *The only non-trivial product in P_n is $a_{\{1\}} a_{\{n\}} = a_{\{1, n\}}$.*

Proof. We first consider when $j_r \leq k_s$. In this case we have $\bigvee (\sigma \cup \tau) = (k_1, k_s) = \bigvee \sigma$, which implies that $a_\sigma a_\tau = 0$. Whence we may assume that $j_r > k_s$. Recall that $a_\sigma a_\tau$ is non-zero when

$$n - 1 + k_s - k_1 + n - 1 + j_r - j_1 = \max\{k_s, j_r\} - k_1 + n - 1,$$

and since $j_r > k_s$, this reduces to

$$j_1 - k_s = n - 1.$$

This is true only when $j_1 = n$ and $k_s = 1$, which proves the lemma. $\square$

Define $T(k, j, s)$ to be the subcomplex of P_n generated by all a_σ where $\bigvee \sigma = (k, j)$ and $|\sigma| = s$. If σ satisfies these properties we say that σ has type (k, j, s) .

Lemma 3.2. *The algebra (P_n, d) has a decomposition $(P_n, d) = \bigoplus_{(k, j)} (T(k, j, *), d)$ and $d(T(k, j, s)) \subset T(k, j, s - 1)$.*

Proof. Let $\sigma = \{k_1, \dots, k_s\}$; the differential is determined by the formula

$$da_\sigma = \sum_{1 < p < s} (-1)^p a_{\sigma \setminus k_p}. \quad \square$$

We restrict our attention to the subalgebras $(T(k, j, *), \hat{d})$, where $\hat{d}$ is the restriction of the differential of P_n to $T(k, j, *)$ and more specifically to the sequence

$$(3.3) \quad \cdots \rightarrow T(k, j, s + 1) \rightarrow T(k, j, s) \rightarrow T(k, j, s - 1) \rightarrow \cdots,$$

where the maps are the restricted differential $\hat{d}_s : T(k, j, s) \rightarrow T(k, j, s - 1)$.

Lemma 3.3. *The differential $\hat{d}_s$ satisfies the inequality*

$$\dim (\text{Im } \hat{d}_s) \geq \binom{j - k - 2}{s - 3}.$$

Proof. There is a one-to-one correspondence between the set of all σ with type (k, j, s) that begin with $k, k+1, \dots$ and the set of σ with type $(k, j, s-1)$ that do not contain $k+1$. These sets have order $\binom{j-k-2}{s-3}$. The one-to-one correspondence is given by inserting (or deleting) the element $k+1$ from σ . $\square$

Lemma 3.4. *The sequence (3.3) is exact for $s \geq 3$.*

Proof. We show that the rank of the kernel of $\hat{d}_s$ is equal to the rank of the image of $\hat{d}_{s+1}$:

$$\begin{aligned} \dim(\text{Ker } \hat{d}_s) &= \dim(T(k, j, s)) - \dim(\text{Im } \hat{d}_s) \\ &\leq \binom{j-k-1}{s-2} - \binom{j-k-2}{s-3} \\ &= \binom{j-k-2}{s-2} \\ &\leq \dim(\text{Im } \hat{d}_{s+1}). \end{aligned}$$

The second and fourth lines follow from Lemma 3.3, and the third line is Pascal's rule. Since $\hat{d}_s$ is a differential, the opposite inequality is also true. $\square$

Proposition 3.5. *The rational cohomology algebra $H^*(\mathcal{P}_n)$ is isomorphic to the quotient of the polynomial algebra $\mathbb{Q}[\alpha_{\{j\}}, \alpha_{\{k, k+1\}}]$ by all products, where $1 \leq j \leq n$ and $1 \leq k \leq n-1$. The generators $\alpha_{\{k\}}$ have degree $2n-3$ and $\alpha_{\{k, k+1\}}$ has degree $2n-2$.*

Proof. By Lemma 3.2 it suffices to prove the theorem for a fixed $T(k, j, *)$. Lemma 3.4 reduces the problem to the cases $s = 1, 2$. By inspection we have $da_{\{k\}} = da_{\{k, j\}} = 0$ and $da_{\{k, i, j\}} = a_{\{k, j\}}$. Therefore, for $s = 2$ the only elements that are not in the image of the differential are $a_{\{k, k+1\}}$. $\square$

Now we state and prove the main theorem.

Theorem 3.6. *The complement of $\mathcal{P}_n$ is rationally formal.*

Proof. We construct a quasi-isomorphism from P_n to its homology. Define the map $F: P_n \rightarrow H(P_n)$ as the extension of

$$F(a_\sigma) = \begin{cases} \alpha_\sigma & \text{if } \sigma = \{k\}, \{k, k+1\}, \\ 0 & \text{if } \sigma \neq \{k\}, \{k, k+1\} \end{cases}$$

to an algebra map. We must check that F does extend to an algebra map and that it is a quasi-isomorphism. By Lemma 3.1 the only product in P_n is $a_{\{1\}}a_{\{n\}} = a_{\{1, n\}}$ and $F(a_{\{1\}})F(a_{\{n\}}) = 0 = F(a_{\{1, n\}})$ since there are no products in $H(P_n)$ by Proposition 3.5. Similarly, $F(d(a_\sigma)) = 0 = d(F(a_\sigma))$ since F is non-zero only on representatives of non-trivial homology classes. Lastly we note that $F(a_\sigma) = \alpha_\sigma = [a_\sigma]$ when $\sigma = \{k\}, \{k, k+1\}$, which shows that F induces the identity map on homology. Thus F is the desired quasi-isomorphism. $\square$

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